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NON ADDITIVE (FUZZY) MEASURES. NON LINEAR (FUZZY) INTEGRALS

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Abstract. We present a survey of recent results concerning generalized measures and integrals: possibly non-additive measures and possibly non-linear integrals with respect to generalized measures. This theory is very recent, develops very quickly and the amount of literature dedicated to it is already huge, due to its multiple applications.

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1. Introduction

Classical measure theory is based upon the concept of additivity (or, which is more, upon countable additivity).

Recently it was seen that more varied tools,other than additive measures, are necessary in order to describe and study a multitude of phenomena. These tools are the generalized measures which are monotone and possibly non additive. A rough discussion pertaining to this type of measures can run as follows: superadditive measures indicate a cooperative action or synergy between the measured items (sets), while subadditive measures indicate inhibitory effects, lack of cooperation or incompatibility between the measured items (sets). Additive measures can express non interaction or indifference.

It is generally accepted that the most important generalized measures are the $\lambda-$measures, introduced by the Japanese scholar Michio Sugeno in his doctoral thesis [9]. He called these measures „fuzzy measures“, but today most people call the normalized $\lambda-$measure *Sugeno measures.* An important result due to Z. Wang [10] states that any $\lambda-$measure can be obtained from a classical measure via a canonical procedure – composition with a special increasing function (see also [11]). We called the generalized measures which can be obtained in such a way *representable measures* in [4]. In the same paper it was pointed out that $\lambda-$measure appear naturally also within the framework of functional equations.

Aside the mathematical interest of the $\lambda-$measures, it is worth noticing that they have an important impact in many domains of activity. For instance, we refer to the mathematical theory of evidence, created by G. Shafer under the influence of A. P. Dempster. This theory is based upon belief and plausibility measures, which are special cases of $\lambda-$measures (see [8]) and [11].

Other interesting facts concerning $\lambda-$measures can be found in [1] and [7].

Lebesgue's philosophy says that, once we have a measure, it is possible to have an integral (with respect to this measure). Following this line of thinking, integrals with respect to generalized measures appeared. The most popular such integrals are the Sugeno and the Choquet integral. The Sugeno integral is very far away from all standard integrals, while the Choquet integral is a direct generalization of the standard abstract Lebesgue integral. Because of the possible non additivity of the measures involved,these integrals are possibly non linear.

Thinking about integrals as information fusion instruments (i.e. aggregation instruments for compressing a multitude of numerical data into a single numerical date) we see that the use of non additive measures in computing the integrals allows us to take into consideration the interaction between the generators of data (see the beginning of this introduction). The non linear integrals can be used in other directions, e. g. multiregression, classification. To be synthetic, we can use nonlinear integrals in a huge lot of data mining operations.

A short presentation of the contents follows. The first part deals with generalized measures, whilst the second part deals with nonlinear integrals.

The first part (paragraph 2) begins with the presentation of general measures and their basic properties and continues with $\lambda-$additive measures. Basic properties of $\lambda-$additive measures are presented, among them the idea of representability. The construction of $\lambda-$measures with preassigned values follows. Here the recent result concerning the existence and uniqueness of $\lambda-$measures on $\mathcal{P}(\mathbb{N})$ with preassigned values appears (see [6]). The first part ends with a recent result concerning the structure of Sugeno measures on the code space (see [5]).

The second part (paragraph 3) begins with the presentation of the Sugeno and Choquet integrals with their basic properties. Some results concerning sequences of positive measurable functions and their Sugeno and Choquet integrals follow. The second part (i.e. the paper) ends with the transformation theorem for the Sugeno integral and with a recent result concerning the parametric continuity of the numerical flow of Sugeno and Choquet integrals attached to the canonical flow of the $\lambda-$measures generated by a probability (for the last topic, see [3] and [2], which is an expanded version of [3]).

2. Non Additive (Fuzzy) Measures

We shall work only with *positive finite* measures.

The set $[0,\infty)$ will be denoted by $\mathbb{R}_+$. The set$\{1,2,\ldots,n,\ldots\}$ will be denoted by $\mathbb{N}$. For any T we write $\mathcal{P}(T)$ to denote the set of all subsets of T. We shall consider *classes of sets*, i. e. sets $\mathcal{T}\subset\mathcal{P}(T)$ such that $\emptyset\in\mathcal{T}$.

Definition 1. *We shall call measure (or generalized measure) a function $\mu:\mathcal{T}\to\mathbb{R}_+$ having the following properties:*

(i) $\mu(\emptyset)=0$.

(ii) For any A, B in $\mathcal{T}$ such that $A\subset B$,one has $\mu(A)\le\mu(B)$ (we say that μ is monotone or increasing).

Definition 2. *We say that a measure $\mu:\mathcal{T}\to\mathbb{R}_+$ is normalized if*

a) $T\in\mathcal{T}$;

b) $\mu(T)=1$.

Recall that a function $\mu:\mathcal{T}\to\mathbb{R}_+$ is additive if

$$\mu(A\cup B)=\mu(A)+\mu(B)$$

whenever A, B are in $\mathcal{T}$, $A\cap B=\emptyset$ and $A\cup B\in\mathcal{T}$. If $\mathcal{T}$ is a ring and μ is additive, then μ is a (generalized) measure.

Definition 3. *We say that a measure $\mu:\mathcal{T}\to\mathbb{R}$ is representable if:*

α) $T\in\mathcal{T}$ and $\mu(T)=A>0$.

β) There exists an additive measure $m:\mathcal{T}\to\mathbb{R}_+$ such that $m(T)=a>0$ and a strictly increasing bijection $h:[0,a]\to[0,A]$ such that $\mu=h\circ m$.

In this case, we say that the pair (m,h) represents μ and h is a transfer function for μ.

Remarks

1. In the previous context, h and h^{-1} are mutually inverse homeomorphisms.
2. Clearly, any additive measure μ is representable (if $\mu(T)=A$, we have the transfer function $h:[0,A]\to[0,A]$, $h(x)=x$ and the pair (μ,h) represents μ).
3. It is possible to have more than one pair (m,h) which represents the same μ. For instance, take $T=\{1,2\}$, $\mathcal{T}=\mathcal{P}(T)$ and $\mu:\mathcal{T}\to\mathbb{R}_+$ given via $\mu(\emptyset)=0$, $\mu(T)=1$, $\mu(\{1\})=\alpha$, $\mu(\{2\})=\beta$, where $0<\alpha<\beta<1$. Let us consider an additive measure $m:\mathcal{T}\to\mathbb{R}_+$ given via $m(\emptyset)=0$, $m(T)=1$, $m(\{1\})=a$, $m(\{2\})=b$, where $0<a<1-a=b<1$. Then, any strictly increasing continuous function $h:[0,1]\to[0,1]$ such that $h(0)=0$, $h(1)=1$,$h(a)=\alpha$, $h(b)=\beta$, generates the pair (m,h) which represents μ.

