Acta Universitatis Apulensis No. 53/2018
ISSN: 1582-5329 pp. 41-54
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2018.53.04

ON A FIBONACCI-LIKE SEQUENCE ASSOCIATED WITH K-LUCAS SEQUENCE

A. A. Wani, V. H. Badshah, S. Halici, P. Catarino

Abstract. In the present article we consider a new generalization of classical Fibonacci sequence and we call it as Fibonacci-Like sequence $\langle V_{k,n}\rangle$ and then we study Fibonacci-Like sequence $\langle V_{k,n}\rangle$ and k-Lucas sequence $\langle L_{k,n}\rangle$ side by side by introducing two special matrices for these two sequences. After that by using these matrices we obtain Binet formulae for Fibonacci-Like sequence and for k-Lucas sequence, we also give Cassini's identity for Fibonacci-Like sequence.

2010 *Mathematics Subject Classification*: 11B37, 11B39.

Keywords: Fibonacci Sequence, Generalized Fibonacci Sequence and k-Lucas Sequence.

1. Introduction

The Fibonacci sequence $\langle F_n\rangle_{n\geq 0}$ is the sequence of integers given by

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0 \text{ and } F_1 = 1 \tag{1.1}$$

Classical Fibonacci numbers have been generalized by lot of the authors in different ways. The two most important generalizations of Fibonacci numbers are k-Fibonacci numbers $\langle F_{k,n}\rangle$ [1] and k-Lucas numbers $\langle L_{k,n}\rangle$ [2].
Especially some authors used matrix technique to study these numbers. In 1979, Silvester [3] obtained some properties of classical Fibonacci numbers by matrix methods, particularly here the author used diagonaliztion of 2×2 matrix to obtain Binet formula for classical Fibonacci numbers. In [4] Kilic introduced Binet formula, sums, combinatorial representations and generating function for the generalized Fibonacci p-numbers by using matrix technique, these numbers are defined by the following recurrence relation:

$$F_p\left(n\right) = F_p\left(n-1\right) + F_p\left(n-p-1\right),\ n > p+1 \tag{1.2}$$

With initial conditions

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1$$

Jun and Choi [5] presented some basic properties for generalized Fibonacci sequence $\langle q_n \rangle_{n\in\mathbb{N}}$. Ying et al. [6] studied the generalized Fibonacci numbers by two different matrix methods via the method of diagonalization and the method of matrix collation. Akyuz and Halici [7] considered Horadam sequence (see [7]) and showed interest towards two cases of Horadam sequence i.e. the sequences $\langle U_n \rangle$ and $\langle V_n \rangle$, which are delineated by

$$\begin{aligned} U_n &= pU_{n-1} - qU_{n-2},\ n \geq 2 \text{ and } U_0 = 0,\ U_1 = 1 \\ V_n &= pV_{n-1} - qV_{n-2},\ n \geq 2 \text{ and } V_0 = 2,\ V_1 = p \end{aligned} \tag{1.3}$$

where p and q are integers with $p > 0$, $q \neq 0$, in [7] the authors derived some combinatorial identities, the determinant and the n^{th} power of 2×2 matrix. In [8] the authors found some well-known equalities and Binet formula for Jacobsthal numbers by matrix methods. Some divisible properties have been obtained for the generalized Fibonacci sequence by Yalciner [9]. Borges et al. [10] obtained Cassini's identities and Binet formulae for k-Fibonacci and k-Lucas sequences by using some tools of matrix algebra. In [11, 12] the authors also used matrix terminology to deduce Cassini's identities and Binet formulae for the k-Pell and k-Pell-Lucas Sequences. Srisawat and Sripad [13] applied matrix technique to investigate some generalizations of Pell and Pell-Lucas numbers as (s,t)-Pell and (s,t)-Pell-Lucas numbers, these numbers are defined respectively by

$$\begin{aligned} \mathcal{P}_n(s,t) &= 2s\mathcal{P}_{n-1}(s,t) + t\mathcal{P}_{n-2}(s,t) \text{ for } n \geq 2 \\ \mathcal{Q}_n(s,t) &= 2s\mathcal{Q}_{n-1}(s,t) + t\mathcal{Q}_{n-2}(s,t) \text{ for } n \geq 2 \end{aligned} \tag{1.4}$$

with initial conditions $\mathcal{P}_0(s,t) = 0,\ \mathcal{P}_1(s,t) = 1$ and $\mathcal{Q}_0(s,t) = 2,\ \mathcal{P}_1(s,t) = 2s$.

