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CERTAIN SUBCLASS OF BESSEL FUNCTIONS WITH RESPECT TO $(j, k)-$SYMMETRIC POINTS

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Abstract. In this paper, the authors introduces new class of analytic functions with respect to $(j, k)-$symmetric points and investigate various inclusion properties for these classes. Integral representation for functions in these classes and some interesting applications involving a familiar integral operator, are also obtained.

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1. Introduction

Observations: Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0, \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{\ z : z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

We denote by $\mathcal{S}^*, \mathcal{C}, \mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are ([3], [6]) which covers most of the topics in a lucid and economical style.

In [9], Rønning introduced a new class of starlike functions related to UCV deined as

$$f(z) \in \mathcal{S}_P \quad \Longleftrightarrow \quad Re\left(\frac{zf'(z)}{f(z)}\right) > \Big|\frac{zf'(z)}{f(z)} - 1\Big|.$$

Note that $f(z) \in UCV \Longleftrightarrow zf'(z) \in \mathcal{S}_P$. The geometrical interpretation of uniformly convex and related functions have been studied by several authors (see [4, 5, 9]).

An analytic function f is said to be subordinate to an analytic function g(written as $f \prec g$)if and only if there exists an analytic function ω with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for } z \in \mathcal{U},$$

such that

$$f(z) = g(\omega(z)) \text{ for } z \in \mathcal{U}.$$

In particular, if g is univalent in $\mathcal{U}$, we have the following equivalence

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

The convolution or Hadamard product of two functions of class $\mathcal{A}$ is denoted and defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where f has the form (1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad z \in \mathcal{U}.$$

Let us consider the following second-order linear homogeneous differential equation (see for details [1] and [2]):

$$z^2\omega''(z) + bz\omega'(z) + \left[dz^2 - v^2 + (1-b)v\right]\omega(z) = 0 \quad (v, b, d \in \mathbb{C}). \tag{2}$$

The function $\omega_{v,b,d}(z)$, which is called the generalized Bessel function of the first kind of order v, it is defined as a particular solution of (2). The function $\omega_{v,b,d}(z)$ has the familiar representation as

$$\omega_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n!\Gamma\left(v+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{3}$$

Here Γ stands for the Euler gamma function. The series (3) permits the study of Bessel, modified Bessel, and spherical Bessel function altogether. It is worth mentioning that, in particular:

(i) For $b = d = 1$ in (3), we obtain the familiar Bessel function of the first kind of order $\mathcal{U}$ defined by

$$\mathcal{J}_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{4}$$

(ii) For $b = 1$ and $d = -1$ in (3), we obtain the modified Bessel function of the first kind of order v defined by

$$\mathcal{I}_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(v+n+1)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{5}$$

(iii) For $b = 2$ and $d = 1$ in (3), the function $\omega_{v,b,d}(z)$ reduces to $\frac{\sqrt{2}}{\sqrt{\pi}}\mathcal{S}_v(z)$, where $\mathcal{S}_v$ is the spherical Bessel function of the first kind of order v, defined by

$$\mathcal{S}_v(z) = \frac{\sqrt{\pi}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(v+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+v} \quad (z \in \mathbb{C}). \tag{6}$$

Now, consider the function $u_{v,b,d}(z) : \mathbb{C} \longrightarrow \mathbb{C}$, defined by the transformation

$$u_{v,b,d}(z) = 2^v \Gamma\left(v + \frac{b+1}{2}\right) z^{\frac{-v}{2}} \omega_{v,b,d}(\sqrt{z}). \tag{7}$$

By using the well-known Pochhammer symbol (or the shifted factorial)$(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler gamma function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

and $(\lambda)_0 = 1$, we obtain for the function $u_{v,b,d}(z)$ the following representation

$$u_{v,b,d}(z) = \sum_{n \geq 0} \frac{\left(\frac{-d}{4}\right)^n}{\left(v + \frac{b+1}{2}\right)_n} \frac{z^n}{n!},$$

where $k = v + \frac{b+1}{2} \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second order linear differential equation

$$4z^2 u''(z) + 2(2v+b+1)zu'(z) + dzu(z) = 0.$$

Now, we introduce the function $\varphi_{v,b,d}(z)$ defined in terms of generalized Bessel function $\omega_{v,b,d}(z)$, by

$$\begin{aligned} \varphi_{v,b,d}(z) &= zu_{v,b,d}(z) \\ &= 2^v \Gamma\left(v + \frac{b+1}{2}\right) z^{1-\frac{v}{2}} \omega_{v,b,d}(\sqrt{z}) \\ &= z + \sum_{n=1}^{\infty} \frac{(-d)^n z^{n+1}}{4^n n!(c)_n}, \quad \textit{where } c = \left(v + \frac{b+1}{2}\right) \\ &= g(c,d,z). \end{aligned}$$