□

Example 1. (Popular transfer functions.) Take the strictly positive numbers a, A.

a) For $\theta > 0$, define $h : [0,a] \to [0,A]$, via $h(x) = A\left(\frac{x}{a}\right)^{\theta}$.

b) For any $0 \neq \lambda \in \left(-\frac{1}{A}, \infty\right)$, define $h_{\lambda} : [0,a] \to [0,A)$, via

$$h_{\lambda}(x) = \frac{(1+\lambda A)^{\frac{x}{a}} - 1}{\lambda}.$$

□

Not all measures are representable.

Example 2. Take $T = \{1,2\}$, $\mathcal{T} = \mathcal{P}(T)$,$\mu : \mathcal{T} \to \mathbb{R}_+$ defined via $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(\{1\}) = \alpha$, $\mu(\{2\}) = 1$, where $0 < \alpha \leq 1$. Then μ is a measure which is not representable.

□

Caution

From now on we shall work only with *non null measures* having the following property:

$$a = \sup \mu \overset{def}{=} \sup \{\mu(A) \mid A \in \mathcal{T}\} < \infty.$$

Hence $0 < a < \infty$. In particular, this property is valid in case $T \in \mathcal{T}$ and in this case $a = \mu(T)$.

□

Definition 4. *For a measure μ as above, we say that a number λ is $\mu-$admissible in case $\lambda \in \left(-\frac{1}{\sup \mu}, \infty\right)$.*

We can introduce now a most important class of (generalized) measures

Definition 5. *Let $\mu : \mathcal{T} \to \mathbb{R}_+$ be a measure and λ a $\mu-$admissible number. We say that μ satisfies the $\lambda-$rule (μ is $\lambda-$additive) if*

$$\mu(E \cup F) = \mu(E) + \mu(F) + \lambda\mu(E)\mu(F)$$

whenever E, F are in $\mathcal{T}$ and $E \cup F \in \mathcal{T}$, $E \cap F = \emptyset$.

In case there exists a $\mu-$admissible number δ such that μ satisfies the $\delta-$rule,we say that μ satisfies some $\lambda-$rule.

From now on, when asserting that μ satisfies the $\lambda-$rule, we tacitly assume that λ is $\mu-$admissible.

Remarks

1. The "necessity" of working with numbers λ which are $\mu-$admissible can be explained as follows. Then, if λ is $\mu-$admissible and $A \in \mathcal{T}$, one has $1+\lambda\mu(A) > 0$. Hence, if E, F are in $\mathcal{T}$ and $E \cup F \in \mathcal{T}$, $E \cap F = \emptyset$, one has

$$\mu(E \cup F) = \mu(E) + \mu(F)(1 + \lambda\mu(E)) \geq \mu(E)$$

(confirmation of the monotony) . □

2. If μ satisfies the $\lambda-$rule,then
– in case $\lambda < 0$, μ is *subadditive*

$$(\mu(E \cup F) \leq \mu(E) + \mu(F), \text{ whenever } E \cup F \in \mathcal{T},\ E \cap F = \emptyset)\,.$$

– in case $\lambda > 0$, μ is *superadditive*

$$(\mu(E \cup F) \geq \mu(E) + \mu(F), \text{ whenever } E \cup F \in \mathcal{T},\ E \cap F = \emptyset)\,.$$

– in case $\lambda = 0$, μ is *additive*

$$(\mu(E \cup F) = \mu(E) + \mu(F), \text{ whenever } E \cup F \in \mathcal{T},\ E \cap F = \emptyset)\,.$$

3. We pointed out in [4] that the measures which satisfy the $\lambda-$rule appear as the most natural non additive measure. □

Here are some computations.

Theorem 1. *Assume $\mathcal{T}$ is a ring and μ satisfies the $\lambda-$ rule. Then, for any E, F in $\mathcal{T}$, one has*

(i) $\mu(E \setminus F) = \dfrac{\mu(E) - \mu(F)}{1 + \lambda\mu(E \cap F)}$

(ii) $\mu(E \cup F) = \dfrac{\mu(E) + \mu(F) + \lambda\mu(E)\mu(F) - \mu(E \cap F)}{1 + \lambda\mu(E \cap F)}$

(iii) $\mu(E \Delta F) = \dfrac{\mu(E) + \mu(F) + \lambda\mu(E)\mu(F) - \lambda\mu(E \cap F)^2 - 2\mu(E \cap F)}{(1 + \lambda\mu(E \cap F))^2}$.

(iv) Assuming, supplementarily, that $\mathcal{T}$ is an algebra, one has

$$\mu\left(\complement_E\right) = \frac{\mu(T) - \mu(E)}{1 + \lambda\mu(E)}.$$

Now, we introduce the *dual* of a normalized measure.

Definition 6. *Assume that $\mathcal{T}$ has the following property: $\complement_A \in \mathcal{T}$ whenever $A \in \mathcal{T}$ (this is true, in particular, if $\mathcal{T}$ is an algebra). For any normalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$, we define the dual of μ, which is the normalized measure $\nu : \mathcal{T} \to \mathbb{R}_+$, acting via*

$$\nu(A) = 1 - \mu(\complement_A)$$

Remarks

1. The dual of the dual of μ is μ. Hence ν is the dual of μ if and only if μ is the dual of ν

2. In case $\mathcal{T}$ is an algebra and μ is additive, it follows that the dual of μ is μ (μ is autodual). □

Theorem 2. *Assume that $\mathcal{T}$ is an algebra, $\mu : \mathcal{T} \to \mathbb{R}_+$ is normalized and satisfies the $\lambda-$ rule (hence $\lambda \in (-1, \infty)$). Then the dual measure ν of μ satisfies the $\lambda'-$ rule, where*

$$\lambda' = -\frac{\lambda}{\lambda + 1}.$$

The properties gathered in the following theorem are often useful

Theorem 3. *Assume $\mathcal{T}$ is a ring and $\mu : \mathcal{T} \to \mathbb{R}_+$ satisfies the $\lambda-$ rule. Let $A \in \mathcal{T}$.*

1. For any $E \in \mathcal{T}$ such that $\mu(E) = 0$, one has (μ is null-additive):

$$\mu(A \cup E) = \mu(A \setminus E) = \mu(A \Delta E) = \mu(A).$$

2. Assume $\mu(A) = \sup \mu$. Then, for any $E \in \mathcal{T}$ such that $A \cap E = \emptyset$, one has $\mu(E) = 0$. Consequently, for any $F \in \mathcal{T}$ one has

$$\mu(F) = \mu(A \cap F).$$

3. Assume $\mathcal{T}$ is an algebra. Then, we have the implication

$$0 < \mu(A) < \mu(T) \Rightarrow 0 < \mu(T \setminus A) < \mu(T).$$

A property which is more restrictive than the $\lambda-$ rule follows

Definition 7. *Let $\mu : \mathcal{T} \to \mathbb{R}_+$ be a measure and λ a $\mu-$ admissible number.*

We say that μ satisfies the finite $\lambda-$ rule if

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{n} (1 + \lambda\mu(E_i)) - 1\right), & if \quad \lambda \neq 0 \\ \sum_{i=1}^{n} \mu(E_i), & if \quad \lambda = 0, \end{cases}$$

whenever $E_i, E_2, \ldots, E_n$ are mutually disjoint sets in $\mathcal{T}$ with $\bigcup_{i=1}^{n} E_i \in \mathcal{T}$.

