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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF SPIRALLIKE FUNCTIONS

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Abstract. In this article, we derive a sharp estimates for the Taylor-Maclaurin coefficients of functions in a certain subclass of spirallike functions. Also, we give several corollaries and consequences of the main results.

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1. Introduction

Let $\mathbb{D}$ be the unit disk $\{z : |z| < 1\}$, $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1}$$

Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

because of the conditions (1). Let S denote class of analytic and univalent functions in $\mathbb{D}$ with the normalization conditions (1).

Definition 1. *For $0 \le \alpha < 1$ let $S^*(\alpha)$ and $S^c(\alpha)$ denote the class of starlike and convex univalent functions of order α, which are defined as the following, respectively*

$$S^*(\alpha) = \left\{ f(z) \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{D} \right\}$$

and

$$S^c(\alpha) = \left\{ f(z) \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{D} \right\}.$$

Observe that $S^{*}(0) = S^{*}$ represent standard starlike functions. A notation of α_starlikeness and α_convexity were generalized onto a complex order α by Nasr and Aouf [7] . Spaĉek [10] extend the class of starlike functions by introducing the class of spirallike functions of type β in $\mathbb{D}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{D}$.

Theorem 1. (*Spaĉek* [10]) *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(z)$ is a spirallike function of type β in $\mathbb{D}$ if and only if*

$$\operatorname{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{D}. \tag{3}$$

We denote the the class of spirallike functions of type β in in $\mathbb{D}$ by $\widetilde{S}^{\beta}$. Libera [6] unified and extended the classes $S^{*}(\alpha)$ and $\widetilde{S}^{\beta}$ by introducing the analytic function class $\widetilde{S}^{\beta}_{\alpha}$ in $\mathbb{D}$ as follows.

Definition 2. (*Libera* [6]) *Let the function $f(z)$ ben in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\alpha \in [0,1)$. We say that $f \in \widetilde{S}^{\beta}_{\alpha}$ if and only if*

$$\operatorname{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \quad (z \in \mathbb{D};\ 0 \le \alpha < 1). \tag{4}$$

From Definition 1 and 2, we have the following inclusions:

$$\widetilde{S}^{0}_{\alpha} = S^{*}(\alpha) \text{ and } \widetilde{S}^{\beta}_{0} = \widetilde{S}^{\beta}.$$

Libera [6] also proved the following coefficients bounds for the functions in the class $\widetilde{S}^{\beta}_{\alpha}$.

Theorem 2. (*Libera* [6]) *If the function $f \in \widetilde{S}^{\beta}_{\alpha}$ is given by* (2) , *then*

$$|a_n| \le \prod_{j=0}^{n-2}\left(\frac{\left|2(1-\alpha)e^{-i\beta}\cos\beta + j\right|}{j+1}\right) \quad \left(n \in \mathbb{N}\backslash\{1\};\ \mathbb{N} := \{1,2,3,\cdots\}\right). \tag{5}$$

The coefficient estimates in (5) *are sharp.*

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. The Hadamard product (convolution) of f and g, denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{D}.$$

Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The Ruscheweyh derivative [8] of the n^{th} order of f, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \tag{6}$$

The Ruscheweyh derivative gave an impulse for various generalization of well known classes of functions. Exemplary, for α $(0 \le \alpha < 1)$ and $n \in \mathbb{N}_0$, Ahuja $[1, 2]$ defined the class of functions, denoted $R_n(\alpha)$, which consist of univalent functions of the form (2) that satisfying the condition

$$\mathrm{Re}\left(\frac{z(D^n f(z))'}{D^n f(z)}\right) > \alpha,\ z \in \mathbb{D}. \tag{7}$$

We denote that $R_0(\alpha) = S^*(\alpha)$. The class $R_n(0) = R_n$ was studied by Singh and Singh $[9]$. With the aid of Ruscheweyh derivative we can generalize the spirallike functions as follows.

Definition 3. *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(z)$ is in the class $\widetilde{R}_n^{\beta}$ if and only if*

$$\mathrm{Re}\left(e^{i\beta}\frac{z(D^n f(z))'}{D^n f(z)}\right) > 0,\ z \in \mathbb{D}. \tag{8}$$

Note that $\widetilde{R}_0^{\beta} = \widetilde{S}_{\beta}$.

