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$*$-RING N-HOMOMORPHISMS BETWEEN TOPOLOGICAL ALGEBRAS

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Abstract. We first obtain some results on $*$-ring n-homomorphisms between certain topological algebras. Indeed, if θ is a $*$-ring n-homomorphism from a functionally continuous topological $*$-algebra A into a symmetric commutative *lmc* Q-$*$-algebra B such that $M_A \neq \emptyset$ and $M_B \neq \emptyset$, then $\theta(\text{Rad}A)^{n-1} \subseteq \text{Rad}B$. We also show that there exists a decomposition of M_B under certain conditions. Finally, we show that if θ is a $*$-ring n-homomorphism from a Banach $*$-algebra onto a unital commutative C^*-algebra B, then there exists $k > 0$ such that $\|\theta(x)\| \leq k\|x\|$ for every $x \in A$.

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1. Introduction

We first present the notations, definitions and known results, which are related to our work. For further details one can refer, for example, to [2] and [3].

A locally multiplicatively convex (*lmc*) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms.

The set of all characters (non-zero complex-valued homomorphisms) of a algebra A is denoted by S_A. If A is a complex topological algebra, then the set of all continuous characters of A is denoted by M_A and it is called the topological spectrum, or simply spectrum of A. We always endow M_A with the Gelfand topology.

A $*$-algebra is an algebra with an involution. A topological(Banach) $*$-algebra is a topological(Banach) algebra and a $*$-algebra. An *lmc* $*$-algebra is a $*$-algebra which is also an *lmc* algebra with a family of seminorms $\mathcal{P} = (p_\alpha)$, such that $p_\alpha(a^*) = p_\alpha(a)$ for every α and all $a \in A$. A topological algebra A is a Q-algebra if the set of all quasi invertible elements of A, (q$-InvA$), is open in A. An *lmc*

Q-$*$-algebra is an *lmc* $*$-algebra that is also a Q-algebra. Let A be a $*$-algebra, we say that A is symmetric if $\widehat{x^*} = \overline{\widehat{x}}$ for every $x \in A$, where $\widehat{x}$ is Gelfand transform of x, and $\bar{\ }$ denotes the complex conjugate.

A left ideal I of an algebra A is a modular left ideal if there exists $u \in A$ such that $A(e_A - u) \subseteq I$, where $A(e_A - u) = \{x - xu : x \in A\}$. The Jacobson radical, Rad$A$, of A is the intersection of all maximal modular left ideals of A.

Let A and B be algebras and $n \geq 2$ be an integer. A map $\theta : A \to B$ is called a ring n-homomorphism if $\theta(a_1 + a_2) = \theta(a_1) + \theta(a_2)$ and

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2)\cdots\theta(a_n),$$

for all elements $a_1, a_2, ..., a_n \in A$. If θ is also a linear mapping, then it is called an n-homomorphism. A ring 2-homomorphism is then just a ring homomorphism in the usual sense. Obviously, each ring homomorphism is a ring n-homomorphism for every $n \geq 2$, but the converse is not true, in general. For example, if φ is a ring homomorphism, then $\theta = -\varphi$ is a ring 3-homomorphism which is not a ring homomorphism. For certain properties of n-homomorphisms one may refer to [1, 4, 5, 6, 7, 10, 12]. Let A and B be $*$-algebras and $\theta : A \to B$ be a ring n-homomorphism, we say that θ is a $*$-ring n-homomorphism if $\theta(x^*) = (\theta(x))^*$ for all $x \in A$.

In 1954 Kaplansky [8] proved that if θ is a ring isomorphism between semisimple complex Banach algebras, then θ can be decomposed into a linear part, a conjugate linear part and a non-continues part on a finite dimensional ideal. In 1999 Šemrl [11] proved that if X and Y are compact Hausdorff spaces and if $\theta : C(X) \to C(Y)$ is a $*$-ring homomorphism, then there exist a clopen decomposition $\{Y_{-1}, Y_0, Y_1\}$ of Y and a continuous map $\Phi : Y_{-1} \cup Y_1 \to X$ such that for every $f \in C(X)$

$$\theta(f)(y) = \begin{cases} \overline{f(\Phi(y))}, & y \in Y_{-1}, \\ 0, & y \in Y_0, \\ f(\Phi(y)), & y \in Y_1. \end{cases}$$