In case there exists a $\mu-$admissible number δ such that μ satisfies the finite $\delta-$ rule, we say that μ satisfies some finite $\lambda-$rule.

Remarks

1. Clearly, to say that μ satisfies the finite $0-$ rule means to say that μ is finitely additive.

2. Clearly, if μ satisfies the finite $\lambda-$ rule it follows that μ satisfies the $\lambda-$ rule (and not conversely). In case $\mathcal{T}$ is a semiring, the converse assertion is true. □

The analogue of the $\sigma-$ additivity property for measures satisfying the $\lambda-$ rule is given in

Definition 8. *Let $\mu : \mathcal{T} \to \mathbb{R}_+$ be a measure and λ be μ admissible. We say that μ satisfies the $\sigma - \lambda-$rule if*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(1+\lambda\mu(E_i)) - 1\right), & if \quad \lambda \neq 0 \\ \sum_{i=1}^{\infty}\mu(E_i), & if \quad \lambda = 0, \end{cases}$$

whenever the mutually disjoint $E_i \in \mathcal{T}$ are such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{T}$.

In case there exists a $\mu-$ admissible δ such that μ satisfies the $\sigma - \delta$ rule, we say that μ satisfies some $\sigma - \lambda-$ rule.

Remarks

1. Clearly, the case $\lambda = 0$ means that μ is $\sigma-$additive (is a classical measure).

2. Because, for a number λ which is $\mu-$ admissible one has $1 + \lambda\mu(A) > 0$ for any $A \in \mathcal{T}$, it follows that the infinite product in the definition must be convergent. □

Definition 9. *Let $\mu : \mathcal{T} \to \mathbb{R}_+$ be a measure.*

1. If λ is $\mu-$ admissible and μ satisfies the $\sigma - \lambda-$ rule, we say that μ is a $\lambda-$measure. If $T \in \mathcal{T}$, $\mu(T) = 1$ and μ is a $\lambda-$ measure we say that μ ***is a*** *$\lambda-$* ***Sugeno measure****. Clearly, the $0-$ Sugeno measures are the probabilities (on $\mathcal{T}$).*

2. In case there exists a $\mu-$ admissible δsuch that μ is a $\delta-$ measure, we say that μ is a some $\lambda-$ measure.

A some $\lambda-$ measure with $T \in \mathcal{T}$ and $\mu(T) = 1$ is called a ***Sugeno measure****.*

Example 3. *Take $T = \mathbb{N}$, $\mathcal{T} = \{\phi\} \cup \{\{n\} \mid n \in \mathbb{N}\}$ and $(a_n)_n$ a bounded sequence of numbers $a_n > 0$. Define $\mu : \mathcal{T} \to \mathbb{R}_+$ via $\mu(\phi) = 0$ and $\mu(\{n\}) = = a_n$. Then μ is a $\lambda-$ measure for any $\lambda \in \left(-\frac{1}{A}, \infty\right)$ where $A = \sup_n a_n$*

□

This example leads us in a natural way to the following question: when is it possible for a measure μ to satisfy the $\lambda-$ rule and the $\lambda'-$ rule for two different μ admissible numbers λ and λ'?

A partial answer is given in

Theorem 4. *Assume $\mathcal{T}$ is an algebra and $\mu : \mathcal{T} \to \mathbb{R}_+$ is a measure. The following assertions are equivalent:*

1. *There exist two different $\mu-$ admissible numbers λ and λ' such that μ is both a $\lambda-$ measure and a λ' measure.*
2. *μ is a $\lambda-$ measure for any $\mu-$ admissible λ*
3. *μ is $\sigma-$ additive and T is an atom of μ (i.e. for any $A \in \mathcal{T}$ one has either $\mu(A) = 0$ or $\mu(A) = \mu(T)$).*

The next fundamental result is essentially due to Z.Wang (see [10]). It asserts that any measure which satisfies some $\lambda-$rule is representable.

Theorem 5. *Assume $T \in \mathcal{T}$ and let $\mu : \mathcal{T} \to \mathbb{R}_+$ be a measure which satisfies some $\lambda-$ rule. Then μ is representable.*

Remarks concerning the preceding enunciation

1. We exhibit, for a measure $\mu : \mathcal{T} \to \mathbb{R}_+$ with $\mu(T) = A > 0$ and for $\lambda \in \left(-\dfrac{1}{A}, \infty\right) \setminus \{0\}$, a transfer function. Namely, for μ as above and for an arbitrary $0 < a < \infty$, there exists an additive measure $m : \mathcal{T} \to \mathbb{R}_+$ with $m(T) = a$ and the transfer function $h_\lambda : [0, a] \to [0, A]$ given via

$$h_\lambda(x) = \frac{(1 + \lambda A)^{\frac{x}{a}} - 1}{\lambda}$$

such that $\mu = h_\lambda \circ m$.

Namely, the measure m is obtained from μ via the formula $m = h_\lambda^{-1} \circ \mu$. We have $h_\lambda^{-1} = [0, A] \to [0, a]$, given via

$$h_\lambda^{-1}(y) = \frac{a \ln(1 + \lambda y)}{\ln(1 + \lambda A)}.$$

In case $a = A = 1$ (the most popular situations) the formulae from above $(0 \neq \lambda > -1)$ become, for $h_\lambda, h_\lambda^{-1} : [0, 1] \to [0, 1]$:

$$h_\lambda(x) = \frac{(1 + \lambda)^x - 1}{\lambda} \quad \text{and } h_\lambda^{-1}(y) = \frac{\ln(1 + \lambda y)}{\ln(1 + \lambda)},$$

2. The connection between μ and m defined at Remark 1 is very straight: μ satisfies the finite $\lambda-$rule if and only if m is finitely additive; μ satisfies the $\sigma-\lambda-$rule if and only if m is $\sigma-$ additive. Working for $a = A = 1$, we can see that μ is a $\lambda-$ Sugeno measure if and only if m is a classical probability. We say that Sugeno measures are represented by probabilities. □

The reciprocal of Theorem 5 is false. Namely, there exist representable measures which do not satisfy any $\lambda-$rule, as the following example shows.

Example 4. *Take $T = [0,1]$, $\mathcal{T}$ = the Lebesgue measurable subsets of T and $m : \mathcal{T} \to \mathbb{R}_+$ the Lebesgue measure on T. Define $\mu : \mathcal{T} \to \mathbb{R}_+$ given via $\mu(A) = (m(A))^2$. Hence $\mu = h \circ m$ where $h : [0,1] \to [0,1]$ is the transfer function given via $h(t) = t^2$. So μ is representable.*

One can check that μ does not satisfy the $\lambda-$ rule for any $\lambda \in (-1, \infty)$. □

The final part of this paragraph will be dedicated to the structure of $\lambda-$measures on some special spaces.