2. Preliminaries

First of all, we consider a sequence $\langle G_{k,n} \rangle$ which is defined by the following recurrence relation:

$$G_{k,n+2} = kG_{k,n+1} + G_{k,n},\ n \geq 1 \text{ and } G_{k,0} = a,\ G_{k,1} = b \tag{2.1}$$

where $k \in \mathbb{R}^+$ and $a, b \in \mathbb{Z}^+$.
For the present study we are interested in the two cases of the sequence $\langle G_{k,n} \rangle$:

(i) Fibonacci-Like sequence $\langle V_{k,n} \rangle$ is defined by the following equation:

$$V_{k,n+2} = kV_{k,n+1} + V_{k,n}, \ n \geq 0 \text{ and } V_{k,0} = 2m, \ V_{k,1} = p + mk \tag{2.2}$$

(ii) k-Lucas sequence $\langle L_{k,n} \rangle$[2] is defined recurrently by

$$L_{k,n+2} = kL_{k,n+1} + L_{k,n}, \ n \geq 0 \text{ and } L_{k,0} = 2, \ L_{k,1} = k \tag{2.3}$$

Both the recurrence relations (2.2) and (2.3) have same characteristic equation $x^2 - kx - 1 = 0$. Let r and s be its roots and are given as

$$r = \frac{k + \sqrt{k^2+4}}{2} \quad \text{and} \quad s = \frac{k - \sqrt{k^2+4}}{2} \tag{2.4}$$

We can see easily r and s holds the following properties:

(a) $rs = -1$, $r + s = k$ and $r - s = \sqrt{k^2+4}$

(b) $r^2 - 1 = kr$ and $s^2 - 1 = ks$

(c) $r^2 + 1 = kr + 2 = (r - s)\, r$ and $s^2 + 1 = ks + 2 = -(r - s)\, s$

Also we introduce two special 2×2 matrices V and L which are given by

$$V = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} k^2+2 & k \\ k & 2 \end{bmatrix} \tag{2.5}$$

3. n^{th} Powers of the Matrices

Theorem 1. *For $n \in \mathbb{Z}^+$, we have the following result*

$$V^n = \begin{bmatrix} \dfrac{2mV_{k,n+2} - (p+mk)\, V_{k,n+1}}{m^2k^2 + 4m^2 - p^2} & \dfrac{2mV_{k,n+1} - (p+mk)\, V_{k,n}}{m^2k^2 + 4m^2 - p^2} \\ \dfrac{2mV_{k,n+1} - (p+mk)\, V_{k,n}}{m^2k^2 + 4m^2 - p^2} & \dfrac{2mV_{k,n} - (p+mk)\, V_{k,n-1}}{m^2k^2 + 4m^2 - p^2} \end{bmatrix} \tag{3.1}$$

Proof. We use principle of mathematical induction on n. Certainly the result is true for $n = 1$.

Assume that the result is true for all values j less than or equal n and then

$$V^{n+1} = \begin{bmatrix} \frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{m^2k^2+4m^2-p^2} \\ \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n} - (p+mk)\,V_{k,n-1}}{m^2k^2+4m^2-p^2} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^{n+1} = \left(m^2k^2+4m^2-p^2\right) \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^{n+1} = \left(m^2k^2+4m^2-p^2\right) \begin{bmatrix} ka_1 + a_2 & a_1 \\ ka_3 + a_4 & a_3 \end{bmatrix}$$

Here

$$\begin{aligned} ka_1 + a_2 &= 2m\left(kV_{k,n+2} + V_{k,n+1}\right) - (p+mk)\left(kV_{k,n+1} + V_{k,n}\right) \\ &= 2mV_{k,n+3} - (p+mk)\,V_{k,n+2} \end{aligned}$$

and

$$\begin{aligned} ka_3 + a_4 &= 2m\left(kV_{k,n+1} + V_{k,n}\right) - (p+mk)\left(kV_{k,n} + V_{k,n-1}\right) \\ &= 2mV_{k,n+2} - (p+mk)\,V_{k,n+1} \end{aligned}$$

Hence

$$V^{n+1} = \begin{bmatrix} \frac{2mV_{k,n+3} - (p+mk)\,V_{k,n+2}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{m^2k^2+4m^2-p^2} \\ \frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{m^2k^2+4m^2-p^2} \end{bmatrix}$$

as required.

