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HENSELIAN VALUED QUASILOCAL FIELDS WITH TOTALLY INDIVISIBLE VALUE GROUPS

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Abstract. This paper characterizes the quasilocal fields lying in the class of Henselian valued fields with totally indivisible value groups, which possess finite separable extensions of nontrivial defect. We show that, for any prime number q, a divisible subgroup T in the multiplicative group of complex roots of unity is realizable as the Brauer group of such a quasilocal field of residual characteristic q unless $q = 2$ and the 2-component of T is trivial.

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1. Introduction

This paper is a continuation of [2]. Let K be a field, K^* its multiplicative group, K_{sep} a separable closure of K, $\mathcal{G}_K = \mathcal{G}(K_{\mathrm{sep}}/K)$ the absolute Galois group of K, and for any prime number p, let $\mathrm{cd}_p(\mathcal{G}_K)$ be the cohomological p-dimension of $\mathcal{G}_K$, $K(p)$ the maximal p-extension of K in K_{sep}, and $r(p)_K$ the rank of the Galois group $\mathcal{G}(K(p)/K)$ as a pro-p-group ($r(p)_K = 0$ in case $K(p) = K$). We say that K is primarily quasilocal (abbr, PQL), if every cyclic extension F of K is embeddable as a subalgebra in each central division K-algebra D of Schur index $\mathrm{ind}(D)$ divisible by the degree $[F\colon K]$; K is called quasilocal, if its finite extensions are PQL-fields. The class of quasilocal fields includes the one of local fields and contains p-adically closed fields and Henselian discrete valued fields with quasifinite residue fields (cf. [25], Ch. XIII, Sect. 3, [23], Theorem 3.1 and Lemma 2.9, and [4], Proposition 6.4). The quasilocal property has been fully characterized by [2], Theorem 2.1, in the class of Henselian (valued) fields with totally indivisible value groups, whose finite separable extensions are defectless. Other examples of quasilocal fields, mostly, of nonarithmetic nature (from the perspective of [4], (1.2), (1.3) and Corollary 5.3), can be found in [6].

The present paper proves the existence of quasilocal Henselian fields with totally indivisible value groups, that admit defectful finite separable extensions. It describes, up-to an isomorphism, the abelian torsion groups that can be realized as Brauer groups of such fields.

2. Statement of the main result

A nontrivial (Krull) valuation v of a field K is called Henselian, if it is uniquely, up-to an equivalence, extendable to a valuation v_L on each algebraic field extension L/K. This is the case if and only if the valuation ring $O_v(K) = \{a \in K\colon\ v(a) \geq 0\}$ is Henselian with respect to its (unique) maximal ideal $M_v(K) = \{a \in K\colon\ v(a) > 0\}$ (see (3.1)). Denote by $v(K)$ the value group and by $\widehat{K}$ the residue field of (K, v). We say that $v(K)$ is totally indivisible, if it is p-indivisible, i.e. $v(K) \neq pv(K)$, for every $p \in \mathbb{P}$. As a beginning of our considerations, we introduce the notions of a norm-inertial extension and of a quasiinertial extension, as follows:

Definitions. Let (K, v) be a Henselian field with $\mathrm{char}(\widehat{K}) = p > 0$, M a finite extension of K in $K(p)$, and v_M a valuation of M extending v. The extension M/K is said to be norm-inertial, if the norm group $N(M/K)$ contains all $\theta \in K^*$ with $v(\theta - 1) > 0$. We say that M/K is quasiinertial, if the ring $O_{v_M}(M)$ consists of those $\delta \in M^*$, for which the trace $\mathrm{Tr}_K^M(\delta\mu)$ has value ≥ 0, for each $\mu \in O_v(M)$.

Our next result and [2], Theorem 2.1, give a formally complete characterization of quasilocal Henselian fields with totally indivisible value groups, and attract interest in the algebraic nature of immediate norm-inertial extensions:

Proposition 1. *Let (K, v) be a Henselian field admitting a finite extension in K_{sep} of nontrivial defect, and for each prime p, let G_p be a Sylow pro-p-subgroup of $\mathcal{G}_K$ and K_p the fixed field of G_p. Suppose that $\mathrm{char}(\widehat{K}) = q$ and $v(K) \neq pv(K)$ whenever $G_p \neq \{1\}$. Then K is quasilocal if and only if it satisfies the following:*

(a) *The quotient group $v(K)/qv(K)$ is of order q, $\widehat{K}$ is perfect, $\mathrm{cd}_q(\mathcal{G}_{\widehat{K}}) = 0$, and K_q has an immediate $\mathbb{Z}_q$-extension Y in K_{sep}, such that every finite extension L_q of K_q in K_{sep} with $L_q \cap Y = K_q$ is totally ramified; in addition, finite extensions of K_q in Y are norm-inertial;*

(b) *$r(p)_{K_p} \leq 2$, for each prime $p \neq q$.*

Proposition 1 has been proved as [4], Proposition 6.1. The main result of the present paper (stated without proof in [4], Sect. 6) provides series of examples of quasilocal Henselian real-valued fields satisfying the conditions of this proposition. Before stating it, note that the assumptions of Proposition 1 ensure that K is a nonreal field [17], Theorem 3.16, which implies that the Brauer group $\mathrm{Br}(K)$ is divisible whenever K is quasilocal (cf. [3], I, Theorem 3.1). At the same time, in the quasilocal case, by [4], Theorem 1.1, $\mathrm{Br}(K)$ is embeddable as a subgroup in the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the subgroup of integers. Conversely, divisible subgroups of $\mathbb{Q}/\mathbb{Z}$ are realizable as Brauer groups of quasilocal Henselian fields of the type studied in [2] (see (3.6) and [28], Proposition 2.2). These observations attract interest in the description of the isomorphism classes of Brauer groups of the quasilocal Henselian fields admissible by Proposition 1. Our main result in this direction is contained in the following theorem:

Theorem 1. *Let* (Φ, ω) *be a Henselian discrete valued field with* $\widehat{\Phi}$ *quasifinite and* $\mathrm{char}(\widehat{\Phi}) = q \neq 0$, *and let* T *be a divisible subgroup of* $\mathbb{Q}/\mathbb{Z}$ *with a nontrivial* q*-component* T_q. *Then there is a quasilocal Henselian field* (K, v) *such that:*

(a) $\mathrm{Br}(K)$ *is isomorphic to* T, K/Φ *is a field extension of transcendency degree* 1 *and* v *is a prolongation of* ω;

(b) $v(K)$ *is a totally indivisible Archimedean group,* $\widehat{K}/\widehat{\Phi}$ *is an algebraic extension, and* K *possesses an immediate quasiinertial* $\mathbb{Z}_q$*-extension* I_∞.

Theorem 1 is proved in Section 4. Its proof relies on the characterization of quasiinertial Galois extensions given in Section 2, and on their relations with norm-inertial Galois extensions (these results, in a special case, also play a role in the proof of Proposition 1, see (3.8), Lemma 2 and [4], (3.4)). In addition, we use the quasilocal property of Φ and an easily applicable criterion for the fulfillment of the quasiinertial condition, presented in Section 3. In Section 5 we obtain similarly that if $q > 2$, then divisible subgroups $T \leq \mathbb{Q}/\mathbb{Z}$ with $T_q = \{0\}$ are also realizable as Brauer groups of quasilocal fields admissible by Proposition 1. The case of $q = 2$ is exceptional - then $\mathrm{Br}(K)_2$ is a quasicyclic 2-group whenever K is a quasilocal field satisfying the conditions of Proposition 1 (see Proposition 2).

Note that Brauer groups of quasilocal fields E have influence on a wide spectrum of their algebraic properties. This includes the structure of the continuous character groups of $\mathcal{G}(E(p)/E)$, $p \in \mathbb{P}$ [3], II, Lemmas 2.3 and 3.3, cohomological properties of $\mathcal{G}(E(p)/E)$ and the Sylow pro-p-subgroups of $\mathcal{G}_E$ [3], I, Theorem 8.1, and [5], Sect. 5, finite abelian extensions of E and their norm groups [5] (concerning nonabelian Galois extensions of E, see [6]). Therefore, the description of $\mathrm{Br}(E)$ is a major objective of the study of E, and the present research can be viewed as the final step towards a really complete characterization of quasilocal Henselian fields with totally indivisible value groups.

The basic notation, terminology and conventions kept in this paper are standard and essentially the same as in [3], I, [4] and [5]. Preliminaries on Henselian valuations used in the sequel are included in Section 2. Throughout, Brauer and value groups are additively presented, Galois groups are viewed as profinite with respect to the

Krull topology, and by a profinite group homomorphism, we mean a continuous one. We write $\mathbb{P}$ for the set of prime numbers, and for each $p \in \mathbb{P}$, $\mathbb{Z}_p$ denotes the additive group of p-adic integers and $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group. For any profinite group G, $\mathrm{cd}(G)$ is the cohomological dimension of G, and $\mathrm{cd}_p(G)$, $p \in \mathbb{P}$, are its cohomological p-dimensions. Given a field E, $\mathrm{Br}(E)_p$ is the p-component of the Brauer group $\mathrm{Br}(E)$, and ${}_p\mathrm{Br}(E) = \{\delta \in \mathrm{Br}(E)\colon\ p\delta = 0\}$, where $p \in \mathbb{P}$, $P(E) = \{p \in \mathbb{P}\colon\ E(p) \neq E\}$, and $\Pi(E) = \{p \in \mathbb{P}\colon\ \mathrm{cd}_p(\mathcal{G}_E) > 0\}$. We write $s(E)$ for the class of finite-dimensional central simple E-algebras, $d(E)$ stands for the class of division algebras $D \in s(E)$, and for each $A \in s(E)$, $[A]$ is the similarity class of A in $\mathrm{Br}(E)$. For any field extension E'/E, $I(E'/E)$ denotes the set of its intermediate fields. By a $\mathbb{Z}_p$-extension of E, for some $p \in \mathbb{P}$, we mean a Galois extension E_∞/E with a Galois group $\mathcal{G}(E_\infty/E) \cong \mathbb{Z}_p$. The field E is called p-quasilocal, if $\mathrm{Br}(E)_p = \{0\}$, or $p \notin P(E)$, or every degree p extension of E in $E(p)$ embeds as an E-subalgebra in each $\Delta_p \in d(E)$ of index p. Note that E is PQL if and only if it is p-quasilocal, for each $p \in P(E)$ (cf. [21], Sects. 13.4, 14.4 and 15.3).