Finite $\lambda-$ measures with preassigned values

The preliminary fact used here is

Lemma 6. *Let $a_1, a_2, \ldots, a_n$ strictly positive numbers ($n \geq 2$) and write*

$$S_n = a_1 + a_2 + \ldots + a_n.$$

Let A be a number such that $A > \max_{i=1}^{n} a_i$. Then the equation

$$1 + Ax = \prod_{i=1}^{n} (1 + a_i x) \tag{En}$$

has an unique admissible solution $x = \lambda$. Here "admissible" means:

– if $A < S_n$, then $\lambda \in \left(-\frac{1}{A}, 0\right)$;
– if $A = S_n$, then $\lambda = 0$;
– if $A > S_n$, then $\lambda \in (0, \infty)$.

Using this result, we have

Theorem 7. *Again let us consider the strictly positive numbers $a_1, \ldots, a_n$ ($n \geq 2$), the number $A > \max_{i=1}^{n} a_i$ and equation (En). Write $T \in \{1, 2, \ldots, n\}$.*

Let λ be the admissible solution of (En). Under these conditions, there exists an unique $\lambda-$ measure $\mu : \mathcal{P}(T) \to \mathbb{R}_+$ having preassigned values, i.e.

$$\mu(\{i\}) = a_i \quad \text{for any} \quad i \in T$$
$$\mu(T) = A.$$

Countable $\lambda-$measures with preassigned values

We succeeded (see [6]) to extend the preceding result in the countable case. First, we have the following auxiliary result:

Lemma 8. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers such that*

$$S = \sum_{n=1}^{\infty} a_n < \infty.$$

Let A be a number such that

$$A > \sup_n a_n = \max_n a_n.$$

1. Assume $A < S$. Then there exists $n_0 \in \mathbb{N}$ such that $A < \sum_{i=1}^{n_0} a_i$. Then, for any $n \geq n_0$ the equation

$$\prod_{i=1}^{n} (1 + a_i x) = 1 + Ax$$

has exactly one solution $x_n \in \left(-\frac{1}{A}, 0\right)$.

The sequence $(x_n)_{n\geq n_0}$ is strictly decreasing and let

$$x_\infty = \lim_n x_n = \inf_n x_n \in \left[-\frac{1}{A}, 0\right).$$

2. Assume $A \geq S$. Then, for any $n \in \mathbb{N}$, $n \geq 2$, the equation

$$\prod_{i=1}^{n} (1 + a_i x) = 1 + Ax$$

has exactly one solution $x_n \in (0, \infty)$.

The sequence $(x_n)_{n\geq 2}$ is strictly decreasing and let

$$x_\infty = \lim_n x_n = \inf_n x_n \in [0, \infty).$$

The analogue of Lemma 6 follows:

Lemma 9. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers such that*

$$S = \sum_{n=1}^{\infty} a_n < \infty.$$

Let A be a number such that

$$A > \sup_n a_n = \max_n a_n.$$

Consider the equation

$$\prod_{i=1}^{\infty} (1 + a_i x) = 1 + Ax. \tag{E}$$

I. In case $A < S$ equation (E) *has an unique solution $x_\infty \in \left(-\frac{1}{A}, 0\right)$.*

II. In case $A = S$ equation (E) *has an unique solution $x_\infty \in [0, \infty)$, namely $x_\infty = 0$.*

III. In case $A > S$ equation (E) *has an unique solution $x_\infty \in (0, \infty)$.*

In all cases, x_∞ is obtained as follows

$$x_\infty = \lim_n x_n,$$

where $(x_n)_n$ is the sequence furnished by Lemma 8.

Using this result, we have the following extension of Theorem 7:

Theorem 10. *Let $(a_n)_n$ be a sequence of strictly positive numbers such that*

$$S = \sum_{n=1}^{\infty} a_n < \infty$$

and a number A satisfying the condition

$$A > \sup_n a_n = \max_n a_n.$$

Consider the $\sigma-$ algebra $\mathcal{T} = \mathcal{P}(\mathbb{N})$.

I. ***Existence***

There exists a real number $\lambda > -\frac{1}{A}$ and a $\lambda-$ measure $\mu : \mathcal{T} \to \mathbb{R}_+$ with preassigned values, i.e.

$$\mu(\{n\}) = a_n \quad \text{for any} \quad n \in \mathbb{N}$$
$$\mu(\mathbb{N}) = A.$$

II. ***Uniqueness***

The number λ from the existence part must be a solution of equation (E) *(see Lemma 9) and is uniquely determined if it satisfies the admissibility condition i.e.*

– if $A < S$, then $\lambda \in \left(-\frac{1}{A}, 0\right)$;

– if $A = S$, then $\lambda \in [0, \infty)$ and in this case $\lambda = 0$;

– if $A > S$, then $\lambda \in (0, \infty)$.

Sugeno measures on the code space

This section is based upon [5].

We begin with some basic facts about the code space.

Let $2 \leq p \in \mathbb{N}$. We shall consider p distinct elements (called **letters**) and write $X = \{x_1, x_2, \ldots, x_p\}$ (call X **the alphabet**). In most cases, one takes $X = \{0, 1, \ldots, p-1\}$.

Next, we introduce $X^* =$ **the free monoid generated by** X. Namely X^* is formed with all **words** of the form $x = u_1 u_2 \ldots u_n$ (where $u_i \in X$) with **length** $l(x) = n$ and we consider also **the empty word** e with length $l(e) = 0$. For x, y in X^* we write $x < y$ to denote the fact that x *is a prefix of* y and this means : either $x = e$ or $x = u_1 u_2 \ldots u_n \neq e$ and $y = v_1 v_2 \ldots v_m$ with $m \geq n$, such that $v_1 = u_1$, $v_2 = u_2$, ..., $v_n = u_n$. We accept that $X \subset X^*$ i.e. $x \in X$ may be viewed in X^*.

Now we introduce

$$X^\infty = X^{\mathbb{N}} = \{f : \mathbb{N} \to X\}.$$

Namely, an element $f \in X^\infty$ will be considered as follows:

$$f \equiv x \equiv u_1 u_2 \ldots u_n \ldots,$$

where $u_n = f(n)$ for any $n \in \mathbb{N}$. So, the elements in X^∞ are sequences of elements in X. We call the elements in X^∞**codes** and X^∞ is called **the code space**.

For any $x \in X^*$ we can form the set xX^∞. Namely, if $x = e$, define $eX^\infty = X^\infty$ and for $e \neq x = u_1 u_2 \ldots u_n$, xX^∞ is formed with all sequences $v = v_1 v_2 \ldots v_m \ldots$ such that $v_1 = u_1$, $v_2 = u_2$, ..., $v_n = u_n$. Clearly, one has $xX^\infty \subset yX^\infty \Leftrightarrow y < x$.