Motivated by Selvaraj and Karthikeyan[11], we define the following $D_\lambda^m(c,d)f(z)$: $\mathcal{U} \longrightarrow \mathcal{U}$ by

$$D_\lambda(c,d)f(z) = f(z) * g(c,d,z) \tag{8}$$

$$D_\lambda^1(c,d)f(z) = (1-\lambda)\left(f(z) * g(c,d,z)\right) + \lambda z (f(z) * g(c,d,z))' \tag{9}$$

$$\vdots$$

$$D_\lambda^m(c,d)f(z) = D_\lambda^1\left(D_\lambda^{m-1}(c,d)f(z)\right) \tag{10}$$

If $f \in \mathcal{A}$, then from (9) and (10) we may easily deduce that

$$D_\lambda^m(c,d)f(z) = z + \sum_{n=2}^{\infty} \frac{(1+(n-1)\lambda)^m\,(-d)^{n-1} a_n z^n}{4^{n-1}(n-1)!(c)_{n-1}}, \tag{11}$$

where $m \in N_0 = N \cup \{0\}$ and $\lambda \geq 0$.
It can be easily verified from definition of (11) that

$$\lambda z\left(D_\lambda^m(c,d)f(z)\right)' = D_\lambda^{m+1}(c,d)f(z) - (1-\lambda)D_\lambda^m(c,d)f(z). \tag{12}$$

Let k be a positive integer and $j = 0,1,2,\ldots(k-1)$. A domain D is said to be $(j,k)-$fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in A$ is said to be $(j,k)-$symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{13}$$

where $\varepsilon = exp(2\pi i/k)$. The family of $(j,k)-$symmetrical functions will be denoted by $\mathcal{F}_k^j$. We observe that $\mathcal{F}_2^1, \mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively.

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \frac{f(\varepsilon^v z)}{\varepsilon^{vj}}, \quad (f \in \mathcal{A}; k = 1,2,\ldots; j = 0,1,2,\ldots(k-1)), \tag{14}$$

where v is an integer.

The notation of $(j,k)-$symmetric functions was introduced and studied by Liczberski and Polubinski in [8].

The following identities follow directly from (14):

$$\begin{aligned} f_{j,k}(\varepsilon^v z) =& \varepsilon^{vj} f_{j,k}(z), \\ f'_{j,k}(\varepsilon^v z) =& \varepsilon^{vj-v} f'_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\frac{f'(\varepsilon^v z)}{\varepsilon^{vj-v}}, \\ f''_{j,k}(\varepsilon^v z) =& \varepsilon^{vj-2v} f''_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\frac{f''(\varepsilon^v z)}{\varepsilon^{vj-2v}}. \end{aligned} \tag{15}$$

Motivated by the concept introduced by K.Sakaguchi in [10], recently several subclasses of analytic functions with respect to k-symmetric points were introduced and studied by various authors. In this paper, new subclasses of analytic functions with respect to $(j,k)-$symmetric points are introduced.

Now we define

$$f_{j,k}^{\lambda,m}(c,d;z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}(D_\lambda^m(c,d)f(\varepsilon^v z)), \quad (f\in\mathcal{A}; k=1,2,\ldots; j=0,1,2,\ldots(k-1)). \tag{16}$$

Clearly for $j=k=1$, we have

$$f_{j,k}^{\lambda,m}(c,d;z) = D_\lambda^m(c,d)f(\varepsilon^v z).$$

Definition 1. *The class $\mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$ of function f, analytic in $\mathcal{U}$ given by (1) and satisfying the condition*

$$Re\left|\left\{e^{i\alpha}\left(1+\frac{1}{\gamma}\left(\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d,z)}-1\right)\right)\right\}\right|^2+\beta > \left|\frac{1}{\gamma}\left(\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d,z)}-1\right)\right|^2, (z\in\mathcal{U}), \tag{17}$$

where $-\pi/2<\alpha<\pi/2$, $\gamma\in\mathcal{C}\backslash\{0\}$ and $f_{j,k}^{\lambda,m}$ is defined by the equality (16).