In 2000 T. Miura [9] proved that each $*$-ring homomorphism from a commutative Banach $*$-algebra A, into a non-radical commutative symmetric Banach $*$-algebra B, maps the Jacobson radical of A into the Jacobson radical of B, moreover if A is a non-radical, then there exists a decomposition of M_B. We show that if A and B are topological $*$-algebras such that B is a commutative symmetric *lmc* Q-$*$-algebra, $M_A \neq \emptyset$, $M_B \neq \emptyset$ and $\theta : A \to B$ is a $*$-ring n-homomorphism, then $\theta(\text{Rad}A)^{n-1} \subseteq \text{Rad}B$. We also show that if A and B are topological $*$-algebras such that $M_A \neq \emptyset$, $M_B \neq \emptyset$ and $\theta : A \to B$ is a $*$-ring n-homomorphism, then there exist a decomposition $\{Y_{-1}, Y_0, Y_1\}$ of M_B and a continuous map $\Phi : Y_{-1} \cup Y_1 \to M_A$ such that for $\psi \in Y_{-1} \cup Y_1$, there exists constant $L_\psi \in \mathbb{C}$ such that for every $x \in A$ we have

$$\widehat{\theta(x)}(\psi) = \begin{cases} \overline{\widehat{x}(\Phi(\psi))} L_\psi, & \psi \in Y_{-1}, \\ \widehat{x}(\Phi(\psi)) L_\psi, & \psi \in Y_1. \end{cases}$$

Also Y_{-1}, Y_1 are open subsets in M_B. Finally, we show that if θ is a $*$-ring n-homomorphism from a Banach $*$-algebra onto a unital commutative C^*-algebra B, then there exists $k > 0$ such that $\|\theta(x)\| \leq k\|x\|$ for every $x \in A$.

2. Main Results

A topological algebra A is called functionally continuous if every character on A is continuous, in other words, $S_A = M_A$. Clearly every Banach algebra is functionally continuous. It is also known that Q-algebras are functionally continuous [2, 2.2.28].

The following lemma has been proved by T. Miura [9], for commutative Banach $*$-algebras, but it is also valid for topological $*$-algebras.

Lemma 1. *Let A be a topological $*$-algebra such that A is functionally continuous. If $\theta : A \to \mathbb{C}$ is a $*$-ring homomorphism, then*

$$\theta = 0 \quad \text{or} \quad \theta \in M_A \quad \text{or} \quad \bar{\theta} \in M_A.$$

Theorem 2. *Let A and B be topological $*$-algebras such that A is functionally continuous, B is symmetric, $M_A \neq \emptyset$ and $M_B \neq \emptyset$. If $\theta : A \to B$ is a $*$-ring n-homomorphism, then*

$$\theta(\cap_{\varphi \in M_A} \ker \varphi)^{n-1} \subseteq \cap_{\varphi \in M_B} \ker \varphi.$$

Proof. We assume that $x \in \cap_{\varphi \in M_A} \ker \varphi$ and $\psi \in M_B$. If $\psi(\theta(x)) = 0$, then $\theta(x)^{n-1} \in \ker \psi$, otherwise, $\psi(\theta(x)) \neq 0$. By the equality

$$\begin{aligned} |\psi(\theta(x))|^{2n} &= (\psi(\theta(x))\overline{\psi(\theta(x))})^n = (\psi(\theta(x)\theta(x)^*))^n \\ &= (\psi(\theta(x)\theta(x^*)))^n = \psi((\theta(x)\theta(x^*))^n), \end{aligned}$$

one can see that $\psi(\theta(xx^*)) \neq 0$. Define $S_\psi : A \to \mathbb{C}$ to be

$$S_\psi(y) = \frac{\psi(\theta(xx^* y))}{\psi(\theta(xx^*))}.$$

Now we show that S_ψ is a $*$-ring homomorphism. For every $y_1, y_2 \in A$, we have