3. Preliminaries on Henselian valuations and characterizations of quasiinertial Galois extensions

Let (K, v) be a (nontrivially) valued field, K_v a completion of K relative to the topology induced by v, $v(K)_0 = \{\gamma \in v(K)\colon\ \gamma > 0\}$, $\nabla_0 = \{\alpha \in K\colon\ v(\alpha - 1) > 0\}$ and $\nabla_\gamma(K) = \{\alpha \in K\colon\ v(\alpha - 1) \geq \gamma\}$, for each $\gamma \in v(K)_0$. It is known that v is Henselian if and only if the following condition holds (cf. [12], Sect. 18.1):

(3.1) Given a polynomial $f(X) \in O_v(K)[X]$, and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the (formal) derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c - a) = v(f(a)/f'(a))$.

The fulfillment of (3.1) ensures that the polynomial $f_b(X) = f(X) + b$ has a zero in K whenever $b \in K^*$ and $v(b) > 2v(f'(a))$. Also, the Henselity of v is inherited by v_M, for every algebraic field extension M/K. When $[M\colon K]$ is finite and $M \subseteq K_{\mathrm{sep}}$, these observations, applied to the minimal polynomial $f_\beta(X)$ over K of a primitive element $\beta \in O_{v_M}(M)$ of M/K, prove the following:

(3.2) The norm group $N(M/K)$ contains every $\alpha \in O_v(K)$, for which $v(\alpha - 1) > 2v_M(f'_\beta(\beta))$.

When v is Henselian and L/K is algebraic, v_L is Henselian and extends uniquely to a valuation v_D on each $D \in d(L)$. Denote by $\widehat{D}$ the residue field of (D, v_D) and put $v(D) = v_D(D)$. By the Ostrowski-Draxl theorem [9], $[D\colon K]$, $[\widehat{D}\colon \widehat{K}]$ and the ramification index $e(D/K)$ are related as follows:

(3.3) $[D\colon K]$ is divisible by $[\widehat{D}\colon \widehat{K}]e(D/K)$ and the defect
$d(D/K) = [D\colon K]/([\widehat{D}\colon \widehat{K}]e(D/K))$ is not divisible by any $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$.

The K-algebra D is said to be defectless, if $d(D/K) = 1$, i.e. $[D\colon K] = [\widehat{D}\colon \widehat{K}]e(D/K)$; it is called immediate, if $\widehat{D} = \widehat{K}$ and $e(D/K) = 1$. We say that D/K is totally ramified, if $e(D/K) = [D\colon K]$. When $v(K) \neq pv(K)$, for a given $p \in \mathbb{P}$, (K, v) is subject to the following alternative (see [7], Corollary 6.5):

(3.4) (i) K has a totally ramified proper extension in $K(p)$;

(ii) $\mathrm{char}(K) = 0$, K does not contain a primitive p-th root of unity and the minimal isolated subgroup of $v(K)$ containing $v(p)$ is p-divisible.

A finite extension R of K is said to be inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}$ is separable over $\widehat{K}$; R/K is called tamely ramified, if $\widehat{R}/\widehat{K}$ is separable and $e(R/K)$ is not divisible by $\mathrm{char}(\widehat{K})$. It is well-known that the compositum K_{ur} of inertial extensions of K in K_{sep} is a Galois extension of K, and so is the compositum K_{tr} of tamely ramified extensions of K in K_{sep}. Note also that K_{ur} and K_{tr} have the following properties (see [14], page 135):

(3.5) (i) $v(K_{\mathrm{ur}}) = v(K)$ and finite extensions of K in K_{ur} are inertial;

(ii) K_{tr} contains a primitive m-th root of unity, for each $m \in \mathbb{N}$ not divisible by $\mathrm{char}(\widehat{K})$, finite extensions of K in K_{tr} are tamely ramified, and $v(K_{\mathrm{tr}}) = pv(K_{\mathrm{tr}})$, for every $p \in \mathbb{P}$ different from $\mathrm{char}(\widehat{K})$;

(iii) $\widehat{K}_{\mathrm{ur}}$ is $\widehat{K}$-isomorphic to $\widehat{K}_{\mathrm{sep}}$, $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$, and the natural mapping of $I(K_{\mathrm{ur}}/K)$ into $I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$ is bijective.

When (K, v) is a local field, T is a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$, and $S(T) = \{p \in \mathbb{P}\colon\ T_p \neq \{0\}\}$, there exists $K_T \in I(K_{\mathrm{ur}}/K)$, such that $\mathcal{G}(K_{\mathrm{ur}}/K_T)$ is isomorphic to the topological group product $\prod_{p \in S(T)} \mathbb{Z}_p$. In other words, T is isomorphic to the continuous character group of $\mathcal{G}(K_{\mathrm{ur}}/K_T)$. Since $v(K_T) = v(K)$ and $\mathrm{Br}(\widehat{K}_T) = \{0\}$, this enables one to deduce from Witt's theorem (cf. [29], (3.10)) that $\mathrm{Br}(K_T) \cong T$. It is therefore clear from [4], Corollary 5.3, that

(3.6) An abelian torsion group is realizable as the Brauer group of a quasilocal Henselian field with a totally indivisible value group and defectless finite separable extensions if and only if it is divisible and embeddable in $\mathbb{Q}/\mathbb{Z}$.

Let now (K, v) be a Henselian field with $\mathrm{char}(\widehat{K}) = p > 0$, and let $M \in I(K(p)/K)$ be a finite extension of K. Then:

(3.7) $\nabla_0(M)$ equals the pre-image of $\nabla_0(K)$, under the norm map N_K^M, provided that $\widehat{M} = \widehat{K}$; in this case, $\varphi(\mu)\mu^{-1} \in \nabla_0(M)$ whenever $\mu \in M^*$ and φ is a K-automorphism of M.

With notation being as above, put $\delta_{M/K}(\mu) = v_M(f'_\mu(\mu))$, for each primitive element μ of M/K, where f'_μ is the derivative of the minimal (monic) polynomial f_μ of μ over K. Clearly, $[M\colon K]\delta_{M/K}(\mu) = v(d_\mu)$, d_μ being the discriminant of f_μ. This fact, (3.7) and the the following lemma will be used in the sequel.

Lemma 2. *Let (K, v) be a Henselian field with $\mathrm{char}(\widehat{K}) = p > 0$, M a finite Galois extension of K in $K(p)$, and for each primitive element μ of M/K lying in $O_v(M)$,* *let* $f_\mu(X)$ *be the minimal polynomial of* μ *over* K*, and* $\delta_{M/K}(\mu) = v_M(f'_\mu(\mu))$*. Then* M/K *is quasiinertial if and only if any of the following three equivalent conditions is fulfilled:*

(a) *For each* $\gamma \in v(K)_0$*, there exists* $\lambda_\gamma \in O_v(M)$ *with* $v(\mathrm{Tr}_K^M(\lambda_\gamma)) < \gamma$*;*

(b) *For each* $\gamma' \in v(M)_0$*,* $O_v(M)$ *contains a primitive element* $\mu_{\gamma'}$ *of* M/K *satisfying the inequality* $\delta_{M/K}(\mu_{\gamma'}) < \gamma'$*;*

(c) *There exists* $L \in I(M/K)$*, such that* L/K *and* M/L *are quasiinertial;*

(d) *For any* $\gamma \in v(K)_0$*, there is* $\beta_\gamma \in O_v(M)$*, such that* $v_M(\varphi(\beta_\gamma) - \beta_\gamma) < \gamma$*, for every* $\varphi \in \mathcal{G}(M/K)$ *different from* 1*.*

When M/K *is quasiinertial, so are* M/M_0 *and* M_0/K*, for every* $M_0 \in I(M/K)$*.*