Considering the metrizable and compact topological space $(X, \mathcal{D})$, where $\mathcal{D}$ is the discrete topology, write $(X_n, \mathcal{D}_n) = (X, \mathcal{D})$ for any $n \in \mathbb{N}$. Then $X^\infty = \prod_{n=1}^{\infty} X_n$ and we can consider on X^∞ the topology $\mathcal{U}$ which is the product topology of the topologies $\mathcal{D}_n$. Then $(X^\infty, \mathcal{U})$ is a metrizable and compact topological space. This space is second countable, namely it has the countable base $\mathcal{P} = \{xX^\infty \mid x \in X^*\}$ formed with sets which are both compact and open. The Borel sets of $(X^\infty, \mathcal{U})$ will be denoted by $\mathcal{B}$.

Because $\mathcal{P}$ is a generalized semiring which generates $\mathcal{B}$ if follows that $\sigma-$ additive or Sugeno measures on $\mathcal{B}$ that coincide on $\mathcal{P}$ are identical. In other words, such a measure is known if we know its values on $\mathcal{P}$.

In the sequel, we shall give a concrete (matricial) representation of all Sugeno measures on $\mathcal{B}$ (see [5]).

Notation

For any $\lambda \in (-1, \infty)$ write

$$\mathcal{S}_\lambda = \{\mu : \mathcal{B} \to \mathbb{R}_+ \mid \mu \text{ is a } \lambda- \text{ Sugeno measure}\}.$$

One can prove

Theorem 11. *One has the equality*

$$\bigcap_{\lambda\in(-1,\infty)} \mathcal{S}_\lambda = DIR$$

where DIR is the set of all Dirac measures $\delta_x : \mathcal{B} \to \mathbb{R}_+$, $x \in X^\infty$.

We pass to the matricial description of all $\mathcal{S}_\lambda$ for $\lambda \in (-1,\infty)\setminus\{0\}$. Write $U_p = \{1,2,\ldots,p\}$.

Definition 10. *Let $\lambda \in (-1,\infty)\setminus\{0\}$. A $\lambda-$ distribution is a sequence $(D_\lambda(n))_n$ where*

$$D_\lambda(1) = (a_\lambda(1), a_\lambda(2), \ldots, a_\lambda(p)) = (a_\lambda(i))_{1\le i\le p}$$
$$D_\lambda(2) = (a_\lambda(i,j))_{1\le i\le p, 1\le j\le p}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D_\lambda(n) = (a_\lambda(i_1,i_2,\ldots,i_n))_{1\le i_k\le p}, \quad \text{for } k=1,2,\ldots.n$$

with the following properties:

a) For any $n\in\mathbb{N}$ and any $(i_1,i_2,\ldots,i_n)\in U_p^n$, one has

– if $-1<\lambda<0$ then $0<a_\lambda(i_1,i_2,\ldots,i_n)\le 1$;

– if $\lambda>0$ then $a_\lambda(i_1,i_2,\ldots,i_n)\ge 1$.

b) $\prod_{i=1}^{p} a_\lambda(i) = \lambda+1$.

c) For any $n\in\mathbb{N}$ and any $(i_1,i_2,\ldots,i_n)\in U_p^n$, one has

$$a_\lambda(i_1,i_2,\ldots,i_n) = \prod_{i=1}^{n} a_\lambda(i_1,i_2,\ldots,i_n,i).$$

Notation

For any $\lambda\in(-1,\infty)\setminus\{0\}$ write $\mathcal{D}_\lambda$ = the set of all $\lambda-$ distributions.

The announced "matricial" description of Sugeno measures is given in

Theorem 12. . *Let $\lambda\in(-1,\infty)\setminus\{0\}$. We have the bijection $T_\lambda : \mathcal{D}_\lambda \to \mathcal{S}_\lambda$ described as follows:*

a) Let $(D_\lambda(n))_n \in \mathcal{D}_\lambda$ where $D_\lambda(n) = (a_\lambda(i_1,i_2,\ldots,i_n))$ as previously. Then

$$T_\lambda((D_\lambda(n))_n) = \mu$$

where

$$\mu(x_{i_1},x_{i_2},\ldots,x_{i_n}X^\infty) = \frac{a_\lambda(i_1,i_2,\ldots,i_n)-1}{\lambda}.$$

b) The inverse $Z_\lambda = T_\lambda^{-1} : \mathcal{S}_\lambda \to \mathcal{D}_\lambda$ *acts via*

$$Z_\lambda(\mu) = (D_\lambda(n))_n$$

where

$$D_\lambda(n) = (a_\lambda(i_1, i_2, \ldots, i_n))_{1 \le i_k \le p}$$

is such that

$$a_\lambda(i_1, i_2, \ldots, i_n) = 1 + \lambda\mu(x_{i_1}, x_{i_2}, \ldots, x_{i_n} X^\infty).$$

3. Non Linear (Fuzzy) Integrals

We shall present the Choquet and the Sugeno integrals (with respect to some generalized measure)

Let $(T, \mathcal{T})$ be a measurable space i.e. T is a non empty set and $\mathcal{T} \subset \mathcal{P}(T)$is a $\sigma-$ algebra. We shall work with positive measurable functions, i.e. with functions $f : T \to \mathbb{R}_+$ having the property that $f^{-1}(A) \in \mathcal{T}$ for any Borel set $A \subset [0, \infty)$. For such a function f and for any $\alpha \in [0, \infty)$, let us define the level set

$$F_\alpha(f) \overset{def}{=} F_\alpha = \{t \in T \mid f(t) \ge \alpha\}$$

(i.e. $F_\alpha = f^{-1}([\alpha, \infty))$; we write F_α instead of $F_\alpha(f)$, because in most cases f is understood).

Clearly $F_\alpha \in \mathcal{T}$ for any $\alpha \in [0, \infty)$.

For any $A \in \mathcal{T}$, φ_A is the characteristic (indicator) function of A.

Let us consider a generalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$. Recall that μ is called **continuous** if one has

$$\mu\left(\lim_n A_n\right) = \lim_n \mu(A_n),$$

for any monotone sequence of sets $A_n \in \mathcal{T}$. To be more precise, $(A_n)_n$ can be either **increasing** (in this case $\lim_n A_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{T}$ and we must have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_n \mu(A_n) = \sup_n \mu(A_n))$$

or **decreasing** (in this case $\lim_n A_n = \bigcap_{n=1}^{\infty} A_n \in \mathcal{T}$ and we must have

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_n \mu(A_n) = \inf_n \mu(A_n)).$$

Notice that one has $F_\alpha \supset F_\beta$ if $0 \le \alpha < \beta < \infty$. Consequently, for any $A \in \mathcal{T}$, the function $\varphi : [0,\infty) \to \mathbb{R}_+$ given via

$$\varphi(\alpha) = \mu\left(F_\alpha(f) \cap A\right)$$

is decreasing (here $f : T \to \mathbb{R}_+$ is measurable).