$$\begin{aligned} S_\psi(y_1y_2) =& \frac{\psi(\theta(xx^*y_1y_2))}{\psi(\theta(xx^*))} = \frac{\psi(\theta(xx^*y_1y_2))\psi(\theta(xx^*)^{n-1})}{\psi(\theta(xx^*))^n} \\ =& \frac{\psi(\theta(xx^*y_1y_2(xx^*)^{n-1}))}{\psi(\theta(xx^*))^n} \\ =& \frac{\psi(\theta(xx^*y_1))\psi(\theta(y_2xx^*))(\psi(\theta(xx^*)))^{n-2}}{\psi(\theta(xx^*))^n} \\ =& \frac{\psi(\theta(xx^*y_1))}{\psi(\theta(xx^*))} \cdot \frac{\psi(\theta(y_2xx^*))}{\psi(\theta(xx^*))}. \end{aligned}$$

Also,

$$\begin{aligned} \psi(\theta(y_2xx^*)) =& \frac{\psi((\theta(xx^*))^{n-1})}{\psi((\theta(xx^*))^{n-1})} \cdot \psi(\theta(y_2xx^*)) = \frac{\psi(\theta((xx^*)^{n-1}y_2xx^*))}{\psi((\theta(xx^*))^{n-1})} \\ =& \frac{\psi((\theta(xx^*))^{n-2})\psi(\theta(xx^*y_2))\psi(\theta(xx^*))}{\psi((\theta(xx^*))^{n-1})} = \psi(\theta(xx^*y_2)), \end{aligned}$$

so $S_\psi(y_1y_2) = S_\psi(y_1)S_\psi(y_2)$. On the other hand,

$$\begin{aligned} S_\psi(y^*) =& \frac{\psi(\theta(xx^*y^*))}{\psi(\theta(xx^*))} = \frac{\psi(\theta((yxx^*)^*))}{\psi(\theta((xx^*))^*)} = \overline{\left(\frac{\psi(\theta(yxx^*))}{\psi(\theta(xx^*))}\right)} = \overline{\left(\frac{\psi(\theta(xx^*y))}{\psi(\theta(xx^*))}\right)} \\ =& \overline{S_\psi(y)}. \end{aligned}$$

Now by Lemma 1 we have

$$S_\psi = 0 \quad \text{or} \quad S_\psi \in M_A \quad \text{or} \quad \overline{S_\psi} \in M_A.$$

Since $x \in \cap_{\varphi\in M_A} \ker\varphi$, in any cases, we have $S_\psi(x^{n-1}) = 0$. We also have

$$0 = S_\psi(x^{n-1}) = \frac{\psi(\theta(xx^*))\psi((\theta(x))^{n-1})}{\psi(\theta(xx^*))} = \psi((\theta(x))^{n-1}),$$

so $\theta(x)^{n-1} \in \ker\psi$. Since ψ is arbitrary, hence $\theta(x)^{n-1} \in \cap_{\varphi\in M_B} \ker\varphi$.

Corollary 3. *By the assumptions in Theorem 2, we have*

$$\theta(\mathrm{Rad}A)^{n-1} \subseteq \cap_{\varphi\in M_B} \ker\varphi.$$

Proof. By [3, 4.22(1)], $\mathrm{Rad}A \subseteq \cap_{\varphi\in M_A} \ker\varphi$.

Corollary 4. *In addition to the assumptions in Theorem 2, if B is a commutative lmc Q-$*$-algebra, then $\theta(\mathrm{Rad}A)^{n-1} \subseteq \mathrm{Rad}B$.*

Proof. By [3, 4.22(3)], $\mathrm{Rad}B = \cap_{\varphi \in M_B} \ker \varphi$. and Corollary 3.

Theorem 5. *Let A and B be topological $*$-algebras such that A is functionally continuous, B is symmetric, $M_A \neq \emptyset$ and $M_B \neq \emptyset$. If $\theta : A \to B$ is a $*$-ring n-homomorphism, then there exists a decomposition $\{Y_{-1}, Y_0, Y_1\}$ of M_B and a continuous map $\Phi : Y_{-1} \cup Y_1 \to M_A$ such that for $\psi \in Y_{-1} \cup Y_1$, there exists a constant $L_\psi \in \mathbb{C}$ such that for every $x \in A$ we have*

$$\widehat{\theta(x)}(\psi) = \begin{cases} \overline{\widehat{x}(\Phi(\psi))} L_\psi, & \psi \in Y_{-1}, \\ \widehat{x}(\Phi(\psi)) L_\psi, & \psi \in Y_1. \end{cases}$$

Also Y_{-1}, Y_1 are open subsets in M_B.