Proof. The concluding assertion of the lemma follows from the claimed equivalence of condition (a) and the one that M/K is quasiinertial, together with the inequalities $v_M(y) \le v(\mathrm{Tr}_K^M(y))$, $y \in O_v(M)$, and the transitivity of traces in towers of finite separable extensions (cf. [19], Ch. VIII, Sect. 5). When condition (c) of Lemma 2 holds, the assertion that M/K is quasiinertial is standardly proved by assuming the opposite, using again trace transitivity (specifically, the equality $\mathrm{Tr}_K^M = \mathrm{Tr}_K^L \circ \mathrm{Tr}_L^M$) and the L-linearity of Tr_L^M. Thus (c) turns out to be equivalent to the assumption that M/K is quasiinertial. Let $r \in O_v(M)$ be a primitive element of M/K. It is easily obtained (by applying basic linear algebra, including Cramer's rule and Vandermonde's determinant) that if $r' \in O_v(M) \setminus \{0\}$ and $\mathrm{Tr}_K^M(r'^{-1}r^{j-1}) \in O_v(K)$, $j = 1, \dots, [M\colon K]$, then $2v_M(r') \le v(d_r)$. Hence, the validity of condition (b) of Lemma 2 ensures that M/K is quasiinertial. As to condition (a), it is satisfied in case M/K is quasiinertial (because if $a \in M_v(K) \setminus \{0\}$ and $a' \in O_v(M)$, then $\mathrm{Tr}_K^M(a^{-1}a') \in O_v(K)$ if and only if $v(a) \le v(\mathrm{Tr}_K^M(a'))$). These observations can be summarized by saying that (b)→(c)→(a). We prove that (a)→(d)→(b). Note here that if $v(K)_0$ contains a minimal element, then M/K is inertial if and only if some of conditions (a), (b), (c), (d) is satisfied. Therefore, it suffices to prove that (a)→(d)→(b) in case $v(K)_0$ does not contain a minimal element. We first prove that (a)→(d). Assume that condition (a) holds, $[M\colon K] = p^n$ and α is an element of $O_v(M)$, such that $v(\mathrm{Tr}_K^M(\alpha)) < v(p)$. It is easily verified that α is a primitive element of M/K. Let α_u, $u = 1, \dots, [M\colon K]$, be the roots in M of the minimal polynomial f_α of α over K. We prove the validity of condition (d) by showing that $v_{M'}(\alpha_{u'} - \alpha_{u''}) \le v(\mathrm{Tr}_K^M(\alpha))$, for $1 \le u' < u'' \le p^n$. Suppose first that $[M\colon K] = p$ and φ is a generator of $\mathcal{G}(M/K)$. Then $v_M(\varphi^\nu(\alpha) - \alpha) = v_M(\varphi(\alpha) - \alpha)$, for $\nu = 1, \dots, p-1$. As $\alpha \in O_v(M)$ and $v(\mathrm{Tr}_K^M(\alpha)) < v(p)$, this implies the stated inequality. The proof in general is carried out by induction on n, under the inductive hypothesis that $n \ge 2$, and for some $K' \in I(M/K)$ of degree $[K'\colon K] = p$, $\mathrm{Tr}_{K'}^M$ is subject to analogous inequalities. Since $\mathrm{Tr}_K^M(\alpha) = \mathrm{Tr}_K^{K'}(\mathrm{Tr}_{K'}^M(\alpha))$, whence $v_{K'}(\mathrm{Tr}_{K'}^M(\alpha)) \le v(\mathrm{Tr}_K^M(\alpha)) < v(p)$, this yields $v_{M'}(\alpha_{u'} - \alpha_{u''}) \le v_{K'}(\mathrm{Tr}_{K'}^M(\alpha))$, provided that $u' \neq u''$ and $\alpha_{u'}$, $\alpha_{u''}$ are conjugate over K'. Now take indices u' and u'' so that $\alpha_{u'}$ and $\alpha_{u''}$ be non-conjugate over K'. Then $\alpha_{u''} = \psi(\alpha_{u'})$, for some $\psi \in \mathcal{G}(M/K)$ inducing on K' a generator, say, ψ' of $\mathcal{G}(K'/K)$. Denote by $S_{u'}$ and $S_{u''}$ the sets of roots in M of the minimal polynomials over K' of $\alpha_{u'}$ and $\alpha_{u''}$, respectively. Using the normality of $\mathcal{G}(M/K')$ in $\mathcal{G}(M/K)$, one obtains that if $v_M(\alpha_{u'} - \alpha_{u''}) > v(\mathrm{Tr}_K^M(\alpha))$, then there is a bijection $\epsilon\colon S_{u'} \to S_{u''}$, such that $v_M(\alpha_u - \epsilon(\alpha_u)) > v(\mathrm{Tr}_K^M(\alpha))$ whenever $\alpha_u \in S_{u'}$. Our conclusion, however, contradicts the inequality $v_{K'}(\psi'(\mathrm{Tr}_{K'}^M(\alpha)) - \mathrm{Tr}_{K'}^M(\alpha)) \leq v(\mathrm{Tr}_K^M(\alpha))$ and thereby proves that $v_M(\alpha_{u'} - \alpha_{u''}) \leq v(\mathrm{Tr}_K^M(\alpha))$. Thus the implication (a)$\to$(d) becomes obvious, and since (b)$\to$(c)$\to$(a), it remains to be seen that (d)$\to$(b). The assertion is evident, if the intersection V of the nontrivial isolated subgroups of $v(K)$ is trivial. Suppose now that $V \neq \{0\}$. This means that V is a minimal isolated subgroup of $v(K)$. Hence, V is Archimedean, and by Höelder's theorem (cf. [12], Theorem 2.5.2), it is isomorphic to an ordered subgroup of the additive group $\mathbb{R}$ of real numbers. Identifying V with its isomorphic copy in $\mathbb{R}$, and taking into account that $v(K)_0$ does not contain a minimal element, one concludes that, for each $h \in V \cap v(K)_0$, there exist $h_m \in V \cap v(K)_0$, $m \in \mathbb{N}$, such that $mh_m < h$, for each index m. This observation completes the proof of implication (d)$\to$(b), and of Lemma 2.

Assuming again that (K, v) is a Henselian field with $\mathrm{char}(\widehat{K}) = p \neq 0$, we say that a field $I_\infty \in I(K(p)/K)$ is said to be a norm-inertial extension of K, if finite extensions of K in I_∞ are norm-inertial. The extension I_∞/K is called quasiinertial, if so are finite extensions of K in I_∞. When I_∞/K is Galois, we show that the two notions are related as follows (see [4], (3.4), for a very concise proof in the special case where I_∞/K is a $\mathbb{Z}_p$-extension):

(3.8) (i) I_∞/K is norm-inertial, provided that it is quasiinertial;

(ii) If I_∞/K is an immediate norm-inertial extension and $H \neq pH$ whenever $H \neq \{0\}$ and H is an isolated subgroup of $v(K)$, then I_∞/K is quasiinertial; when this holds, I_∞/I is quasiinertial, for every $I \in I(I_\infty/K)$.

Inertial extensions of K in $K(p)$ are obviously norm-inertial, so it is sufficient to prove (3.8) under the extra hypothesis that $v(K)_0$ does not contain a minimal element. Since quasiinertial finite Galois extensions of K satisfy condition (b) of Lemma 2, this enables one to deduce (3.8) (i) from (3.2) (by the method of proving implication (d)$\to$(b) of the lemma). The latter assertion of (3.8) (ii) is implied by the former one and Lemma 2. We turn to the proof of the former part of (3.8) (ii). We first show that, for each $\gamma \in v(K)_0$, there exists $\gamma' \in v(K)_0$ less than γ and not lying in $pv(K)$. The assertion is obvious, if the subgroup $V \leq v(K)$ defined in the proof of Lemma 2 is trivial, so we assume that $V \neq \{0\}$. This ensures that V embeds in $\mathbb{R}$ as an ordered subgroup, and it follows from our extra hypothesis on $v(K)_0$ that $V \cap v(K)_0$ does not contain a minimal element. Identifying V with its isomorphic copy in $\mathbb{R}$, one also sees that pV is dense in $\mathbb{R}$. These observations imply the existence of $\gamma' \in V$ with the required properties. Let I be a finite Galois extension of K in I_∞, and let θ be an element of $\nabla_0(I)$, such that $N_K^I(\theta) = 1 + \theta_0$, $v(\theta_0) < v(p)$ and $v(\theta_0) \notin pv(K)$. It is easily verified that $1+\theta_0 \notin K^{*p}$, and therefore, θ is a primitive element of I/K. Denote by f_θ the minimal polynomial of θ over K. We show that $v_I(\theta - \theta') \leq v(\theta_0)$, provided that $\theta' \in I$, $\theta' \neq \theta$ and $f_\theta(\theta') = 0$. Assuming the opposite, one concludes that there exist a nontrivial cyclic subgroup $G \leq \mathcal{G}(I/K)$ and some $\bar{\gamma} \in v(K)$, such that $v(\theta_0) < \bar{\gamma} < v(p)$ and $v_I(\theta - \psi(\theta)) \geq \bar{\gamma}$, for every $\psi \in G$. This implies $v_I(N_J^I(\theta) - 1 - (\theta - 1)^{p^h}) \geq \bar{\gamma}$, where J is the fixed field of G and p^h is the order of G. Thus it turns out that $v_I(N_K^I(\theta) - 1 - (\tilde{\theta} - 1)^p) \geq \bar{\gamma}$, for a suitably chosen $\tilde{\theta} \in \nabla_0(I)$. As $N_K^I(\theta) = 1 + \theta_0$ and $v(\theta_0) < \bar{\gamma}$, our conclusion requires that $v_I((\tilde{\theta} - 1)^p) = v(\theta_0)$. This, however, contradicts the assumptions that $v(I) = v(K)$ and $v(\theta_0) \notin pv(K)$, and so proves that the roots of f_θ satisfy the claimed inequality. In view of (3.7), Lemma 2 and the noted property of the set $v(K)_0 \setminus pv(K)$, the obtained result implies the former assertion of (3.8) (ii).

4. Preparation for the proof of Theorem 1

The proof of Theorem 1 relies on the following two lemmas.