A. The Sugeno integral (basics)

Let $(T, \mathcal{T})$ be a measurable space, $\mu : \mathcal{T} \to \mathbb{R}_+$ a generalized measure, $f : T \to \mathbb{R}_+$ a measurable function and $A \in \mathcal{T}$. Write F_α instead of $F_\alpha(f)$ for any $\alpha \in \mathbb{R}_+$.

Definition 11. *The Sugeno integral of f with respect to μ on A is*

$$(S)\int_A f \mathrm{d}\mu \stackrel{def}{=} \sup_{\alpha \in \mathbb{R}_+} \left(\alpha \wedge \mu\left(F_\alpha \cap A\right)\right) \le \mu(T) < \infty.$$

In case $A = T$ we write only

$$(S)\int f \mathrm{d}\mu = \sup_{\alpha \in \mathbb{R}_+} \left(\alpha \wedge \mu\left(F_\alpha\right)\right) \le \mu(T) < \infty$$

(this is the Sugeno integral of f with respect to μ).

Remarks. 1. One has the formula

$$(S)\int_A f \mathrm{d}\mu = (S)\int f\varphi_A d\mu \le (S)\int f \mathrm{d}\mu.$$

2. In case $\mu(T) \le M$ one has

$$(S)\int_A f \mathrm{d}\mu = \sup_{\alpha \in [0,M]} \left(\alpha \wedge \mu\left(F_\alpha \cap A\right)\right)$$

because, for $\alpha > M$, one has

$$\alpha \wedge \mu\left(F_\alpha \cap A\right) = \mu\left(F_\alpha \cap A\right) \le \mu\left(F_M \cap A\right) = M \wedge \mu\left(F_M \cap A\right).$$

3. In case μ is continuous, it follows that the function $\alpha \mapsto \varphi(\alpha) = \mu\left(F_\alpha \cap A\right)$ is decreasing and continuous. Hence, the function $\alpha \mapsto u(\alpha) = \varphi(\alpha) - \alpha$ is strictly decreasing and continuous, with $u(0) \ge 0$ and $\lim\limits_{\alpha\to\infty} u(\alpha) = -\infty$. The unique zero of u, call it α_0, has the property that $\alpha_0 = (S)\int_A f \mathrm{d}\mu$.

4. Consequently, we have the following **geometric interpretation** of the Sugeno integral if $T \subset \mathbb{R}$ is an interval, μ is the Lebesgue measure and $f : T \to \mathbb{R}_+$ is continuous and unimodal. Namely, in this case $(S)\int f \mathrm{d}\mu$ is the edge's length of the largest square between the graph of f and the $x-$ axis.

5. M. Sugeno called his integral "fuzzy integral" (see his doctoral thesis [9]). Now, the name Sugeno integral is much more in use. □

Here are some properties of the Sugeno integral ($f_1, f_2, f : T \to \mathbb{R}_+$ measurable, A, B in $\mathcal{T}$ and $a \in [0, \infty)$).

Theorem 13. *1. If $\mu(A) = 0$ then $(S)\int_A f \mathrm{d}\mu = 0$;*

2. If μ is continous and $(S)\int_A f \mathrm{d}\mu = 0$ then $\mu(A \cap \{t \in T \mid f(t) > 0\}) = 0$;

3. If $f_1 \le f_2$ then $(S)\int_A f_1 \mathrm{d}\mu \le \int_A f_2 \mathrm{d}\mu$;

4. If $A \subset B$ then $(S)\int_A f \mathrm{d}\mu \le (S)\int_B f \mathrm{d}\mu$.

5. $(S)\int_A a d\mu = a \wedge \mu(A)$.

6. $(S)\int_A (f + a)\mathrm{d}\mu \le (S)\int_A f \mathrm{d}\mu + (S)\int_A a \mathrm{d}\mu$.

B. The Choquet integral (basics)

Let us consider a measurable space $(T, \mathcal{T})$, a measurable function $f : T \to \mathbb{R}_+$, a generalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$ and $A \in \mathcal{T}$.

As always, for any $\alpha \in \mathbb{R}_+$ write $F_\alpha \overset{def}{=} F_\alpha(f)$ and consider again the decreasing function $\varphi : [0, \infty) \to [0, \infty)$ given via $\varphi(\alpha) = \mu(F_\alpha \cap A)$. If L is the Lebesgue measure on $\mathbb{R}_+$, it is possible to integrate φ with respect to L and we write

$$\int_0^\infty \mu(F_\alpha \cap A)\, \mathrm{d}\alpha \overset{def}{=} \int \varphi \mathrm{d}L \le \infty.$$

Definition 12. *Within the framework from above, the Choquet integral of f with respect to μ on A is*

$$(C)\int_A f \mathrm{d}\mu \overset{def}{=} \int_0^\infty \mu(F_\alpha \cap A)\, \mathrm{d}\alpha \le \infty.$$

In case $A = T$ we write only $(C)\int f\mathrm{d}\mu = \int\limits_0^\infty \mu(F_\alpha)\,\mathrm{d}\alpha$

(this is the Choquet integral of f with respect to μ).

In case $(C)\int\limits_A f\mathrm{d}\mu < \infty$, *we say that f is Choquet integrable with respect to μ.*

Remarks 1. We have the formula

$$(C)\int\limits_A f\mathrm{d}\mu = (C)\int\limits_A f\varphi_A\mathrm{d}\mu \leq (C)\int f\mathrm{d}\mu.$$

2. The definition of the Choquet integral is a generalization of the usual abstract Lebesgue integral. Indeed, if μ is a classical measure (i.e. μ is $\sigma-$ additive), then we have the equality

$$(C)\int\limits_A f\mathrm{d}\mu = \int\limits_A f\mathrm{d}\mu$$

the last integral being classical. □

Here are some properties of the Choquet integral ($f_1, f_2, f : \mathcal{T} \to \mathbb{R}_+$ measurable, A, B in $\mathcal{T}$ and $a \in [0,\infty)$).

Theorem 14. *1. If $\mu(A) = 0$ then* $(C)\int\limits_A f\mathrm{d}\mu = 0$.

2. If $\mu(A \cap \{t \in T \mid f(t) > 0\}) = 0$, then $(C)\int\limits_A f\mathrm{d}\mu = 0$. Conversely, if μ is continuous and $(C)\int\limits_A f\mathrm{d}\mu = 0$, then $\mu(A \cap \{t \in T \mid f(t) > 0\}) = 0$.

3. If $f_1 \leq f_2$ then $(C)\int\limits_A f_1\mathrm{d}\mu \leq (C)\int\limits_A f_2\mathrm{d}\mu$.

4. If $A \subset B$ then $(C)\int\limits_A f\mathrm{d}\mu \leq (C)\int\limits_B f\mathrm{d}\mu$.

5. $(C)\int\limits_A 1\mathrm{d}\mu = \mu(A)$.

