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WEAK BIHARMONIC AND HARMONIC 1-TYPE CURVES IN SEMI-EUCLIDEAN SPACE $\mathbb{E}_1^4$

G. Öztürk, B.K. Bayram, K. Arslan

Abstract. In the present study we consider weak biharmonic and harmonic 1-type curves in semi-Euclidean space $\mathbb{E}_1^4$. We give the classifications of these type curves.

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1. Introduction

Chen and Ishikawa [1] classified biharmonic curves in semi-Euclidean space $\mathbb{E}_v^n$. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Further, Inoguchi gave a classification of biharmonic curves in semi-Euclidean 3-space. He pointed out that every biharmonic Frenet curve in Minkowski 3-space $\mathbb{E}_1^3$ is a helix whose curvature κ and torsion τ satisfy $\kappa^2 = \tau^2$ [2, 3]. In the classification of biharmonic curves in Minkowski 3-space due to Chen and Ishikawa [1], there exists biharmonic spacelike curves with null principle normal.

In [4] B. Kılıç et al. studied weak biharmonic submanifolds which satisfy the condition $\Delta^D \overrightarrow{H} = 0$. In [5] the authors studied submanifolds in Euclidean m-space $\mathbb{E}^m$ which satisfy the condition $\Delta^D \overrightarrow{H} + \lambda \overrightarrow{H} = 0$ and they called these manifolds harmonic 1-type.

In [6] the authors showed that the indefinite Cornu spirals are the only non standard curves in a semi-Riemannian manifold that are biharmonic in the normal bundle. As for surfaces, they dealed with the semi-Riemannian Hopf cylinders and they showed that the biharmonicity of them strongly depends on the biharmonicity of the curves to which are associated.

[7] the authors studied weak biharmonic curves whose mean curvature vector fields are in the kernel of normal Laplacian $\nabla^\perp$ in the Lorentzian 3-space L^3. They gave some characterizations and results for a Frenet curve in the same space.

In the present study we consider weak biharmonic and harmonic 1-type curves in semi-Euclidean space $\mathbb{E}_1^4$. We give the classifications of these type curves.

2. Basic Concepts

Let $\mathbb{E}_1^4$ be the Minkowski 4-space with natural Lorentz metric $\langle , \rangle = -dx^2 + dy^2 + dz^2 + dw^2$. Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be an arclength parametrized curve in the semi-Euclidean space $\mathbb{E}_1^4$.

Let $k_1 > 0$, k_2, k_3 be the Frenet curvatures of γ and $\left\{V_1 = \gamma', V_2, V_3, V_4\right\}$ be a Frenet frame along γ. Then the Frenet equations of γ can be written as,

$$\begin{array}{c} \nabla_{V_1} V_1 = \varepsilon_2 k_1 V_2 \\ \nabla_{V_1} V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3 \\ \nabla_{V_1} V_3 = -\varepsilon_2 k_2 V_2 + \varepsilon_4 k_3 V_4 \\ \nabla_{V_1} V_4 = -\varepsilon_3 k_3 V_3 \end{array} \tag{1}$$

where $\varepsilon_i = \langle V_i, V_i \rangle$, $i = 1, 2, 3, 4$ are the causal characters of V_1, V_2, V_3, V_4, respectively and ∇ is the Levi-civita connection on $\mathbb{E}_1^4$. A unit speed curve is said to be spacelike or timelike if its causal character is 1 or -1, respectively.

Let h be the second fundamental form associated to γ. Then the mean curvature vector field $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = trace\,(h) = \varepsilon_1 h\,(V_1, V_1) = \varepsilon_1 \nabla_{V_1} V_1 = \varepsilon_1 \varepsilon_2 k_1 V_2. \tag{2}$$

The Laplacian operator along γ is given by

$$\Delta = -\varepsilon_1 \nabla_{V_1} \nabla_{V_1}. \tag{3}$$

3. Weak Biharmonic and Harmonic 1-Type Curves

Let γ be a unit speed curve in Minkowski 4-space $\mathbb{E}_1^4$. The Laplacian operator along γ associated the connection in the normal bundle is defined by

$$\Delta^D = -\varepsilon_1 D_{V_1} D_{V_1}. \tag{4}$$

A straightforward computation leads to

$$\begin{aligned} \Delta^D \overrightarrow{H} &= -\varepsilon_1 D_{V_1} D_{V_1} \overrightarrow{H} = \left(-\varepsilon_2 k_1'' + \varepsilon_3 k_1 k_2^2\right) V_2 - \\ &\quad \varepsilon_2 \varepsilon_3 \left(2k_1' k_2 + k_1 k_2'\right) V_3 - \varepsilon_2 \varepsilon_3 \varepsilon_4 k_1 k_2 k_3 V_4. \end{aligned} \tag{5}$$

Definition 1. *Let γ be a unit speed curve in Lorentzian $\mathbb{E}_v^n$ satisfying the condition*

$$\Delta^D \overrightarrow{H} = 0, \tag{6}$$

then γ is called a weak biharmonic curve [4].

By the use of (5) we obtain the following result.

Lemma 1. *Let γ be a unit speed curve in Minkowski 4-space $\mathbb{E}_1^4$. If γ is of weak biharmonic then we have*

$$\begin{array}{c} \varepsilon_2 k_1^{''} - \varepsilon_3 k_1 k_2^2 = 0 \\ 2k_1^{'} k_2 + k_1 k_2^{'} = 0 \\ k_1 k_2 k_3 = 0. \end{array} \tag{7}$$

The following result provides some simple characterizations of weak biharmonic curves in Minkowski 4-space $\mathbb{E}_1^4$.