Lemma 3. *Let (E, v) be a Henselian field with* $\mathrm{char}(\widehat{E}) = p \neq 0$ *and* $v(E) = pv(E)$. *Assume that $p \in P(E)$, $r(p)_E \in \mathbb{N}$, and in case* $\mathrm{char}(E) = 0$, *E contains a primitive p-th root of unity ε. Then:*

(a) *$\widehat{E}$ is perfect and* $\mathrm{Br}(E)_p = \{0\}$*;*

(b) *$\mathcal{G}(E(p)/E)$ is a free pro-p-group; in particular, every cyclic extension L of E in $E(p)$ lies in $I(L_\infty/E)$, for some $\mathbb{Z}_p$-extension L_∞/E, $L \subseteq E(p)$;*

(c) *If E is perfect and $v(E) \leq \mathbb{R}$, then finite extensions of E in $E(p)$ are quasiinertial, whence every $\mathbb{Z}_p$-extension of E is quasiinertial.*

Proof. The assumption on $r(p)_E$ and [7], Lemma 4.1, imply that $\widehat{E}$ is perfect. We show that $\mathrm{Br}(E)_p = \{0\}$ and $\mathcal{G}(E(p)/E)$ is a free pro-p-group. When $\mathrm{char}(E) = p$, this is a special case of [15], Proposition 4.4.8, and [24], Ch. II, Proposition 2, respectively. If $\varepsilon \in E$, the two assertions are equivalent (by Galois cohomology, see [26], page 265, [24], Ch. I, 4.2, and [30], page 725), so they are contained in [10], Proposition 3.4 (or [7], Proposition 5.1). This indicates that $\mathcal{G}(E(p)/E) \cong \mathcal{G}_Y$, for some field Y of characteristic p [18], (4.8) (see also [2], Remark 2.6). The obtained result, combined with Galois theory and Witt's lemma (see [8], Sect. 15), completes the proof of Lemma 3 (a) and (b). Since the class of free pro-p-groups is closed under taking open subgroups (cf. [24], Ch. I, 4.2 and Proposition 14), it becomes clear from Lemma 2 that it suffices for the proof of Lemma 3 (c) to show that every degree p extension F of E in $E(p)$ is quasiinertial. If F/E is inertial, there is nothing to prove, so we assume that this is not the case. As $v(E) = pv(E)$ and $[F\colon E] = p$, this means that F/E is immediate. Let ψ be a generator of $\mathcal{G}(F/E)$. Clearly, the claimed property of F/E can be deduced from the following assertion:

(4.1) F contains elements λ_n, $n \in \mathbb{N}$, such that $0 < v_F(\lambda_n) < v_F(\psi(\lambda_n) - \lambda_n) < 1/n$, for each index n.

Our objective is to prove (4.1). Suppose first that $\mathrm{char}(E) = p$ and E is perfect. Then the Artin-Schreier theorem implies the existence of a sequence $t = \{t_n \in M_v(E)\colon\ n \in \mathbb{N}\}$, such that $t_{n+1}^p = t_n \neq 0$ and the polynomial $X^p - X - t_n^{-1}$ is irreducible over E with a root $\xi_n \in F$, for each index n. Observing that $\xi_n^{-1} = t_n \prod_{j=1}^{p-1}(\xi_n + j)$ and $v_F(\xi_n) = p^{-1}v(t_n^{-1})$, one obtains by direct calculations that $v_F(\xi_n^{-1}) = p^{-1}v(t_n)$ and $v_F(\psi(\xi_n^{-1}) - \xi_n^{-1}) = 2v_F(\xi_n^{-1})$. Therefore, $\nabla_0(F)$ contains the elements $\lambda_n = \xi_n\psi(\xi_n^{-1})$, $n \in \mathbb{N}$, and $v_F(\lambda_n - 1) = p^{-1}v(t_n)$, for every index n. The obtained result proves (4.1) in the case where $\mathrm{char}(E) = p$ and E is perfect. Assume now that $\varepsilon \in E$. In view of Kummer theory and the equality $\nabla_0(K)K^{*p} = K^*$ (cf. [10], Lemma 3.3), F is generated over K by a p-th root of the sum $1 + \pi$, for some $\pi \in M_v(K)$. We prove Lemma 3 (c) together with the following statement:

(4.2) There exists a sequence $\pi_n \in M_v(K)$, $n \in \mathbb{N}$, such that $(1+(\varepsilon-1)^p\pi_n^{-1})K^{*p} = (1 + \pi)K^{*p}$ and $1/n > v(\pi_n) > v(\pi_{n+1})$, for each $n \in \mathbb{N}$.

As $r(p)_K \in \mathbb{N}$, Kummer theory ensures the existence of a number $d \in \mathbb{R}$, $d > 0$, such that the cosets λK^{*p}, $\lambda \in K^*$, have representatives in ∇_d; in particular, one may assume without loss of generality that $v(\pi) \geq d$. In addition, it is not difficult to see that, for each $n \in \mathbb{N}$, $M_v(K)$ contains elements $a_{n,j}\colon\ j = 1, \dots, n$, such that $v(a_{n,1}^p) = v(\pi)$, $v(\pi - \sum_{j=1}^n a_{n,j}^p) > pv(a_{n,n})$ and $v(a_{n,j}) \geq (d/p) + v(a_{n,(j-1)})$, provided that $j \geq 2$. Also, it is known that $\nabla_{\bar{p}}(K) \subset K^{*p}$, where $\bar{p} = (p/(p-1))v(p)$, which implies the elements $a_{n,j}\colon\ n, j \in \mathbb{N}, j \leq n$, can be chosen so that $(1+\pi)K^{*p} = (1 + \sum_{j=1}^n a_{n,j}^p)K^{*p}$, for every sufficiently large n. Note also that $\nabla_{v(p)}(K)$ contains the element $(1 + \sum_{j=1}^n a_{n,j}^p)(1 - \sum_{j=1}^n a_{n,j})^p$. Thus it turns out that there exist $b_n \in M_v(K)$, $n \in \mathbb{N}$, such that $(1 + pb_n)K^{*p} = (1 + \pi)K^{*p}$ and $v(b_{n+1}^p) = v(pb_n)$, for each index n. As F/K is immediate and $v(p) = (p - 1)v(\varepsilon - 1)$, this implies $v(b_n) < v(\varepsilon - 1)$, for each $n \in \mathbb{N}$, which enables one to prove that the sequence $v(b_n)$, $n \in \mathbb{N}$, increases and converges to $v(\varepsilon - 1)$. Hence, b_n, $n \in \mathbb{N}$, possesses a subsequence b'_n, $n \in \mathbb{N}$, such that $v(b'_n) > v(\varepsilon - 1) - (1/n)$, for each index n. It is therefore clear that the sequence $\pi_n = (\varepsilon - 1)^p/(pb'_n)$, $n \in \mathbb{N}$, satisfies (4.2). Consider now the polynomials $f_n(X)$, $g_n(X)$, $h_n(X)$, $t_n(X) \in K[X]$, $n \in \mathbb{N}$, defined by the rule $f_n(X) = X^p - X - \pi_n^{-1}$, $g_n(X) = \pi_n^p f_n(X/\pi_n) = X^p - \pi_n^{p-1}X - \pi_n^{p-1}$, $h_n(X) = (\varepsilon - 1)^{-p}[((\varepsilon - 1)X + 1)^p - 1 - (\varepsilon - 1)^p\pi_n^{-1}]$ and $t_n(X) = \pi_n^p h_n(X/\pi_n)$, for each n. It is easily verified that $t_n(X) \in O_v(K)[X]$, $t_n(X)$ is monic and the coefficients of the difference $t_n(X) - g_n(X)$ are divisible by $\varepsilon - 1$ (in $O_v(K)$). These observations enable one to deduce from (3.1) that there exists $N_0 \in \mathbb{N}$, such that $f_n(X)$, $g_n(X)$, $h_n(X)$ and $t_n(X)$ are irreducible over K and have roots in F, for each $n > N_0$. They also show that N_0 can be chosen so that $v_F(\eta_n^{-1}) = (1/p)v(\pi_n)$ and $v_F(\psi(\eta_n^{-1}) - \eta_n^{-1}) = (2/p)v(\pi_n) < 2/(pn)$ whenever $n > N_0$, $\eta_n \in F$ and $f_n(\eta_n) = 0$. When both conditions hold, the sequence $\lambda_n = \eta_{n+N_0}^{-1}$, $n \in \mathbb{N}$, satisfies (4.1), which proves Lemma 3 (c).

Lemma 4. *Let (E, w) be a Henselian field with* $\mathrm{char}(\widehat{E}) = q > 0$ *and* $w(E) \neq qw(E)$. *Assume that $w(E)$ is Archimedean and* $\mathrm{Br}(E')_q = \{0\}$, *where $E' \in I(E_{\mathrm{sep}}/E)$ is the root field over E of the binomial $X^q - 1$. Then:*

(a) *$\widehat{E}$ is perfect, $w(E)/qw(E)$ is of order q, $q \in P(E)$ and finite extensions of E in $E(q)$ are totally ramified; in particular, $q \notin P(\widehat{E})$;*

(b) *For any cyclic extension Φ of E in $E(q)$, there exists $\Gamma_0 \in I(E'(q)/E)$, such that $E'(q)/\Gamma_0$ is a $\mathbb{Z}_q$-extension and $\Phi \cap \Gamma_0 = E$.*

Proof. The assertion that $q \in P(E)$ follows from the fact that (E, w) satisfies condition (3.4) (i). Since, by [17], Theorem 3.16, E is a nonreal field, this assertion and [31], Theorem 2, indicate that $E(q)$ contains as a subfield a $\mathbb{Z}_q$-extension Γ of E; in particular, $[E(q)\colon E] = \infty$. Let L be a finite extension of E in $E(q)$, and let $[L\colon E] = q^k$. It is clear from [3], I, Lemma 4.2, and the triviality of $\mathrm{Br}(E)_q$ (following from the condition on $\mathrm{Br}(E')_q$) that $N(L/E) = E^*$. Hence, by the Henselity of w, $q^k w(L) = w(E)$, which implies in conjunction with (3.3) and the inequality $w(E) \neq qw(E)$ that $\widehat{\Phi} = \widehat{E}$ and $w(E)/q^k w(E)$ is a cyclic group of order q^k. These observations, combined with (3.5) (iii), prove Lemma 4 (a). They also enable one to deduce from Galois theory the existence of a field $E_1 \in I(E(q)/E)$, such that $[E_1\colon E] \leq q$, $E_1 \cap \Phi = E$ and $\Phi E_1 = \Gamma E_1$. Clearly, $\Gamma E_1/E_1$ is a $\mathbb{Z}_q$-extension. For the rest of the proof of Lemma 4 (b), it suffices to observe that the set $Y(\Phi) = \{Y \in I(E'(q)/E_1)\colon\ Y \cap \Phi = E\}$, partially ordered by inclusion, satisfies the conditions of Zorn's lemma, to take as Γ_0 any maximal element of $Y(\Phi)$, and using the projectivity of $\mathbb{Z}_q$ as a profinite group (cf. [24], Ch. I, 5.9), to prove that $\Gamma_0\Phi_1 = E'(q)$ and $\mathcal{G}(\Gamma_0\Phi_1/\Gamma_0) \cong \mathbb{Z}_q$.