Corollary 2. *For* $\lim_{n\to\infty} \frac{V_{k,n+1}}{V_{k,n}} = r$*, we have*

$$r^2 - kr - 1 = 0 \tag{3.2}$$

Proof. To prove the required result, we should use the concept of limits. Since

$$\lim_{n\to\infty} \frac{V^n}{V_{k,n-1}} = \lim_{n\to\infty} \begin{bmatrix} \frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{(m^2k^2+4m^2-p^2)\,V_{k,n-1}} & \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{(m^2k^2+4m^2-p^2)\,V_{k,n-1}} \\ \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{(m^2k^2+4m^2-p^2)\,V_{k,n-1}} & \frac{2mV_{k,n} - (p+mk)\,V_{k,n-1}}{(m^2k^2+4m^2-p^2)\,V_{k,n-1}} \end{bmatrix}$$

Now

$$\lim_{n\to\infty} \frac{2mV_{k,n+2}-(p+mk)\,V_{k,n+1}}{V_{k,n-1}} = r^2\,(2mr-p-mk)$$
$$\lim_{n\to\infty} \frac{2mV_{k,n+1}-(p+mk)\,V_{k,n}}{V_{k,n-1}} = r\,(2mr-p-mk)$$
$$\lim_{n\to\infty} \frac{2mV_{k,n}-(p+mk)\,V_{k,n-1}}{V_{k,n-1}} = 2mr-p-mk$$

Therefore,

$$\lim_{n\to\infty} \frac{V^n}{V_{k,n-1}} = \left(m^2k^2+4m^2-p^2\right)^{-1} \begin{bmatrix} r^2\,(2mr-p-mk) & r\,(2mr-p-mk) \\ r\,(2mr-p-mk) & 2mr-p-mk \end{bmatrix}$$

$$\begin{aligned} &\left(m^2k^2+4m^2-p^2\right)^{-1} \begin{bmatrix} r^2\,(2mr-p-mk) & r\,(2mr-p-mk) \\ r\,(2mr-p-mk) & 2mr-p-mk \end{bmatrix} \\ &= \left(m^2k^2+4m^2-p^2\right)^{-1} \begin{bmatrix} (kr+1)\,(2mr-p-mk) & r\,(2mr-p-mk) \\ r\,(2mr-p-mk) & 2mr-p-mk \end{bmatrix} \end{aligned}$$

If we equate determinant on both sides, we obtain

$$\begin{aligned} 0 &= kr+1-r^2 \\ r^2-kr-1 &= 0 \end{aligned}$$

Hence the result.

Theorem 3. *For $n \in \mathbb{N}$, the following result holds*

$$L^n = \begin{cases} (r-s)^{n-1}\,U^n & if\ n\ is\ odd \\ (r-s)^n\,V^n & if\ n\ is\ even \end{cases} \tag{3.3}$$

where $U^n = \begin{bmatrix} L_{k,n+1} & L_{k,n} \\ L_{k,n} & L_{k,n-1} \end{bmatrix}$

Proof. To prove the result, we use induction on n. First, we consider odd n. Let $n = 1$, we have

$$L = \begin{bmatrix} k^2+2 & k \\ k & 2 \end{bmatrix} = \begin{bmatrix} L_{k,2} & L_{k,1} \\ L_{k,1} & L_{k,0} \end{bmatrix}$$

Let us suppose that the result is true for all odd values i less than or equal n and then

$$\begin{aligned} L^{n+2} &= (r-s)^{\frac{n-1}{2}}\, U^n L^2 \\ &= (r-s)^{\frac{n+1}{2}}\, U^n V^2 \\ &= (r-s)^{\frac{n+1}{2}}\, U^{n+2} \end{aligned}$$

as required.
Now we consider even n. Let $n=2$, we have

$$\begin{aligned} L^2 &= \begin{bmatrix} k^4+5k^2+4 & k^3+4k \\ k^3+4k & k^2+4 \end{bmatrix} \\ &= (r-s)\begin{bmatrix} k^2+1 & k \\ k & 1 \end{bmatrix} \\ &= (r-s)\, V^2 \end{aligned}$$

Assume that the result is true for all even values j less than or equal n and then

$$\begin{aligned} L^{n+2} &= (r-s)^{\frac{n}{2}}\, V^n L^2 \\ &= (r-s)^{\frac{n+2}{2}}\, V^n V^2 \\ &= (r-s)^{\frac{n+2}{2}}\, V^{n+2} \end{aligned}$$

as needed.