Definition 4. *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also, let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\alpha \in [0, 1)$. Then $f(z)$ is in the class $\widetilde{R}_n^{\beta}(\alpha)$ if and only if*

$$\mathrm{Re}\left(e^{i\beta}\frac{z(D^n f(z))'}{D^n f(z)}\right) > \alpha\cos\beta,\ z \in \mathbb{D}. \tag{9}$$

Also, note that $\widetilde{R}_0^{\beta}(\alpha) = \widetilde{S}_{\alpha}^{\beta}$, $\widetilde{R}_0^{\beta}(0) = \widetilde{S}^{\beta}$ and $\widetilde{R}_0^{0}(\alpha) = S^*(\alpha)$.

Definition 5. *Let $\alpha \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let f be an univalent function of the form (2) such that $D^n f(z) \neq 0$ for $z \in \mathbb{D}\setminus\{0\}$. We say that f belongs to $\widetilde{R}_n^{\beta}(\alpha, \lambda)$ if and only if*

$$\mathrm{Re}\left(e^{i\beta}\frac{z(D^n f(z))'}{(1-\lambda)D^n f(z) + \lambda z(D^n f(z))'}\right) > \alpha\cos\beta,\ z \in \mathbb{D}. \tag{10}$$

Definition 6. *Let $f(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. We say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{D}$ and we denote*

$$f(z) \prec g(z) \quad (z \in \mathbb{D}),$$

if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$, with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ \ (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

After the proof of the Bieberbach Conjecture [3] (which is also known as de Branges Theorem [4]), many authors were interested in other interesting subclasses of normalized analytic function class $\mathcal{A}$. In this paper, we obtain sharp coefficient bounds for functions in the class $\widetilde{R}_n^{\beta}(\alpha, \lambda)$ and we give a necessary and sufficient condition such that $f \in \mathcal{A}$ belongs to $\widetilde{R}_n^{\beta}(\alpha, \lambda)$.

2. Main Results

In this section, we obtain coefficient conditions for functions in the class given by Definition 5. Also, we get sharp estimates for functions belong to $\widetilde{R}_n^{\beta}(\alpha, \lambda)$.

Theorem 3. *Let $\alpha \in [0,1)$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f(z)$ is in the form (2) such that $D^n f(z) \neq 0$ for $z \in \mathbb{D}\setminus\{0\}$. Then, $f(z)$ belongs to the class $\widetilde{R}_n^{\beta}(\alpha, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty} \left\{ (k-1)(1 - \lambda(\alpha + i\tan\beta)) + 2e^{2i\beta} - \lambda(1-\alpha)\left(1 - e^{2i\beta}\right)(k-1) \right\} A_k z^k \neq 0 \tag{11}$$

$$(z \in z \in \mathbb{D}\setminus\{0\}),$$

where

$$A_k = (1 + (k-1)\lambda)\frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!}a_k, \ \ k \in \mathbb{N}\setminus\{1\}.$$

Proof. Let the function $f \in S$ be defined by (2). Define a function

$$h(z) = D^n f(z) = z + \sum_{k=2}^{\infty} A_k z^k, \ z \in \mathbb{D}. \tag{12}$$

Consider the function

$$p(z) = \frac{e^{i\beta} \sec\beta \left(\frac{h(z)}{(1-\lambda)h(z)+\lambda z h'(z)} \right) - i\tan\beta - \alpha}{1-\alpha}$$

is an analytic function which satisfies $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$, then $f \in \widetilde{R}_n^{\beta}(\alpha, \lambda)$ if and only if

$$p(z) \neq \frac{1-e^{2i\beta}}{1+e^{2i\beta}}$$

or,

$$\frac{e^{i\beta}\sec\beta z h'(z) - (\alpha + i\tan\beta)\left((1-\lambda)h(z) + \lambda z h'(z)\right)}{(1-\alpha)\left((1-\lambda)h(z) + \lambda z h'(z)\right)} \neq \frac{1-e^{2i\beta}}{1+e^{2i\beta}}.$$

By using the series expantion of $h(z)$ which is given by (12), we get the following

$$\left(1+e^{2i\beta}\right) \sum_{k=1}^{\infty} \left[(k-1)(1-\alpha\lambda - i\lambda\tan\beta) + (1-\alpha)\right] A_k z^k$$

$$\neq (1-\alpha)\left(1-e^{2i\beta}\right) \sum_{k=1}^{\infty} \left(1+(k-1)\lambda\right) A_k z^k$$

for $z \neq 0$. It is equivalent to

$$\sum_{k=1}^{\infty} \left\{ (k-1)\left(1-\lambda(\alpha + i\tan\beta)\right) + 2e^{2i\beta} - (1-\alpha)\left(1-e^{2i\beta}\right)(k-1)\lambda \right\} A_k z^k \neq 0,$$

which completes the proof of Theorem 3. ∎

Now, we prove our coefficient estimates for functions which belong to the class $\widetilde{R}_n^{\beta}(\alpha, \lambda)$.