Remark 1. *Retaining assumptions and notation as in Lemma 4, put $\Gamma_* = E(q) \cap \Gamma_0$ and denote by Γ_n the extension of Γ_0 in $E'(q)$ of degree q^n, for each $n \in \mathbb{N}$. Observing that E'/E is cyclic and $[E'\colon E] \mid (q-1)$, one obtains that $E'(q)/E$ is Galois and Γ_0 contains a primitive q-th root of unity unless char$(E) = q$. Note further that $[\Gamma_*\colon E] = \infty$. Indeed, Lemma 4 (a) ensures that $\widehat{E}$ is an infinite perfect field (with $r_q(\widehat{E}) = 0$), so it it follows from [7], Remark 4.2 and Lemma 4.3, that $r(q)_E = \infty$. By Galois theory, this means that there are infinitely many degree q extensions of E in Γ_*, whence $[\Gamma_*\colon E] = \infty$, as claimed. In addition, it follows from (3.3) and Lemma 4 (a) that $\widehat{\Gamma}_* = \widehat{E}$, $\widehat{E}'(q) = \widehat{E}'$, $w(E'\Gamma_*) = w(E') + w(\Gamma_*)$, $w(\Gamma_*) = qw(\Gamma_*)$*

and $w(E'\Gamma_*) = qw(E'\Gamma_*)$. *Observing also that* $E'(q) = (E'\Gamma_*)(q) = \Gamma_0(q)$, *one deduces from Lemma 3 that* $E'(q)/\Gamma_0$ *is an immediate quasiinertial* $\mathbb{Z}_q$*-extension.*

5. Proof of Theorem 1

Throughout this Section, we assume that (Φ, ω) and T satisfy the conditions of Theorem 1, and $\overline{\Phi}$ is an algebraic closure of $\Phi_{\rm sep}$. Put $S(T) = \{p \in \mathbb{P}\colon\ T_p \neq \{0\}\}$, $S_q(T) = S(T) \setminus \{q\}$, $S'(T) = \mathbb{P} \setminus S_q(T)$, and denote by U the maximal extension of Φ in $\Phi_{\rm ur}$ whose finite subextensions have degrees not divisible by any $p \in S_q(T)$. The assumptions on Φ, ω and $\widehat{\Phi}$ and the definition of U indicate that U/Φ and $\Phi_{\rm ur}/U$ are Galois extensions with $\mathcal{G}(U/\Phi)$ and $\mathcal{G}(\Phi_{\rm ur}/U)$ isomorphic to the topological group products $\prod_{\pi' \in S'(T)} \mathbb{Z}_{\pi'}$ and $\prod_{\pi_q \in S_q(T)} \mathbb{Z}_{\pi_q}$, respectively; this implies $q \notin \Pi(\widehat{U})$, whence $\widehat{U}$ is infinite. As Φ is quasilocal, the obtained result proves (in conjunction with [3], I, Proposition 4.4, Lemma 8.2 and Corollary 8.5) that $\mathrm{Br}(U_1)_{\pi'} = \{0\}$, for every $U_1 \in I(\overline{\Phi}/U)$ and each $\pi' \in S'(T)$. At the same time, it follows from (3.4) and the equality $\omega(U) = \omega(\Phi)$ that $\Phi(q) \notin I(U/\Phi)$, which ensures that $q \in P(U)$. Observing that ω_U is discrete and Henselian, one obtains from [28], Proposition 2.2, that finite extensions of U in $\Phi_{\rm sep}$ are defectless. Since $\widehat{\Phi}$ is perfect, U does not possess inertial proper extensions in $U(q)$, and we have $\mathrm{Br}(U_1)_q = \{0\}$, $U_1 \in I(\overline{\Phi}/U)$, one also concludes that finite extensions of U in $U(q)$ are totally ramified and $\mathcal{G}(U(q)/U)$ is a free pro-q-group (cf. [24], Ch. I, 4.2, and Ch. II, 3.1). Note further that $r(q)_U = \infty$; since ω_U is Henselian and discrete, and $\widehat{U}$ is infinite, this follows from [22], (2.7) (as well as from Remark 1 and the fact that $\mathrm{Br}(U)_q = \{0\}$). The rest of our proof relies on the observation that the set Σ of all $\Theta \in I(\Phi_{\rm sep}/U)$, such that $\Theta \cap \Phi_{\rm ur} = U$ and the degrees of finite extensions of U in Θ are not divisible by q, is nonempty and satisfies the conditions of Zorn's lemma with respect to the partial ordering by inclusion. Fix a maximal element $\Theta' \in \Sigma$ and put $\omega' = \omega_{\Theta'}$. Then it follows from Galois theory, statement (3.3), the projectivity of $\mathcal{G}(\Phi_{\rm ur}/U)$ as a profinite group, and the triviality of the groups $\mathrm{Br}(U_1)_q$, $U_1 \in I(\overline{\Phi}/U)$, that Θ' satisfies the following:

(5.1) (i) $\Phi_{\rm ur}\Theta' = \Phi_{\rm tr}$; in particular, finite extensions of U in Θ' are tamely totally ramified, $\omega'(\Theta') \neq q\omega'(\Theta')$ and $\omega'(\Theta') = p\omega'(\Theta')$, for each $p \in \mathbb{P} \setminus \{q\}$.

(ii) Finite extensions of Θ' in $\Theta'(q)$ are totally ramified.

(iii) $\mathcal{G}(\Theta'(q)/\Theta')$ is a free pro-q-group, $r(q)_{\Theta'} = \infty$ and $\mathrm{Br}(\Theta'')_q = \{0\}$, for every $\Theta'' \in I(\overline{\Phi}/\Theta')$.

The former assertion of (5.1) (iii) and [31], Theorem 2, imply the existence of a $\mathbb{Z}_q$-extension Γ of Θ' in $\Phi_{\rm sep}$. Put $\Gamma_0 = \Theta'$, and for each $n \in \mathbb{N}$, let Γ_n be the extension of Θ' in Γ of degree q^n. It follows from Galois theory and the assumption on $\widehat{\Phi}$ that the compositum $U' = \Theta'\Gamma\Phi_{\rm ur}$ is a Galois extension of Θ'

with $\mathcal{G}(U'/\Theta') \cong \prod_{\pi \in S(T)} \mathbb{Z}_\pi$. This implies $\mathrm{cd}(\mathcal{G}(U'/\Theta')) = 1$, which means that $\mathcal{G}(U'/\Theta')$ is a projective profinite group (cf. [24], Ch. I, 4.2 and 5.9). Note also that the set $\widetilde{\Sigma} = \{\widetilde{\Theta} \in I(\overline{\Phi}/\Theta') \colon \widetilde{\Theta} \cap U' = \Theta'\}$, partially ordered by inclusion, satisfies the conditions of Zorn's lemma. Let $\widetilde{K}$ be a maximal element of $\widetilde{\Sigma}$, $\tilde{v} = \omega_{\widetilde{K}}$ and $\tilde{k}$ the residue field of $(\widetilde{K}, \tilde{v})$. It is easily verified that $\widetilde{K}$ and $\tilde{k}$ are perfect fields, and it follows from the projectivity of $\mathcal{G}(U'/\Theta')$ that $\overline{\Phi} = U'\widetilde{K}$. Hence, by Galois theory and the equality $\widetilde{K} \cap U' = \Theta'$, $\mathcal{G}_{\widetilde{K}} \cong \mathcal{G}(U'/\Theta')$. Our argument, together with the former part of (5.1) (iii), also proves that there exists a $\mathbb{Z}_q$-extension of Θ' in $\widetilde{K}$. Since ω is discrete, this enables one to deduce the former part of the following assertion from (5.1) (i), (ii) and (3.3):

(5.2) $\tilde{v}(\widetilde{K}) = \mathbb{Q}$, $\tilde{k}/\widehat{\Phi}$ is an algebraic extension and $\Gamma\widetilde{K}/\widetilde{K}$ is immediate. Moreover, $\widetilde{K}(q) = \Gamma\widetilde{K}$, $\Gamma\widetilde{K}/\widetilde{K}$ is a $\mathbb{Z}_q$-extension with $[\Gamma_n\widetilde{K} \colon \Gamma_{n-1}\widetilde{K}] = q^n$, for each $n \in \mathbb{N}$, and $\Phi_{\mathrm{ur}}\widetilde{K}(q)/\Phi_{\mathrm{ur}}\widetilde{K}$ is a quasiinertial $\mathbb{Z}_q$-extension.

As Γ/Θ' is a $\mathbb{Z}_q$-extension, $\widetilde{K} \cap U' = \Theta'$, and Φ_{ur} contains a primitive q-th root of unity unless $\mathrm{char}(\Phi) = q$, the latter part of (5.2) follows at once from the former one, Galois theory and Lemma 3 (c). Taking into account that the degrees of finite extensions of $\widetilde{K}$ in $\Phi_{\mathrm{ur}}\widetilde{K}$ are not divisible by q ($\mathcal{G}(\Phi_{\mathrm{ur}}\widetilde{K}/\widetilde{K}) \cong \mathcal{G}(\Phi_{\mathrm{ur}}/U) \cong \prod_{\pi_q \in S_q(T)} \mathbb{Z}_{\pi_q}$), and using trace transitivity in towers of finite separable extensions, one concludes that (5.2) can be supplemented as follows:

(5.3) The $\mathbb{Z}_q$-extension $\Gamma\widetilde{K}/\widetilde{K}$ is quasiinertial.