Proof. For every $\psi \in M_B$, we define function $T_\psi : A \to \mathbb{C}$ such that

$$T_\psi(x) = \psi(\theta(x)) = \hat{\theta(x)}(\psi).$$

We see that T_ψ is a $*$-ring n-homomorphism, if $T_\psi = 0$, then we define $S_\psi = 0$, otherwise, there exists $a_\psi \in A$ such that $T_\psi(a_\psi) \neq 0$. Considering the following relation

$$\begin{aligned} |\psi(\theta(a_\psi))|^{2n} =& (\psi(\theta(a_\psi))\overline{\psi(\theta(a_\psi))})^n = (\psi(\theta(a_\psi)\theta(a_\psi)^*))^n \\ =& (\psi(\theta(a_\psi)\theta(a_\psi^*)))^n = \psi((\theta(a_\psi)\theta(a_\psi^*))^n), \end{aligned}$$

it is easy to see that $T_\psi(a_\psi {a_\psi}^*) = \psi(\theta(a_\psi a_\psi^*)) \neq 0$. Now consider the function $S_\psi : A \to \mathbb{C}$ defined by $S_\psi(x) = \frac{T_\psi(a_\psi {a_\psi}^* x)}{T_\psi(a_\psi {a_\psi}^*)}$. Now by the same method used in Theorem 2 we can show that S_ψ is a $*$-ring homomorphism. By Lemma 1, $S_\psi = 0$ or $S_\psi \in M_A$ or $\overline{S_\psi} \in M_A$. We define Y_{-1}, Y_0 and Y_1 as follows

$$\begin{aligned} Y_{-1} =& \{\psi \in M_B \ : \ \overline{S_\psi} \in M_A\}, \\ Y_0 =& \{\psi \in M_B \ : \ S_\psi = 0\}, \\ Y_1 =& \{\psi \in M_B \ : \ S_\psi \in M_A\}. \end{aligned}$$

It is easy to check that the set $\{Y_{-1}, Y_0, Y_1\}$ is a decomposition of M_B and Y_{-1}, Y_1 are open subset in M_B. Now we define function $\Phi : Y_{-1} \cup Y_1 \to M_A$ as follows

$$\Phi(\psi) = \begin{cases} \overline{S_\psi}, & \psi \in Y_{-1}, \\ S_\psi, & \psi \in Y_1. \end{cases}$$

One can see that Φ is continuous by weak* topology and

$$\widehat{\theta(a_\psi {a_\psi}^* x)}(\psi) = \widehat{x}(\Phi(\psi))\psi(\theta(a_\psi {a_\psi}^*)).$$

By the equality

$$\begin{aligned}|\psi(\theta(a_\psi))|^{4n} =&(\psi(\theta(a_\psi))\overline{\psi(\theta(a_\psi))})^{2n} = (\psi(\theta(a_\psi)\theta(a_\psi)^*))^{2n}\\ =&(\psi(\theta(a_\psi)\theta(a_\psi^*)))^{2n} = \psi((\theta(a_\psi)\theta(a_\psi^*))^{2n}),\end{aligned}$$

we can show that $T_\psi((a_\psi a_\psi{}^*)^2) = \psi(\theta((a_\psi a_\psi{}^*)^2)) \neq 0$. Now for every $\psi \in Y_1$ and $x \in A$ we have

$$\begin{aligned}\widehat{\theta(a_\psi a_\psi{}^* x)}(\psi) =&\psi(\theta(a_\psi a_\psi{}^* x)) = \frac{\psi(\theta(a_\psi a_\psi{}^* x))\psi(\theta(a_\psi a_\psi{}^*)^{n-1})}{\psi(\theta(a_\psi a_\psi{}^*)^{n-1})}\\ =&\frac{\psi(\theta(a_\psi a_\psi{}^* x(a_\psi a_\psi{}^*)^{n-1})}{\psi(\theta(a_\psi a_\psi{}^*)^{n-1})}\\ =&\frac{\psi(\theta(a_\psi a_\psi{}^*))\psi(\theta(x))\psi(\theta(a_\psi a_\psi{}^*))^{n-3}\psi(\theta((a_\psi a_\psi{}^*)^2))}{\psi(\theta(a_\psi a_\psi{}^*)^{n-1})}\\ =&\frac{\psi(\theta(x))\psi(\theta((a_\psi a_\psi{}^*)^2))}{\psi(\theta(a_\psi a_\psi{}^*))}.\end{aligned}$$