Theorem 2. *Let γ be a weak biharmonic curve in Minkowski 4-space $\mathbb{E}_1^4$ and let s be its arclength function. Then:*
i) γ is a straight line,
ii) γ is a pseudo circle,
iii) γ is a cornu spiral in $\mathbb{E}_1^2$ with $k_1 = cs + d$,
iv) γ is a spherical cornu spiral in $\mathbb{E}_1^3$ with the non-constant curvatures

$$k_1 = \pm\sqrt{\frac{c_1^2(s+c_2)^2 + a}{c_1}}, \quad k_2 = \frac{cc_1}{c_1^2(s+c_2)^2 + a} \tag{8}$$

where $a = \frac{\varepsilon_3 c^2}{\varepsilon_2}$ and c_1, c_2 are integral constants.

Proof. If we assume that $k_1 = 0$, then the equations (7) are satisfied. So, γ is a straight line. If k_1 is a non-zero constant and $k_2 = 0$, then the equations (7) are also satisfied. So, γ is a pseudo circle. Further, If k_1 is a non-constant function and $k_2 = 0$, then from the equations (7), we find $k_1^{''} = 0$ and the solution of differential equation is $k_1 = cs + d$. So, γ is a cornu spiral in $\mathbb{E}_1^2$. Finally, if k_1 and k_2 are non-constant functions, then from the equations (7), we find $k_3 = 0$, $k_1 = \pm\sqrt{\frac{c_1^2(s+c_2)^2+a}{c_1}}$ and $k_2 = \frac{cc_1}{c_1^2(s+c_2)^2+a}$. So, γ is a spherical cornu spiral in $\mathbb{E}_1^3$.

Conversely, if γ is a curve given with arc-length parameter s and if one of $(i), (ii), (iii)$, or (iv) holds, then γ is of weak biharmonic.

Definition 2. *Let γ be a unit speed curve in Lorentzian $\mathbb{E}_v^n$ satisfying the condition*

$$\Delta^D \overrightarrow{H} + \lambda \overrightarrow{H} = 0, \ \lambda \neq 0 \tag{9}$$

then γ is called harmonic 1-type curve [5].

By the use of (2) and (6) we obtain the following result.

Lemma 3. *Let γ be a unit speed curve in Minkowski 4-space $\mathbb{E}_1^4$. If γ is of harmonic 1-type then we have*

$$\begin{aligned} -\varepsilon_2 k_1'' + \varepsilon_3 k_1 k_2^2 - \lambda\varepsilon_3 k_1 &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned} \tag{10}$$

The following result provides some simple characterizations of harmonic 1-type curves in Minkowski 4-space $\mathbb{E}_1^4$.

Theorem 4. *Let γ be a harmonic 1-type curve in Minkowski 4-space $\mathbb{E}_1^4$ and let s be its arclength function. Then:*

i) γ is a straight line,

ii) γ is a curve in $\mathbb{E}_1^2$ with the curvature $k_1 = c_1 \sin(\sqrt{a}s) + c_2 \cos(\sqrt{a}s)$,

iii) γ is a a helix in $\mathbb{E}_1^3$ with the curvatures k_1 =constant and $k_2 = \pm\sqrt{\lambda}$,

iv) γ is a curve in $\mathbb{E}_1^3$ with the non-constant curvatures

$$k_1 = \pm\frac{1}{2}\frac{\sqrt{c_1\varepsilon_2\varepsilon_3\lambda\left(4\varepsilon_2\varepsilon_3^2c^2\lambda - c_1^2\varepsilon_2\left(e^{-2\sqrt{-a}s}\right)^2 - 4c_1^2c_2\varepsilon_2\sqrt{-a}e^{-2\sqrt{-a}s} + 4c_1^2c_2^2\varepsilon_3\lambda\right)}}{c_1\varepsilon_2\varepsilon_3\lambda e^{\frac{\varepsilon_3\lambda s}{\varepsilon_2\sqrt{-a}}}}. \tag{11}$$

and

$$k_2 = \frac{c}{k_1^2} \tag{12}$$

where $a = \frac{\varepsilon_3\lambda}{\varepsilon_2}$ and c, c_1, c_2 are integral constants.

Proof. If we assume that $k_1 = 0$, then the equations (10) are satisfied. So, γ is a straight line. If k_1 is a non-constant function and $k_2 = 0$, then from the equations (10), we find $\varepsilon_2 k_1'' + \lambda\varepsilon_3 k_1 = 0$ and the solution of differential equation is $k_1 = c_1 \sin\left(\frac{\sqrt{\lambda}\sqrt{\varepsilon_3}s}{\sqrt{\varepsilon_2}}\right) + c_2 \cos\left(\frac{\sqrt{\lambda}\sqrt{\varepsilon_3}s}{\sqrt{\varepsilon_2}}\right)$. If k_1 and k_2 are non-zero constant, then from the equations (10), we find $k_3 = 0$ and $k_2 = \pm\sqrt{\lambda}$. If we take k_1 and k_2 are non-constant functions, then from the equations (10), we find $k_3 = 0$, $k_2 = \frac{c}{k_1^2}$ and the equation (11).

Conversely, if γ is a curve given with arc-length parameter s and if one of $(i), (ii), (iii)$, or (iv) holds, then γ is of harmonic 1-type .

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Günay ÖZTÜRK
Department of Mathematics
Kocaeli University
Kocaeli, Turkey
e-mail: *ogunay@kocaeli.edu.tr*

Bengü (Kılıç) Bayram
Department of Mathematics
Balıkesir University
Balıkesir, Turkey
e-mail: *benguk@balikesir.edu.tr*

Kadri Arslan
Department of Mathematics
Uludağ University
16059, Bursa, Turkey
e-mail: *arslan@uludag.edu.tr*