We are now in a position to construct a quasilocal Henselian field of the type required by Theorem 1. Fix a positive number $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ and a rational function field $\widetilde{K}(X)$ in one indeterminate over $\widetilde{K}$. It is easily verified that $\tilde{v}$ is uniquely extendable to a valuation $\tilde{v}_\gamma$ of $\widetilde{K}(X)$ satisfying the equality $\tilde{v}_\gamma(X) = \gamma$, and it follows from the choice of γ that $\tilde{v}_\gamma(\widetilde{K}(X))$ is an Archimedean group equal to the sum of $\mathbb{Q}$ and $\langle\gamma\rangle$. In addition, it becomes clear that $\tilde{v}_\gamma(\widetilde{K}(X))$ is isomorphic (as an abstract group) to the direct sum $\mathbb{Q} \oplus \langle\gamma\rangle$, and the residue field of $(\widetilde{K}(X), \tilde{v}_\gamma)$ coincides with $\tilde{k}$. Note also that $\bar{v}_\gamma(\overline{\Phi}(X)) = \tilde{v}_\gamma(\widetilde{K}(X))$, where $\bar{v}_\gamma$ is the valuation of $\overline{\Phi}(X)$ naturally extending $\tilde{v}_{\overline{\Phi}}$ and $\tilde{v}_\gamma$. Now take a Henselization (K, v) of $(\widetilde{K}(X), \tilde{v}_\gamma)$ so that $K \subset \overline{\Phi}(X)_{\mathrm{sep}}$, and fix an algebraic closure $\overline{K}$ of K including $\overline{\Phi}(X)_{\mathrm{sep}}$ as a subfield. It is well-known that $(K, v)/(\widetilde{K}(X), \tilde{v}_\gamma)$ is immediate. The obtained properties of $\tilde{v}_\gamma(\widetilde{K}(X))$ and the equality $\tilde{v}_\gamma(\widetilde{K}(X)) = v(K)$ indicate that $v(K)/pv(K)$ is of order p and $v(\gamma) \notin pv(K)$, for any $p \in \mathbb{P}$; in particular, $v(K)$ is totally indivisible. We show that K, v and $I_\infty = \Gamma K$ are admissible by Theorem 1. As a first step towards this, we prove the following:

(5.4) (i) $\widetilde{K}$ is algebraically closed in K and $\overline{\Phi}K/K$ is a Galois extension with $\mathcal{G}(\overline{\Phi}K/K) \cong \mathcal{G}_{\widetilde{K}} \cong \prod_{p \in S(T)} \mathbb{Z}_p$; in addition, $v(\overline{\Phi}K) = v(K)$, $\Gamma K/K$ is an immediate $\mathbb{Z}_q$-extension, and $[\Gamma_n K \colon K] = q^n$, for each $n \in \mathbb{N}$;

(ii) $\Gamma\Omega/\Omega$ is a quasiinertial $\mathbb{Z}_q$-extension, for every finite extension Ω of K in $\overline{K}$.

Let $K(\sqrt[q]{X})$ be an extension of K in $\overline{K}$ obtained by adjunction of a q-th root of X. It is clear from the definition of $\tilde{v}_\gamma$ and the immediacy of the valued extension $(K, v)/(\widetilde{K}(X), \tilde{v}_\gamma)$ that $K(\sqrt[q]{X})/K$ is totally ramified and $[K(\sqrt[q]{X})\colon K] = q$. Since $\widetilde{K}$ is perfect and $K \in I(\widetilde{K}(X)_{\rm sep}/\widetilde{K}(X))$, it is also clear that in case ${\rm char}(\Phi) = q$, $K(\sqrt[q]{X})$ is the unique purely inseparable extension of K in $\overline{K}$ of degree q. Note further that the inclusion of $v(K) = \tilde{v}_\gamma(\widetilde{K}(X))$ in $\mathbb{R}$ guarantees that $\widetilde{K}(X)_{\tilde{v}_\gamma}$ is Henselian with respect to its valuation v_γ continuously extending $\tilde{v}_\gamma$. As $(\widetilde{K}(X)_{\tilde{v}_\gamma}, v_\gamma)$ is immediate over $(\widetilde{K}(X), \tilde{v}_\gamma)$, these facts show that K is $\widetilde{K}(X)$-isomorphic to the (relative) algebraic closure of $\widetilde{K}(X)$ in $\widetilde{K}(X)_{\tilde{v}_\gamma}$ (cf. [12], Sect. 18.3). At the same time, it follows from the definition of the valuation $\bar{v}_\gamma$ of $\overline{\Phi}(X)$ that an element $\rho \in \overline{\Phi}$ lies in $\widetilde{K}(X)_{v_\gamma}$ if and only if $\rho \in \widetilde{K}_{\tilde{v}}$. Taking also into account that $\widetilde{K}$ is algebraically closed in $\widetilde{K}_{\tilde{v}}$ (because $\widetilde{K}$ is perfect and $\tilde{v}$ is Henselian), one concludes that $\widetilde{K}$ is algebraically closed in K. In view of Galois theory, this means that $\overline{\Phi}K/K$ is a Galois extension with $\mathcal{G}(\overline{\Phi}K/K) \cong \mathcal{G}_{\widetilde{K}}$. These observations prove the former part of (5.4) (i), so we turn to the proof of the latter one. Using the equalities $\bar{v}_\gamma(\overline{\Phi}(X)) = v_\gamma(\widetilde{K}(X)) = v(K)$, and replacing $\widetilde{K}$ by any of its finite extensions in $\overline{\Phi}$, one obtains further that $v(\overline{\Phi}K) = v(K)$. As ${\rm cd}_{p'}(\mathcal{G}_{\tilde{k}}) = 0$, for every $p' \in \mathbb{P} \setminus S(T)$, this result implies in conjunction with (3.3) and (5.2) that $\Gamma K/K$ is immediate and $\Gamma \cap K = \Theta'$, so (5.4) (i) is proved. As to (5.4) (ii), it can be deduced from Galois theory and Lemma 2, since $\Gamma\widetilde{K}/\widetilde{K}$ is quasiinertial (by (5.3)), $v(K) \leq \mathbb{R}$, v extends $\tilde{v}$ upon K, $v(K)$ is Archimedean and $\widetilde{K}$ is algebraically closed in K.

Next we show that ${\rm Br}(K)_p \neq \{0\}$ if and only if $p \in S(T)$. Suppose first that $p \notin S(T)$. Then $p \dagger [\widetilde{M}\colon \widetilde{K}]$, for any finite extension $\widetilde{M}$ of $\widetilde{K}$, which implies ${\rm Br}(K)_p \cap {\rm Br}(\overline{\Phi}K/K) = \{0\}$ (cf. [21], Sect. 13.4). On the other hand, $\overline{\Phi}K/\overline{\Phi}$ is a field extension of transcendency degree 1, so it follows from Tsen's theorem (see [21], Sect. 19.4) that ${\rm Br}(\overline{\Phi}K) = \{0\}$. It is therefore easy to see that ${\rm Br}(K) = {\rm Br}(\overline{\Phi}K/K)$ and ${\rm Br}(K)_p = \{0\}$. Assume now that $p \in S(T)$. Then it follows from Galois theory and (5.4) that $I(\overline{\Phi}K/K)$ contains a cyclic extension Y_p of K of degree p. Moreover, by (5.4) (i), $v(Y_p) = v(K)$, whence the uniqueness of v_{Y_p} implies $N(Y_p/K) \subseteq \{\lambda \in K^*\colon\ v(\lambda) \in pv(K)\}$. Since $v(K) \neq pv(K)$, this means that ${\rm Br}(Y_p/K) \neq \{0\} \neq {\rm Br}(K)_p$.

It remains to be proved that K is quasilocal and ${\rm Br}(K) \cong T$. Assuming as above that $p \in S(T)$, let G_p be a Sylow pro-p-subgroup of $\mathcal{G}_K$ and K_p the fixed field of G_p. We show that K_p is p-quasilocal with ${\rm Br}(K_p) \cong \mathbb{Z}(p^\infty)$. The equality $v(K) = v_\gamma(\widetilde{K}(X))$ and the isomorphism $v(K_p)/pv(K_p) \cong v(K)/pv(K)$ guarantee that $v(K_p)/pv(K_p)$ is of order p. When $p \neq q$, this enables one to deduce from (5.4) and [11], Lemma 1.2, that K_p^*/K_p^{*p} is a group of order p^2. As K_p contains a primitive p-th root of unity and $\mathrm{Br}(K)_p \cap \mathrm{Br}(K_p/K) = \{0\}$, the obtained results and Galois cohomology (see [30], Lemma 7, [20], (11.5), and [24], Ch. I, 4.2) prove that G_p is a Demushkin group, $r(p)_{K_p} = 2$ and $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty)$. Hence, by [3], I, Lemma 3.8, K_p is p-quasilocal. It remains to be seen that K_q is q-quasilocal and $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$. As $\tilde{k}$ is perfect, $\mathrm{cd}_q(\mathcal{G}_{\tilde{k}}) = 0$ and $\widehat{K} = \tilde{k}$, $\widehat{K}_q$ is an algebraic closure of $\tilde{k}$, so $\widehat{Z} = \widehat{K}_q$, for each $Z \in I(K_{\mathrm{sep}}/K_q)$. In addition, it follows from Tsen's theorem that $\mathrm{Br}(K_q) = \mathrm{Br}(\Gamma K_q/K_q)$. Applying (5.4), (3.8) and Lemma 2, one also sees that $\nabla_0(\Gamma_1) \subseteq N(\Gamma_n K_q/\Gamma_1 K_q)$, for each $n \in \mathbb{N}$. As $\Gamma_1 K_q/K_q$ is immediate, this enables one to deduce from (3.7) and Hilbert's Theorem 90 that an element $\theta \in K_q^*$ lies in $N(\Gamma_\nu K_q/\Gamma_1 K_q)$, for a given index ν, if and only if $\theta^q \in N(\Gamma_\nu K_q/K_q)$. Since $\mathrm{Br}(\Gamma K_q/K_q) = \cup_{n=1}^{\infty} \mathrm{Br}(\Gamma_n K_q/K_q)$, these observations and the canonical isomorphisms $\mathrm{Br}(\Gamma_n K_q/K_q) \cong K_q^*/N(\Gamma_n K_q/K_q)$, $n \in \mathbb{N}$ (cf. [21], Sect. 15.1, Proposition b), prove that ${}_q\mathrm{Br}(K_q) = \mathrm{Br}(\Gamma_1 K_q/K_q)$. The obtained result, combined with the fact that $\widehat{K}_q$ is algebraically closed and $v(K_q)/qv(K_q)$ is of order q, proves that $N(\Gamma_1 K_q/K_q) = \{\mu \in K_q^* \colon\ v(\mu) \in qv(K_q)\}$, ${}_q\mathrm{Br}(K_q)$ is of order q and $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$. Let now Λ be an extension of K_q in K_{sep}, such that $[\Lambda \colon K_q] = q$ and $\Lambda \neq \Gamma_1 K_q$, and let $V_q(\Lambda) = \{\lambda \in \Lambda \colon\ v_\Lambda(\lambda) \in qv(\Lambda)\}$. Applying (5.4) and (3.7), and arguing as in the proof of the isomorphism $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$, one obtains consecutively the following results:

(5.5) (i) $V_q(\Lambda) \subseteq N(\Gamma_1\Lambda/\Lambda)$; $\tau(\lambda')\lambda'^{-1} \in N(\Gamma_1\Lambda/\Lambda)$, for each $\lambda' \in \Lambda^*$ and every generator τ of $\mathcal{G}(\Lambda/K_q)$;

(ii) $\mathrm{Br}(\Gamma_1\Lambda/\Lambda) = {}_q\mathrm{Br}(\Lambda) \neq \{0\}$; hence $N(\Gamma_1\Lambda/\Lambda) \neq \Lambda^*$.

As $\widehat{\Lambda}$ is algebraically closed and $v(\Lambda)/qv(\Lambda)$ has order q, one also proves that

(5.6) (i) $N(\Gamma_1\Lambda/\Lambda) = V_q(\Lambda)$ and $\Gamma_1\Lambda/\Lambda$ is immediate.

(ii) $K^* \subseteq N(\Gamma_1\Lambda/\Lambda)$, provided that Λ is totally ramified over K_q; when this holds, $\mathrm{Br}(\Gamma_1/K_q) \subseteq \mathrm{Br}(\Lambda/K_q) = {}_q\mathrm{Br}(K_q)$.

In view of (5.5) (ii) and (5.6) (ii), it suffices, for the proof of the q-quasilocality of K_q, to show that Λ/K_q is totally ramified. Assuming the opposite, one gets from (3.3) and the equality $\widehat{\Lambda} = \widehat{K}_q$ that Λ/K_q is immediate. Fix a generator τ of $\mathcal{G}(\Lambda/K_q)$, denote by τ' the Γ_1-automorphism of $\Gamma_1\Lambda$ extending τ, and put $D_\rho = (\Lambda/K_q, \tau, \rho)$, $\Delta_\rho = (\Gamma_1\Lambda/\Gamma_1, \tau', \rho)$, for some $\rho \in K_q^*$. Clearly, $\Delta_\rho \cong D_\rho \otimes_{K_q} \Gamma_1$ over Γ_1. Hence, the equality $\mathrm{Br}(\Gamma_1/K_q) = {}_q\mathrm{Br}(K_q)$ requires that $[\Delta_\rho] = 0$ in $\mathrm{Br}(\Gamma_1)$. On the other hand, (5.6) (i) and the assumption on Λ/K_q imply $\Gamma_1\Lambda/\Gamma_1$ is immediate. This shows that if $v(\rho) \notin qv(K_q)$, then $D_\rho \in d(K_q)$ and $\Delta_\rho \in d(\Gamma_1)$, whence $[\Delta_\rho] \neq 0$. The observed contradiction proves that Λ/K_q is totally ramified, so K_q is q-quasilocal (with $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$).

It is now easy to complete the proof of Theorem 1. Indeed, it follows from [3], I, Lemma 8.3, and the p-quasilocal property of the fields K_p, $p \in \Pi(K)$, that K is quasilocal. As K is nonreal and $S(T) = \{p \in \mathbb{P} \colon\ \mathrm{Br}(K)_p \neq \{0\}\}$, this result, [4], Lemma 3.3 (i) (see also [3], I, Theorem 3.1), and the isomorphisms $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty)$, $p \in S(T)$, yield $\mathrm{Br}(K) \cong T$. Theorem 1 is proved.

6. Complements to Theorem 1

First we show that, in residual characteristic 2, Theorem 1 and [4], Theorem 1.1, fully describe the isomorphism classes of Brauer groups of quasilocal Henselian fields admissible by Proposition 1.

Proposition 2. *Let (K, v) be a quasilocal Henselian field satisfying the conditions of Proposition 1, and let $\mathrm{char}(\widehat{K}) = 2$. Then there exists an immediate norm-inertial $\mathbb{Z}_2$-extension Γ/K; in particular, $\mathrm{Br}(K)_2 \cong \mathbb{Z}(2^\infty)$.*

Proof. Proposition 1 and our assumptions show that $\widehat{K}$ is perfect and $\mathrm{cd}_2(\mathcal{G}_{\widehat{K}}) = 0$. In view of (3.3) and (3.5), this ensures that $\mathrm{cd}_2(\mathcal{G}(K_{\mathrm{tr}}/K)) = 0$, K_{tr} is the fixed field of a Sylow pro-2-subgroup of $\mathcal{G}_K$, and K_{tr} has a $\mathbb{Z}_2$-extension Y in K_{sep}. In addition, it follows from the uniqueness of Y and the normality of K_{tr}/K that Y/K is a Galois extension. Note also that $\mathcal{G}(Y/K_{\mathrm{tr}}) \cong \mathbb{Z}_2$ and $\mathcal{G}(Y/K_{\mathrm{tr}})$ is a normal Sylow pro-2-subgroup of $\mathcal{G}(Y/K)$. These observations indicate that $\mathcal{G}(Y/K_{\mathrm{tr}})$ is included in the centre of $\mathcal{G}(Y/K)$. It is therefore clear from Galois theory and Burnside's theorem (cf. [13], Theorem 14.3.1, and [24], Ch. I, 5.9) that $\mathcal{G}(Y/K)$ possesses a closed normal subgroup N, such that $\mathcal{G}(Y/K_{\mathrm{tr}})N = \mathcal{G}(Y/K)$ and $\mathcal{G}(Y/K_{\mathrm{tr}}) \cap N = \{1\}$. This means that $\mathcal{G}(Y/K) \cong \mathcal{G}(Y/K_{\mathrm{tr}}) \times N$, the fixed field Γ of N is a $\mathbb{Z}_2$-extension of K, $\Gamma K_{\mathrm{tr}} = Y$ and $\Gamma \cap K_{\mathrm{tr}} = K$. As Y/K_{tr} is immediate and finite extensions of K in K_{tr} are of odd degrees, one deduces from (3.3) and Proposition 1 that Γ/K is immediate and norm-inertial. Hence, by [4], Theorem 1.1, $\mathrm{Br}(K)_2 \cong \mathbb{Z}(2^\infty)$.

Theorem 1 and Proposition 2 can be complemented as follows:

Proposition 3. *Let (Φ, ω) satisfy the conditions of Theorem 1, for some $q > 2$, and let T_q be a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$ with $T_q = \{0\}$. Then there exists a valued extension (K, v) of (Φ, ω), such that v is Henselian, $v(K)$ is totally indivisible and Archimedean, $\widehat{K} = \widehat{\Phi}_{\mathrm{sep}}$, K/Φ has transcendency degree 1, $\mathrm{Br}(K) \cong T$, and K admits a defectful finite extension in K_{sep}.*

Proof. It is clearly sufficient to consider only the special case where $\mathrm{char}(\Phi) = q$ or Φ contains a primitive q-th root of unity. Our argument goes along the same lines as the proof of Theorem 1, so we omit the details and note only its main steps. Our starting point are the following statements:

(6.1) For any integer $m \geq 2$ dividing $q - 1$, Φ_{sep} contains as a subfield a totally ramified Galois extension Ψ_m of Φ, such that $[\Psi_m \colon \Phi] = qm$ and $\mathcal{G}(\Psi_m/\Phi)$ is a metacyclic group with a cyclic non-normal subgroup of order m. For example, if $\mathrm{char}(\Phi) = q$ and π is a generator of $M_\omega(\Phi)$, then one may take as Ψ_m the field $\Phi(\xi_m)$, where $\xi_m \in \Phi_{\mathrm{sep}}$ is a root of the polynomial $g_m(X) = (X^q - X)^m - \pi^{-1}$. When $\mathrm{char}(\Phi) = 0$, Ψ_m can be chosen among the subfields of the root field $\widetilde{\Psi}_m \in I(\overline{\Phi}/\Phi)$ of the polynomial $h_m(X) = (X^q - 1)^m - \pi$, under the same hypothesis on π.