So

$$\psi(\theta(x)) = \widehat{\theta(a_\psi a_\psi{}^* x)}(\psi)\frac{\psi(\theta(a_\psi a_\psi{}^*))}{\psi(\theta((a_\psi a_\psi{}^*)^2))} = \widehat{x}(\Phi(\psi))\frac{\psi(\theta(a_\psi a_\psi{}^*))}{\psi(\theta((a_\psi a_\psi{}^*)^2))}.$$

If $L_\psi = \dfrac{\psi(\theta(a_\psi a_\psi{}^*)^2)}{\psi(\theta((a_\psi a_\psi{}^*)^2))}$, then we have $\widehat{\theta(x)}(\psi) = \widehat{x}(\Phi(\psi))L_\psi$. By similar discussion, for every $\psi \in Y_{-1}$ and $x \in A$, we can say that there exists $L_\psi \in \mathbf{C}$ such that $\widehat{\theta(x)}(\psi) = \overline{\widehat{x}(\Phi(\psi))}L_\psi$.

Theorem 6. *Let A be a Banach $*$-algebra and B be a unital symmetric commutative Banach $*$-algebra such that $\|b\|_B = \|\hat{b}\|_\infty$ holds for every $b \in B$. If $\theta : A \to B$ is a surjective $*$-ring n-homomorphism, then there exists $k > 0$ such that $\|\theta(x)\| \leq k\|x\|$, for every $x \in A$.*

Proof. By hypothesis, there exists $a \in A$ such that $\theta(a) = 1_B$, so for every $\psi \in M_B$, $\psi(\theta(a)) = 1$. Now consider a_ψ, L_ψ and S_ψ defined in previous theorem. So we have

$$|\psi(\theta(x))| = |S_\psi(x)L_\psi| \leq \|x\||L_\psi|,$$

in other words,

$$|\psi(\theta(x))\psi(\theta((a_\psi a_\psi{}^*)^2))| \leq \|x\||\psi(\theta(a_\psi a_\psi{}^*))^2|.$$

Considering the following relation

$$\begin{aligned}\theta((a_\psi {a_\psi}^*)^2)) =&\theta((a_\psi {a_\psi}^*)^2))\theta(a)^{n-1} = \theta((a_\psi {a_\psi}^*)^2 a^{n-1}) = \theta((a_\psi {a_\psi}^*)^2 a^{n-1})\theta(a)^{n-1}\\ =&\theta((a_\psi {a_\psi}^*)^2 a^{2n-2}) = \theta(a_\psi {a_\psi}^*)^2\theta(a)^{n-3}\theta(a^{n+1})\\ =&\theta(a_\psi {a_\psi}^*)^2\theta(a)^{n-1}\theta(a^2)\\ =&\theta(a_\psi {a_\psi}^*)^2\theta(a^2),\end{aligned}$$

we conclude that $|\psi(\theta(x))\psi(\theta(a^2))| \leq \|x\|$. Now, by the equality

$$1 = \psi(\theta(a))^{2n} = \psi(\theta(a^n)\theta(a)^n) = \psi(\theta(a^{2n-1})\theta(a)),$$

we can see that $\psi(\theta(a^2)) \neq 0$. Hence, if $k = |\dfrac{1}{\psi(\theta(a^2))}|$, then we have

$$\|\theta(x)\| \leq k\|x\|.$$

So the desired result is obtained.

Corollary 7. *Let A be a Banach $*$-algebra and B be a unital commutative C^*-algebra. If $\theta : A \to B$ is a surjective $*$-ring n-homomorphism, then there exists $k > 0$ such that $\|\theta(x)\| \leq k\|x\|$, for every $x \in A$.*

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