Theorem 4. *Let $\alpha \in [0,1)$ and $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $f(z)$ is in the form (2) such that $D^n f(z) \neq 0$ for $z \in \mathbb{D}\setminus\{0\}$. If $f(z)$ belongs to the class $\widetilde{R}_n^{\beta}(\alpha, \lambda)$ then*

$$|a_k| \leq \frac{\Gamma(n+1)}{\Gamma(n+k)(1-\lambda)^{k-1}} \prod_{j=0}^{k-2} \left| j(1-\lambda) + 2(1-\alpha)e^{i\beta}\cos\beta(1+\lambda j) \right| \tag{13}$$

$$\left(n \in \mathbb{N}\setminus\{1\};\ \mathbb{N} := \{1, 2, 3, \cdots\}\right).$$

This result is sharp.

Proof. Since $f \in \widetilde{R}_n^{\beta}(\alpha, \lambda)$ there exists a Schwarz function $w(z)$, which is already introduced in Definition 6, such that

$$e^{i\beta} \sec\beta \left(\frac{z\left(D^n f(z)\right)'}{(1-\lambda) D^n f(z) + \lambda z \left(D^n f(z)\right)'} \right) - i \tan\beta = \frac{1 + (1-2\alpha) w(z)}{1 - w(z)}.$$

Consider the function $h(z)$ defined by (12). Then, we get

$$\sum_{k=2}^{\infty} \left[k e^{i\beta} \sec\beta - (1 + i\tan\beta)(1 - (k-1)\lambda) \right] A_k z^k$$
$$= \left(\sum_{k=1}^{\infty} \left[k e^{i\beta} \sec\beta + (1 - 2\alpha - i\tan\beta)(1 + (k-1)\lambda) \right] A_k z^k \right) w(z). \quad (14)$$

The last equation (14) may be written (for $n \in \mathbb{N}$) as follows:

$$\sum_{k=2}^{m} \left[k e^{i\beta} \sec\beta - (1 + i\tan\beta)(1 - (k-1)\lambda) \right] A_k z^k + \sum_{k=m+1}^{\infty} b_k z^k$$
$$= \left(\sum_{k=1}^{m-1} \left[k e^{i\beta} \sec\beta + (1 - 2\alpha - i\tan\beta)(1 + (k-1)\lambda) \right] A_k z^k \right) w(z). \quad (15)$$

The last sum on the left-hand side of (15) is convergent in $\mathbb{D}$ for $m = 2, 3, \cdots$.

Since, by hypothesis, $|w(z)| < 1$ $(z \in \mathbb{D})$, it is not difficult to find by appealing to Parseval's Theorem that

$$\sum_{k=1}^{m-1} \left| k e^{i\beta} \sec\beta \left(1 - 2\alpha - i\tan\beta\right)(1 + (k-1)\lambda) \right|^2 |A_k|^2$$
$$\geq \sum_{k=2}^{m} \left| k e^{i\beta} \sec\beta - (1 + i\tan\beta)(1 - (k-1)\lambda) \right|^2 |A_k|^2$$

or

$$\sum_{k=1}^{m-1} 4(1-\alpha)\left(k - \alpha\left(1 + (k-1)\lambda\right)\right) |A_k|^2 \geq \frac{(m-1)^2 (1-\lambda)^2}{\cos^2\beta} |A_m|^2 \quad (16)$$

where $A_1 = 1$.

We claim that

$$|A_m| \leq \frac{1}{(m-1)!\,(1-\lambda)^{m-1}} \prod_{j=0}^{m-2} \left| j(1-\lambda) + 2(1-\alpha)\cos\beta e^{i\beta}(1 + j\lambda) \right|. \quad (17)$$

For $m=2$, we get from (16)

$$|A_2| \leq \frac{2(1-\alpha)\cos\beta}{1-\lambda},$$

which is equivalent to (17) . (17) is obtained for larger m from inequality (16) by the principle of the mathematical induction.