6. $(C)\int\limits_A af\mathrm{d}\mu = a(C)\int\limits_A f\mathrm{d}\mu$.

7. $(C)\int\limits_A (f+a)\mathrm{d}\mu = (C)\int\limits_A f\mathrm{d}\mu + a\mu(A)$.

C. Sugeno and Choquet integrals for sequences of measurable functions

C.1. Sugeno integral

We shall work with a generalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$ which is **continuous**.

Theorem 15. *(Monotone Convergence). Let $(f_n)_n$ be a sequence of positive measurable functions which is monotone and let $f = \lim\limits_n f_n$.*

Then, for any $A \in \mathcal{T}$ one has

$$(S)\int_A f\mathrm{d}\mu = \lim_n (S)\int_A f_n\mathrm{d}\mu.$$

(in case $(f_n)_n$ is increasing, this means $(S)\int_A f\mathrm{d}\mu = \sup\limits_n (S)\int_A f_n\mathrm{d}\mu$ and in case $(f_n)_n$ is decreasing this means $(S)\int_A f\mathrm{d}\mu = \inf\limits_n (S)\int_A f_n\mathrm{d}\mu$).

The analogue of Fatou's Lemma follows:

Theorem 16. *For any sequence $(f_n)_n$ of positive measurable functions and any $A \in \mathcal{T}$ one has*

$$(S)\int_A \liminf_n f_n\mathrm{d}\mu \le \liminf_n (S)\int_A f_n\mathrm{d}\mu.$$

Theorem 17. *(Uniform Convergence). Assume the sequence $(f_n)_n$ of positive measurable functions converges uniformly to f. Then, for any $A \in \mathcal{T}$ one has*

$$(S)\int_A f\mathrm{d}\mu = \lim_n (S)\int_A f_n\mathrm{d}\mu.$$

Remark In the case of the Sugeno integral, the Sugeno-mean convergence coincides with the usual convergence in measure. More precisely: the fact that the sequence $(f_n)_n$ of positive measurable functions (S) converges in mean to the positive measurable f (i.e. $(S)\int |f_n - f|\,\mathrm{d}\mu \underset{n}{\to} 0$) is equivalent to the fact that for any $a > 0$ one has

$$\mu\{t \in T \mid |f_n(t) - f(t)| \ge a\} \underset{n}{\to} 0. \qquad \square$$

C.2. Choquet integral

Again we shall work with a generalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$ which is **continuous**.

We shall use the following concepts, concerning a sequence $(f_n)_n$ of positive measurable functions and a positive measurable function f.

Definition 13. *We say that $(f_n)_n$ $(C)-$convergence in mean to f if*

$$(C)\int |f_n - f|\,\mathrm{d}\mu \underset{n}{\to} 0.$$

We say that $(f_n)_n$ converges in measure to f if, for any $a > 0$, one has

$$\mu\left(\{t \in T \mid |f_n(t) - f(t)| \geq a\}\right) \underset{n}{\to} 0.$$

Theorem 18. *We have the implications:*

$$((f_n)_n \text{ converges uniformly to } f) \Rightarrow ((f_n)_n \ (C) - \text{converges in mean to } f) \Rightarrow$$

$$\Rightarrow ((f_n)_n \ \text{ converges in measure to } \ f).$$

Theorem 19. *(Dominated Convergence). Let $(f_n)_n$ be a sequence of positive measurable functions, f and g positive measurable functions and $A \in \mathcal{T}$. Assume that $f_n \underset{n}{\to} f$ pointwise, $f_n \leq g$ for any n and g is Choquet integrable (with respect to μ). Then*

$$(C)\int_A f_n \mathrm{d}\mu \underset{n}{\to} (C)\int_A f \mathrm{d}\mu.$$

and f_n, f are all Choquet integrable (with respect to μ).

Theorem 20. *(Uniform Convergence). Let $(f_n)_n$ be a sequence of positive measurable functions, which converges uniformly to f and f_n, f are Choquet integrable with respect to μ and let $A \in \mathcal{T}$. Then*

$$(C)\int_A f_n \mathrm{d}\mu \underset{n}{\to} (C)\int_A f \mathrm{d}\mu.$$

D. Sugeno and Choquet integrals: special properties

The first result of this section shows that any Sugeno integral on an abstract space can be computed as a Sugeno integral on $\mathbb{R}_+$ with respect to Lebesgue measure.

So, let us consider a measurable space $(T, \mathcal{T})$ and a generalized measure $\mu : \mathcal{T} \to \mathbb{R}_+$. We consider also the measurable space $(\mathbb{R}_+, \mathcal{B}_+)$, where $\mathcal{B}_+$ stands for the Borel sets of $\mathbb{R}_+$ and the Lebesgue measure $L : \mathcal{B}_+ \to \overline{\mathbb{R}_+}$.

Theorem 21. *(Transformation Theorem) For any positive measurable function* $f : T \to \mathbb{R}_+$ *and any* $A \in \mathcal{T}$*, one has*

$$(S)\int_A f\mathrm{d}\mu = (S)\int \varphi \mathrm{d}L,$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is the decreasing function which acts via*

$$\varphi(t) = \mu\left(A \cap F_t(f)\right).$$

Remarks. 1. Clearly $\varphi(t) \leq \mu(T)$ for any $t \in [0,\infty)$. According to the definition

$$(S)\int \varphi \mathrm{d}L = \sup_{\alpha\in[0,\infty)} \left(\alpha \wedge L\left(F_\alpha(\varphi)\right)\right).$$

For any $\alpha \in [0,\infty)$, one has

$$F_\alpha(\varphi) = \{u \in [0,\infty) \mid \varphi(u) \geq \alpha\} = \{u \in [0,\infty) \mid \mu\left(A \cap F_u(f)\right) \geq \alpha\}$$

hence $F_\alpha(\varphi) = \emptyset$ for $\alpha > \mu(T)$.

It follows that

$$(S)\int \varphi \mathrm{d}L = \sup_{\alpha\in[0,\mu(T)]} \left(\alpha \wedge L\left(F_\alpha(\varphi)\right)\right).$$

It is worth noticing that, for any $0 < \alpha < \infty$, the set $F_\alpha(\varphi)$ is a bounded interval with left extremity 0 in $[0,\infty)$, hence only the finite values of the measure L are used in the computation of $(S)\int \varphi \mathrm{d}L$. Actually, we allowed here for L to have infinite values (which are not used!) in computing $(S)\int \varphi \mathrm{d}L$.

2. The formula in Theorem 3.12 is often written as follows

$$(S)\int_A f\mathrm{d}\mu = (S)\int_0^\infty \mu\left(A \cap F_\alpha\right)\mathrm{d}\alpha. \qquad \square$$

The second result of this section pertains to the parametric continuity of a "flow" of Sugeno and Choquet integrals with respect to a "flow" of Sugeno measures, which is canonically generated by a fixed probability. The precise facts follow.