Remark 1. *If we let $j=k=1$ and $\alpha=\beta=0$, $\gamma=1$ in (17), the class $\mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$ reduces to the function class $\mathcal{S}_p$.*

2. Integral Representation

Theorem 1. *If $f\in\mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$, then $f_{j,k}^{\lambda,m}\in\mathcal{S}^*$.*

Proof. Let $f\in\mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$. For $\omega=u+iv$, the inequality (17) can be rewritten as

$$u > \frac{1}{2}\left(v^2+1-\frac{\beta}{\cos^2\alpha}\right).$$

Setting

$$\mathcal{G} = \left\{u+iv : u > \frac{1}{2}\left(v^2+1-\frac{\beta}{\cos^2\alpha}\right)\right\}.$$

From the equivalent subordination condition proved by N.Xu and D.Yang in [15], we may rewrite the conditions (17) in the form

$$1+\frac{1}{\gamma}\left(\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}-1\right) \prec e^{-i\alpha}\left[h(z)\cos\alpha+i\sin\alpha\right],$$

where

$$h(z) = 1 - \frac{\beta}{2\cos^2\alpha} + \frac{2}{\pi^2}\left(log\frac{1+\sqrt{(z+\theta)(1+\theta z)}}{1-\sqrt{(z+\theta)(1+\theta z)}}\right)^2 \text{ with}$$

$$\theta = \left(\frac{e^\mu - 1}{e^\mu + 1}\right)^2, \mu = \sqrt{\beta}\pi/2\cos\alpha.$$

$$1 + \frac{1}{\gamma}\left(\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} - 1\right) = e^{-i\alpha}\left[h(\omega(z))\cos\alpha + i\sin\alpha\right].$$

The function $e^{-i\alpha}\left[h(\omega(z))\cos\alpha + i\sin\alpha\right]$ is univalent and convex in $\mathcal{U}$, where $\omega(z)$ is a Schwarz function, analytic in $\mathcal{U}$ with $\omega(0) = 0$.

$$\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \tag{18}$$

If we replace z by $\varepsilon^v z$ in (18), then (18) will be of the form

$$\frac{\varepsilon^v z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{f_{j,k}^{\lambda,m}(c,d;\varepsilon^v z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \tag{19}$$

Using (13) in (19), we get

$$\frac{\varepsilon^v z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{\varepsilon^{vj} f_{j,k}^{\lambda,m}(c,d;z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1$$

$$\frac{\varepsilon^{v-vj} z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \tag{20}$$

Let $v = 0, 1, 2, \cdots, k-1$ in (20) respectively and summing them, we get

$$\frac{\sum\limits_{v=0}^{k-1}\varepsilon^{v-vj} z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \sum_{v=0}^{k-1}\gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1,$$

or equivalently

$$\frac{z[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \frac{\gamma}{k}\sum_{v=0}^{k-1}\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1,$$

that is $f_{j,k}^{\lambda,m} \in \mathcal{S}^*$.

Theorem 2. *If $f \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$, then we have*

$$f_{j,k}^{\lambda,m}(c,d;z) = z\ exp\left\{\frac{\gamma}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v t))\cos\alpha + i\sin\alpha\right] - 1}{t}\ dt\right\}, \quad (21)$$

where $f_{j,k}^{\lambda,m}(z)$ is defined by (16), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0, |\omega(z)| < 1$.

Proof. Let $f \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$. In view of (18), we have

$$\frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1, \quad (22)$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting z by $\varepsilon^v z$ in the equality (22) respectively $(v = 0, 1, 2, \cdots, k-1, \varepsilon^k = 1)$, we have

$$\frac{\varepsilon^v z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{f_{j,k}^{\lambda,m}(c,d;\varepsilon^v z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \quad (23)$$

Using (13) in (23) can be rewritten in the form

$$\frac{\varepsilon^{v-vj} z[D_\lambda^m(c,d)f(\varepsilon^v z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \gamma\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \quad (24)$$

Let $v = 0, 1, 2, \cdots, k-1$ in (24) respectively and summing them, we get

$$\frac{z[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} = \frac{\gamma}{k}\sum_{v=0}^{k-1}\left(e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \quad (25)$$

From the equality (25), we get

$$\frac{[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)} - \frac{1}{z} = \frac{\gamma}{k}\sum_{v=0}^{k-1}\frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v z))\cos\alpha + i\sin\alpha\right] - 1}{z}.$$

Integrating this equality, we get

$$\log\left\{\frac{f_{j,k}^{\lambda,m}(c,d;z)}{z}\right\} = \frac{\gamma}{k}\sum_{v=0}^{k-1}\int_0^z \frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v \zeta))\cos\alpha + i\sin\alpha\right] - 1}{\zeta}\ d\zeta,$$

$$= \frac{\gamma}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v t))\cos\alpha + i\sin\alpha\right] - 1}{t}\ dt,$$

or equivalently,

$$f_{j,k}^{\lambda,m}(c,d;z) = z\ exp\left\{\frac{\gamma}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v t))\cos\alpha + i\sin\alpha\right]-1}{t}\ dt\right\}.$$

This completes the proof of Theorem 2.