Fix m as in (6.1), put $\Psi'_m = \Psi_m K_{\mathrm{tr}}$ and let $\widetilde{K} \in I(\overline{\Phi}/\Phi_{\mathrm{ur}})$ be maximal with respect to the property that $\widetilde{K} \cap \Psi'_m = \Phi_{\mathrm{ur}}$. Observing that $\mathcal{G}(\Psi'_m/\Phi_{\mathrm{ur}})$ is a pro-supersolvable group and $\mathcal{G}(\Psi'_m\widetilde{K}/\widetilde{K}) \cong \mathcal{G}(\Psi_m/\Phi_{\mathrm{ur}})$, and applying Galois theory and Huppert's theorem (cf. [13], Theorem 10.5.8), one obtains that $\mathcal{G}_{\widetilde{K}}$ is pro-supersolvable. Therefore, by [13], Theorem 10.5.1, $\mathcal{G}(L/\widetilde{K})$ is supersolvable, for each finite Galois extension $L/\widetilde{K}$. Hence, for any $p \in \mathbb{P}$, L possesses a subfield $L_{[p]}$ that is a Galois extension of $\widetilde{K}$ with $\mathcal{G}(L_{[p]}/\widetilde{K})$ isomorphic to a Hall Π-subgroup of $\mathcal{G}(L/\widetilde{K})$, where $\Pi = \{\pi \in \mathbb{P}\colon\ \pi \leq p\}$ (cf. [13], Sect. 9.3 and Corollary 10.5.2). Let H_p be a Sylow p-subgroup of $\mathcal{G}(L_{[p]}/\widetilde{K})$. We show that H_p is cyclic. The group $\mathcal{G}(L_{[p]}/\widetilde{K})$ is supersolvable which implies that it includes H_p as a normal subgroup. The Frattini subgroup $\Phi(H_p)$ of H_p is characteristic in H_p, so it is normal in $\mathcal{G}(L_{[p]}/\widetilde{K})$, and by Galois theory, the fixed field, say Λ_p, of $\Phi(H_p)$ is a Galois extension of $\widetilde{K}$. Let $\overline{H}_p$ be a Sylow p-subgroup of $\mathcal{G}(\Lambda_p/\widetilde{K})$. Then $\mathcal{G}(\Lambda_p/\widetilde{K}) \cong \mathcal{G}(L_{[p]}/\widetilde{K})/\Phi(H_p)$, $\overline{H}_p \cong H_p/\Phi(H_p)$ and $\overline{H}_p$ is an abelian normal subgroup of $\mathcal{G}(\Lambda_p/\widetilde{K})$ of period p. Also, $\mathcal{G}(\Lambda_p/\widetilde{K})$ is supersolvable, and by [13], Corollary 10.5.2, it has a normal subgroup of order p, the greatest prime divisor of $[\Lambda_p\colon \widetilde{K}]$. Regarding $\overline{H}_p$ as an $\mathbb{F}_p$-vector space, and considering the action on $\overline{H}_p$ by conjugation of some Hall Π_p-subgroup of $\mathcal{G}(\Lambda_p/\widetilde{K})$, for $\Pi_p = \Pi \setminus \{p\}$, one obtains from Maschke's theorem that if $\overline{H}_p$ is noncyclic, then it decomposes into the direct product of normal subgroups of $\mathcal{G}(\Lambda_p/\widetilde{K})$ of order p. In view of Galois theory, this leads to the conclusion that if $\overline{H}_p$ is noncyclic, then there exist degree p extensions Λ_1 and Λ'_1 of $\widetilde{K}$ in Λ_p, such that $[\Lambda_1\Lambda'_1\colon \widetilde{K}] = p^2$. Therefore, Λ_1 and Λ'_1 are not $\widetilde{K}$-isomorphic. Our conclusion, however, contradicts the maximum condition on $\widetilde{K}$ and so proves that $\overline{H}_p$ is cyclic. It is now easy to see that H_p has a unique maximal subgroup, whence it is cyclic as well. Summing-up the obtained results, one proves that:

(6.2) $\widetilde{K}$ is perfect, $\widetilde{K}_{\mathrm{tr}} = \Phi_{\mathrm{tr}}\widetilde{K}$ and $\mathcal{G}(\widetilde{K}_{\mathrm{tr}}/\widetilde{K}) \cong \mathcal{G}(\Phi_{\mathrm{tr}}/\Phi_{\mathrm{ur}})$; $\overline{\Phi}/\widetilde{K}_{\mathrm{tr}}$ is a quasiinertial $\mathbb{Z}_q$-extension and the Sylow pro-p-subgroups of $\mathcal{G}_{\widetilde{K}}$ are isomorphic to $\mathbb{Z}_p$, for each $p \in \mathbb{P}$; the Sylow pro-q-subgroup of $\mathcal{G}_{\widetilde{K}}$ is normal and equals the closure of the commutator subgroup of $\mathcal{G}_{\widetilde{K}}$.

As in the proof of Theorem 1, let $\widetilde{K}(X)$ be the rational function field in an indeterminate X with coefficients in $\widetilde{K}$, and $\tilde{v}_\gamma$ the valuation of $\widetilde{K}(X)$ extending ω so that $\tilde{v}_\gamma(X) = \gamma$, where γ is a given element of $\mathbb{R} \setminus \mathbb{Q}$. Fix a Henselization (K_0, v_0) of $(\widetilde{K}(X), \tilde{v}_\gamma)$, put $N(T) = \{p \in \mathbb{P} \setminus \{q\}\colon\ T_p = \{0\}\}$, and let K be an extension of K_0 in $K_{0,\mathrm{sep}}$ maximal with respect to the property that $K \cap \overline{\Phi} = \widetilde{K}$ and finite extensions of K_0 in K are cyclic and totally ramified of degrees not divisible by any $p' \in \mathbb{P} \setminus N(T)$. Note that (K, v) has the properties required by Proposition 3, where v is a prolongation of v_0 on K. It follows from the definition of (K, v) that $\widehat{K} = \widehat{\Phi}_{\mathrm{sep}}$, and for each $p \in N(T)$, $v(K)/pv(K)$ is of order p and $K(p)/K$ is a $\mathbb{Z}_p$-extension; also, $v(K)/pv(K)$ has order p^2 and $r_p(K) = 2$ in case $p \in \mathbb{P} \setminus N(T)$ and $p \neq q$. Since K contains a primitive m-th root of unity, for any $m \in \mathbb{N}$ not divisible by q, these observations show that $\mathrm{Br}(K)_p = \{0\}$, $p \in \mathbb{N}(T)$, and $\mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$, $p \in \mathbb{P} \setminus N(T)$, $p \neq q$. The assertion that K is quasilocal and satisfies with v the conditions of Proposition 1 is proved similarly to Theorem 1, so what remains to be seen is that $\mathrm{Br}(K)_q = \{0\}$. Let Θ_m be the extension of Φ in Ψ_m of degree m. Then Θ_m/Φ and $\Theta_m K/K$ are cyclic extensions of degree m, and $\Theta_m K$ has an immediate $\mathbb{Z}_q$-extension Γ in K_{sep} that is a Galois extension of K. This implies $\mathrm{Br}(\Theta_m K)_q \cong \mathbb{Z}(q^\infty)$. Using (6.1) and regarding ${}_q\mathrm{Br}(\Theta_m K)$ as a module over the group algebra $\mathbb{F}_q[\mathcal{G}(\Theta_m K/K)]$, one obtains that if τ_m is a generator of $\mathcal{G}(\Theta_m K/K)$, then $\tau_m b = fb$, $b \in {}_q\mathrm{Br}(\Theta_m K)$, for some $f \in \mathbb{F}_q^*$, $f \neq 1$. As $m \mid q-1$, $f^m = 1$ and $m > 1$, this observation shows that ${}_q\mathrm{Br}(\Theta_m K)$ is included in the kernel of the corestriction homomorphism $\mathrm{Br}(\Theta_m K) \to \mathrm{Br}(K)$, which enables one to deduce from the basic restriction-corestriction formula for Brauer groups (cf. [27]) that $\mathrm{Br}(K)_q = \{0\}$. Proposition 3 is proved.

Remark 2. *Take $(\widetilde{K}, \tilde{v})$ and $\widetilde{K}(X)$ as in the proof of Theorem 1, and let $\tilde{v}_0$ be a restricted Gauss valuation of $\widetilde{K}(X)$ extending $\tilde{v}$ (see [12], Example 4.3.2). Then, by [4], Proposition 6.5, there exists a quasilocal Henselian field (K, v), such that $K \in I(\widetilde{K}(X)_{\mathrm{sep}}/\widetilde{K})$, v extends $\tilde{v}_0$, $\widehat{K} = \widehat{K}_{\mathrm{sep}} \neq \widehat{K}^q$, $K_{\mathrm{sep}} = K(q)$, K possesses an immediate quasiinertial $\mathbb{Z}_q$-extension, and $Br(K)$ is a divisible hull of the (infinite) quotient group $\widehat{K}^*/\widehat{K}^{*q}$. As shown at the end of [4], Sect. 6, for any global field Ψ with char$(\Psi) = 0$ or q, this enables one to find, by the method of proving Theorem 1, field extensions K_t/Ψ, $t \in \mathbb{N}$, such that K_t is quasilocal, $\mathcal{G}_{K_t}$ is a pro-q-group, the transcendency degree of K_t/Ψ is equal to t, the class $d(K_t) \setminus \{K_t\}$ consists of division algebras of infinite genus, in the sense of [1], and $[K_t\colon K_t^q] = q$ in case char$(\Psi) = q$. More precisely, by [3], I, Corollaries 8.5 and 8.6, the genus of any $D_t \in d(K_t)$ equals both the set $\{[D'_t] \in \mathrm{Br}(K_t)\colon\ D'_t \in d(K_t), \mathrm{ind}(D'_t) = \mathrm{ind}(D_t)\}$ and the equivalence class of D_t, in the sense of [16], Definition 2.1. When char$(\Psi) = 0$, these results ensure that $\mathcal{G}_{K_t}$ is a pro-q-group of Demushkin type with an infinite (Galois) cohomology group $H^2(\mathcal{G}_{K_t}, \mathbb{F}_q)$ (see [3], I, Lemma 3.8, and [5], Proposition 5.1).*

Remark 2 attracts interest in the open problem of whether a global field Ψ admits a transcendental finitely-generated extension Ψ', such that $\mathcal{G}_{\Psi'}$ possesses a pro-p-subgroup P of Demushkin type for which $H^2(P, \mathbb{F}_p)$ is a noncyclic finite group.

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