Let $(T, \mathcal{T})$ be a measurable space. A suitable interpretation of the already mentioned Z.Wang's theorem ([10] and [11]) leads to the following theorem:

Theorem 22. *Denote by $\mathcal{M}$ the set of all probabilities $\mu : \mathcal{T} \to \mathbb{R}_+$ (i.e.μ are $\sigma-$additive and $\mu(T) = 1$). For any $\lambda \in (-1, \infty)$, denote by $\mathcal{S}_\lambda$ the set of all $\lambda-$ Sugeno measures $m : \mathcal{T} \to \mathbb{R}_+$ (of course, $\mathcal{S}_0 = \mathcal{M}$).*

Then, for fixed $\lambda \in (-1, \infty)$, there exists a bijection $B(\lambda) : \mathcal{M} \to \mathcal{S}_\lambda$ acting via

$$B(\lambda)(\mu) = m(\lambda, \mu)$$

where, for any $A \in \mathcal{T}$, one has

$$m(\lambda, \mu)(A) = \begin{cases} \dfrac{(\lambda+1)^{\mu(A)} - 1}{\lambda}, & \text{if} \quad \lambda \neq 0 \\ \mu(A), & \text{if} \quad \lambda = 0. \end{cases}$$

Remark. Remembering the representability facts, let us consider, for any $\lambda \in (-1, \infty)$, the transfer function $h_\lambda : [0, 1]) \to [0, 1]$ given via

$$h_\lambda(t) = \begin{cases} \dfrac{(1+\lambda)^t - 1}{\lambda}, & \text{in case} \quad \lambda \neq 0 \\ t, & \text{in case} \quad \lambda = 0. \end{cases}$$

It is seen that $B(\lambda)(\mu) = h_\lambda \circ \mu$ in all cases. Then $B(\lambda)^{-1} : \mathcal{S}_\lambda \to \mathcal{M}$ acts via $B(\lambda)^{-1}(m) = h_\lambda^{-1} \circ m$ where $h_\lambda^{-1} : [0, 1] \to [0, 1]$ is defined via

$$h_\lambda^{-1}(y) = \begin{cases} \dfrac{\ln(1+\lambda y)}{\ln(1+\lambda)}, & \text{in case} \quad \lambda \neq 0 \\ y, & \text{in case} \quad \lambda = 0 \end{cases}$$

(see Theorem 5 and the afferent remarks). □

Now, we can state the promised parametric continuity theorems (see [2] and [3]).

Theorem 23. *Let $(T, \mathcal{T})$ be a measurable space, $f : T \to \mathbb{R}_+$ a measurable function and $\mu : \mathcal{T} \to \mathbb{R}_+$ a probability.*

Then, for any $A \in \mathcal{T}$, the function $V : (-1, \infty) \to \mathbb{R}_+$, given via

$$V(\lambda) = (S) \int_A f \mathrm{d}m(\lambda, \mu)$$

is continuous.

Theorem 24. *Let $(T, \mathcal{T})$ be a measurable space, $\mu : \mathcal{T} \to \mathbb{R}_+$ a probability and $f : T \to \mathbb{R}_+$ a $\mu-$integrable function. Then:*

1. The function f is Choquet integrable with respect to $m(\lambda, \mu)$ for any $\lambda \in (-1, \infty)$.

2. For any $A \in \mathcal{T}$, the function $W : (-1, \infty) \to \mathbb{R}_+$ given via

$$W(\lambda) = (C) \int_A f \mathrm{d}m(\lambda, \mu)$$

is continuous.

We also can study the asymptotic behaviour, i.e. we can try to extend the previous continuity results to "the marginal values" $\lambda = -1$ and $\lambda = \infty$ (see[2] and [3]).

For the same fixed probability $\mu : \mathcal{T} \to \mathbb{R}_+$ **the asymptotic (marginal) measures** $m(-1, \mu) : \mathcal{T} \to \mathbb{R}_+$ and $m(\infty, \mu) : \mathcal{T} \to \mathbb{R}_+$ are defined as follows:

$$m(-1, \mu)(A) = \lim_{\lambda \to -1} m(\lambda, \mu)(A)$$
$$m(\infty, \mu)(A) = \lim_{\lambda \to \infty} m(\lambda, \mu)(A).$$

Theorem 25. *The marginal measures are computed as follows (for any $A \in \mathcal{T}$):*

$$m(-1, \mu)(A) = \begin{cases} 0, & if \quad \mu(A) = 0 \\ 1, & if \quad \mu(A) > 0 \end{cases}$$
$$m(\infty, \mu)(A) = \begin{cases} 0, & if \quad \mu(A) < 1 \\ 1, & if \quad \mu(A) = 1, \end{cases}$$

being both atomic (T is an atom) generalized measures.

For any $-1 < \lambda_1 < \lambda_2 < \infty$, one has $m(-1, \mu) \geq m(\lambda_1, \mu) \geq m(\lambda_2, \mu) \geq m(\infty, \mu)$.

Also: $m(-1, \mu)$ is countably subadditive and $m(\infty, \mu)$ is countably superadditive.

Supplementarily: 1) *In case μ is atomic (T is an atom), one has $m(-1, \mu) = m(\infty, \mu) = m(\lambda, \mu) = \mu$ for any $\lambda \in (-1, \infty)$.*

2) *In case μ is not atomic:*

a) $m(-1, \mu)$ is "$-1-$additive" (i.e.

$$m(-1, \mu)(A \cup B) = m(-1, \mu)(A) + m(-1, \mu)(B) - m(-1, \mu)(A)m(-1, \mu)(B),$$

whenever A, B are in $\mathcal{T}$ and $A \cap B = \emptyset$) and $m(-1, \mu)$ is not $\lambda-$additive for any $\lambda \in (-1, \infty)$;

b) $m(\infty, \mu)$ is not $\lambda-$additive for any $\lambda \in [-1, \infty)$.

The result concerning the asymptotic behaviour follows.

Theorem 26. *Assume $f : T \to \mathbb{R}_+$ is measurable. Then, for any $A \in \mathcal{T}$ one has:*

$$\lim_{\lambda \to -1} (S) \int_A f \mathrm{d}m(\lambda, \mu) = (S) \int_A f \mathrm{d}m(-1, \mu);$$

$$\lim_{\lambda \to \infty} (S) \int_A f \mathrm{d}m(\lambda, \mu) = (S) \int_A f \mathrm{d}m(\infty, \mu).$$

Theorem 27. *Assume $f : T \to \mathbb{R}_+$ is $\mu-$integrable. Then, for any $A \in \mathcal{T}$ one has*

$$\lim_{\lambda \to -1} (C) \int_A f \mathrm{d}m(\lambda, \mu) = (C) \int_A f \mathrm{d}m(-1, \mu);$$

$$\lim_{\lambda \to \infty} (C) \int_A f \mathrm{d}m(\lambda, \mu) = (C) \int_A f \mathrm{d}m(\infty, \mu);$$

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