Theorem 3. *Let* $f \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$*. Then we have*

$$\begin{aligned} D_\lambda^m(c,d)f(z) = \int_0^z exp\left\{\frac{\gamma}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v \zeta}\frac{e^{-i\alpha}\left[h(\omega(\varepsilon^v t))\cos\alpha + i\sin\alpha\right]-1}{t}\ dt\right\} \\ \cdot\bigg(\gamma\left(e^{-i\alpha}\left[h(\omega(\zeta)\cos\alpha + i\sin\alpha\right]-1+1\right)\bigg)d\zeta, \end{aligned} \tag{26}$$

where $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0)=0, |\omega(z)|<1.$

3. Inclusion properties of the class $\mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$

In this section, we will prove inclusion property associated with generalized Bernardi integral operator given by

$$L_\mu[f](z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt,\ \ (f\in\mathcal{A}, \mu>-1). \tag{27}$$

To establish our results in this section, We need the following Lemmas.

Lemma 4. *Let* h *be convex in* $\mathcal{U}$*, with* $Re[\beta h(z)+\gamma]>0$*. If* $p(z)$ *is analytic in* $\mathcal{U}$ *with* $p(0)=h(0)$*, then*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \qquad \Longrightarrow \qquad p(z)\prec h(z).$$

Lemma 5. *Let* h *be convex in* $\mathcal{U}$*, with* $Re[\beta h(z)+\gamma]>0$*. If* $f\in\mathcal{A}$ *and* F *is given by (27), then*

$$\frac{zf'(z)}{f(z)} \prec h(z) \qquad \Longrightarrow \qquad \frac{zF'(z)}{F(z)}\prec h(z).$$

Theorem 6. *Let* $0\le\lambda\le 1$ *and* $h(z)$ *be convex univalent function, then*

$$\mathcal{S}_{j,k}^{\lambda,m+1}(c,d,\alpha,\beta,\gamma) \subset \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma).$$

Proof. Let $f \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$ and set

$$l(z) = \frac{z[D_\lambda^m(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}, \qquad m(z) = \frac{z[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}, \tag{28}$$

we observe that $l(z)$ and $m(z)$ are analytic in $\mathcal{U}$ with $l(0) = m(0) = 1$. Then by applying (12) in $l(z)$, we obtain

$$l(z)f_{j,k}^{\lambda,m}(c,d;z) = \frac{1}{\lambda}D_\lambda^{m+1}(c,d)f(z) - \frac{(1-\lambda)}{\lambda}D_\lambda^m(c,d)f(z). \tag{29}$$

Differentiating both sides of equation (29) with respect to z, we get after simple computation

$$zl'(z) + \left(\frac{(1-\lambda)}{\lambda} + \frac{z[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}\right)l(z) = \frac{1}{\lambda}\frac{z[D_\lambda^{m+1}(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}. \tag{30}$$

Using the relation between (12) and (16), we can easily deduce that

$$z[f_{j,k}^{\lambda,m}(c,d;z)]' + \frac{(1-\lambda)}{\lambda}f_{j,k}^{\lambda,m}(c,d;z) = \frac{1}{\lambda}f_{j,k}^{\lambda,m+1}(c,d;z). \tag{31}$$

Using (31)in (30), we have

$$l(z) + zl'(z)\left(\frac{(1-\lambda)}{\lambda} + \frac{z[f_{j,k}^{\lambda,m}(c,d;z)]'}{f_{j,k}^{\lambda,m}(c,d;z)}\right)^{-1} = \frac{z[D_\lambda^{m+1}(c,d)f(z)]'}{f_{j,k}^{\lambda,m+1}(c,d;z)}.$$

From the definition of $f \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$, we have

$$l(z) + \frac{zl'(z)}{\frac{(1-\lambda)}{\lambda} + m(z)} \prec \gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha + i\sin\alpha\right] - 1\right) + 1. \tag{32}$$

In view of Lemma 5, the assertion of the Theorem would follow once we prove that $m(z) \prec \gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha + i\sin\alpha\right] - 1\right) + 1, (z \in \mathcal{U})$.