Fix m, $m \geq 3$, and suppose that (13) holds for $k = 2, 3, \cdots, m-1$. Then from (16) we get the following inequality

$$|A_m|^2 \leq \frac{4(1-\alpha)\cos^2\beta}{(m-1)^2(1-\lambda)^2}\left\{1-\alpha+\sum_{k=2}^{m-1} B(k,j,\alpha)\right\} \tag{18}$$

where

$$B(k,j,\alpha) = \frac{(1+(k-1)\lambda)(k-\alpha(k-1)\lambda)}{\left((k-1)!(1-\lambda)^{k-1}\right)^2}\prod_{j=0}^{k-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2.$$

We must show that the square of the right side of (17) is equal to the right side of (18) ; that is

$$\frac{\prod_{j=0}^{m-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2}{\left[(m-1)!(1-\lambda)^{m-1}\right]^2} = \frac{4(1-\alpha)\cos^2\beta}{(m-1)^2(1-\lambda)^2}\left\{1-\alpha+\sum_{k=2}^{m-1} B(k,j,\alpha)\right\} \tag{19}$$

for $m = 3, 4, \cdots$. After necessary calculations we can show that (19) is true for $m = 3$ and proves our claim for $m = 3$. Assume that (19) is valid for all k, $3 < k \leq m-1$; then from (16) and (18) we obtain

$$|A_m|^2 \leq \frac{4(1-\alpha)\cos^2\beta}{(m-1)^2(1-\lambda)^2}\left\{1-\alpha+\sum_{k=2}^{m-2} B(k,j,\alpha)+B(m-1,j,\alpha)\right\}$$

$$|A_m|^2 \le \frac{4(1-\alpha)\cos^2\beta}{(m-1)^2(1-\lambda)^2}\{1-\alpha$$

$$+\sum_{k=2}^{m-2}\frac{(1+(k-1)\lambda)(k-\alpha(k-1)\lambda)}{\left((k-1)!(1-\lambda)^{k-1}\right)^2}\prod_{j=0}^{k-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2$$

$$\left.+\frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{\left((m-2)!(1-\lambda)^{m-2}\right)^2}\prod_{j=0}^{m-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2\right\}$$

$$=\frac{\prod_{j=0}^{m-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2}{\left((m-2)!(1-\lambda)^{m-2}\right)^2}\left\{\frac{(m-2)^2}{(m-1)^2}\right.$$

$$\left.+4(1-\alpha)\cos^2\beta\frac{(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)}{(m-1)^2(1-\lambda)^2}\right\}$$

$$=\frac{\prod_{j=0}^{m-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2}{\left((m-1)!(1-\lambda)^{m-1}\right)^2}\left\{(m-2)^2(1-\lambda)^2\right.$$

$$\left.+4(1-\alpha)\cos^2\beta(1+(m-2)\lambda)(m-1-\alpha(m-2)\lambda)\right\}$$

$$=\frac{1}{\left((m-1)!(1-\lambda)^{m-1}\right)^2}\prod_{j=0}^{m-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2.$$

From equality (6) we get the desired result. ∎

3. Corollaries and Consequences

By choosing appropriate values of values of n, λ, β and α in Theorem 4, we obtain the corresponding results for several subclasses of S.

Corollary 5. *If $\lambda = 0$, we get the following result for function $f \in \widetilde{R}_n^\beta(\alpha)$*

$$|a_k| \le \frac{\Gamma(n+1)}{\Gamma(n+k)}\prod_{j=0}^{k-2}\left|j+2(1-\alpha)e^{i\beta}\cos\beta\right|.$$

Corollary 6. *If* $n = 0$ *and* $\lambda = 0$*, we obtain* (5) *which is stated in Theorem 2.*

Corollary 7. *If* $n = 0$, $\beta = 0$ *and* $\lambda = 0$*, we obtain the following result for functions belong to* $S^*(\alpha)$

$$|a_k| \leq \prod_{j=0}^{k-2} \frac{|j + 2(1-\alpha)|}{j+1} .$$

Corollary 8. *If* $\lambda = 0$ *and* $\alpha = 0$*, we get the following result for function* $f \in \widetilde{R}_n^{\beta}$

$$|a_k| \leq \frac{\Gamma(n+1)}{\Gamma(n+k)} \prod_{j=0}^{k-2} \left| j + 2e^{i\beta} \cos\beta \right| .$$

Corollary 9. *If* $n = 0$, $\lambda = 0$ *and* $\alpha = 0$*, we get the following result for spirallike functions of type* β *in* $\mathbb{D}$

$$|a_k| \leq \prod_{j=0}^{k-2} \frac{\left| j + 2e^{i\beta} \cos\beta \right|}{j+1} .$$

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