It follows from $m(z)$ and (31) that

$$\frac{(1-\lambda)}{\lambda} + m(z) = \frac{f_{j,k}^{\lambda,m+1}(c,d;z)}{\lambda f_{j,k}^{\lambda,m}(c,d;z)}. \tag{33}$$

By logarithmical differentiation of equation (33), we get

$$m(z) + \frac{zm'(z)}{(1-\lambda) + \lambda m(z)} = \frac{z[f_{j,k}^{\lambda,m+1}(c,d;z)]'}{f_{j,k}^{\lambda,m+1}(c,d;z)}. \tag{34}$$

Using Theorem 1 in equality (34), we have

$$m(z)+\frac{zm'(z)}{(1-\lambda)+\lambda m(z)}\prec\gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha+i\sin\alpha\right]-1\right)+1,\quad(z\in\mathcal{U}).\tag{35}$$

In view of Lemma (4), we deduce that

$$m(z)\prec\gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha+i\sin\alpha\right]-1\right)+1.$$

This implies that

$$\mathcal{S}_{j,k}^{\lambda,m+1}\left(c,d,\alpha,\beta,\gamma\right)\subset\mathcal{S}_{j,k}^{\lambda,m}\left(c,d,\alpha,\beta,\gamma\right).$$

Theorem 7. *Let* $f\in\mathcal{A}$ *and* $F=L_\mu[f]$*, where* $L_\mu[f]$ *is defined by (27). If* $f\in\mathcal{S}_{j,k}^{\lambda,m}\left(c,d,\alpha,\beta,\gamma\right)$ *then* $F\in\mathcal{S}_{j,k}^{\lambda,m}\left(c,d,\alpha,\beta,\gamma\right)$*.*

Proof. From the definition of F and the linearity of the operator $D_\lambda^m(c,d)f(z)$, we have

$$z(D_\lambda^m(c,d)L_\mu[f](z))'=(\mu+1)(D_\lambda^m(c,d)f(z))-\mu(D_\lambda^m(c,d)L_\mu[f](z)).\tag{36}$$

From (36), we have

$$(\mu+1)f_{j,k}^{\lambda,m}(c,d;z)=\mu F_{j,k}^{\lambda,m}(c,d;z)+z(F_{j,k}^{\lambda,m}(c,d;z))'.\tag{37}$$

If we let

$$\omega(z)=\frac{z(F_{j,k}^{\lambda,m}(c,d;z))'}{F_{j,k}^{\lambda,m}(c,d;z)},$$

then ω is analytic in $\mathcal{U}$ and $\omega(0)=1$. From (37), we observe that

$$\mu+\omega(z)=(\mu+1)\frac{f_{j,k}^{\lambda,m}(c,d;z)}{F_{j,k}^{\lambda,m}(c,d;z)},\tag{38}$$

Differentiating both sides of (38) with respect to z, we obtain

$$\omega(z)+\frac{z\omega'(z)}{\mu+\omega(z)}=\frac{z(f_{j,k}^{\lambda,m}(c,d;z))'}{f_{j,k}^{\lambda,m}(c,d;z)}.$$

By Theorem 1, we have

$$\omega(z)+\frac{z\omega'(z)}{\mu+\omega(z)}\prec\gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha+i\sin\alpha\right]-1\right)+1,$$

which on using Lemma 4 implies $\omega(z) \prec \gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha + i\sin\alpha\right] - 1\right) + 1$.
Now suppose that

$$q(z) = \frac{z(D_\lambda^m(c,d))'}{F_{j,k}^{\lambda,m}(c,d;z)},$$

then $q(z)$ is analytic in $\mathcal{U}$, with $q(0) = 1$, and it follows from (36) that

$$F_{j,k}^{\lambda,m}(c,d;z)q(z) = (\mu+1)(D_\lambda^m(c,d)f(z)) - \mu D_\lambda^m(c,d)F(z). \tag{39}$$

Differentiating both sides of (39), we get

$$zq'(z) + (\mu + \omega(z))q(z) = (\mu+1)\frac{z(D_\lambda^m(c,d)f(z))'}{F_{j,k}^{\lambda,m}(c,d;z)}. \tag{40}$$

Now from (38) and (40), we can deduce that

$$q(z) + \frac{zq'(z)}{\mu + \omega(z)} = \frac{z(D_\lambda^m(c,d)f(z))'}{F_{j,k}^{\lambda,m}(c,d;z)} \prec \gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha + i\sin\alpha\right] - 1\right) + 1.$$

Hence an application of Lemma 5 yields $q(z) \prec \gamma\left(e^{-i\alpha}\left[h(z)\cos\alpha + i\sin\alpha\right] - 1\right) + 1$, which shows that $F \in \mathcal{S}_{j,k}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$.

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