Lemma 4. *For $n \geq 0$, we have*

$$V^n \begin{bmatrix} V_{k,1} \\ V_{k,0} \end{bmatrix} = \begin{bmatrix} V_{k,n+1} \\ V_{k,n} \end{bmatrix} \tag{3.4}$$

Proof. Here we shall use induction on n. Indeed the result is true for $n=0$. Suppose the result is true for all values i less than or equal n and then

$$\begin{aligned} V^{n+1}\begin{bmatrix} V_{k,1} \\ V_{k,0} \end{bmatrix} &= VV^n \begin{bmatrix} V_{k,1} \\ V_{k,0} \end{bmatrix} \\ &= \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} V_{k,n+1} \\ V_{k,n} \end{bmatrix} \\ &= \begin{bmatrix} kV_{k,n+1}+V_{k,n} \\ V_{k,n+1} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} V_{k,n+2} \\ V_{k,n+1} \end{bmatrix}$$

as required.

4. Binet Formulae

In this section we present Binet formulae for k-Lucas sequence $\langle L_{k,n} \rangle$ and for Fibonacci-Like sequence $\langle V_{k,n} \rangle$. The most worth noticing point is here that we obtain Binet formula for k-Lucas in a different way that did the authors in [2, 10].

Theorem 5. *For $n \in \mathbb{Z}_0$, the n^{th} terms for $\langle L_{k,n} \rangle$ and $\langle V_{k,n} \rangle$ are respectively given by*

$$L_{k,n} = r^n + s^n \tag{4.1}$$

$$V_{k,n} = p\frac{r^n - s^n}{r - s} + m\left(r^n + s^n\right) \tag{4.2}$$

where r and s are determined from equation (2.4).

Proof. To prove the needed result, we diagonalize the matrix L. Sine L is a square matrix, by the Cayley Hamilton theorem the characteristic equation of L is given by

$$det\left(L - xI_2\right) = 0$$

$$\begin{bmatrix} k^2 + 2 - x & k \\ k & 2 - x \end{bmatrix} = 0$$

$$x^2 - \left(k^2 + 4\right)x + \left(k^2 + 4\right) = 0$$

$$x^2 - (r - s)\,x + (r - s) = 0 \tag{4.3}$$

This is the characteristic equation of L. Let u and v be the eigen values of equation (4.3) and are given by

$$\begin{aligned} u &= \frac{k\,(r - s) + (r - s)^2}{2} \\ &= (r - s)\left[\frac{k + (r - s)}{2}\right] \\ &= r\,(r - s) \end{aligned}$$

and

$$\begin{aligned} v &= \frac{-k(r-s)+(r-s)^2}{2} \\ &= -(r-s)\left[\frac{k-(r-s)}{2}\right] \\ &= -s(r-s) \end{aligned}$$

Now the eigen vector corresponding to eigen value u is given by the following equation:

$$(L-uI_2)A$$

where A is the column vector of order 2×1. Then

$$\begin{bmatrix} k^2+2-u & k \\ k & 2-u \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}=0$$

$$\begin{bmatrix} \left(k^2+4-u\right)A_1+kA_2 \\ kA_1+(2-u)A_2 \end{bmatrix}=0$$

Consider the system

$$\begin{aligned} \left(k^2+4-u\right)A_1+kA_2 &= 0 \\ kA_1+(2-u)A_2 &= 0 \end{aligned} \tag{4.4}$$

We assume that $A_2=l$ in equation (4.4), we achieve

$$A_1=\frac{(u-2)l}{k}=\frac{\left(r^2-1\right)l}{k}=lr$$

Thus the eigen vectors corresponding to u are of kind $\begin{bmatrix} lr \\ l \end{bmatrix}$. For particular $l=1$, the eigen vector assigning to u is $\begin{bmatrix} r \\ 1 \end{bmatrix}$. Similarly the eigen vector assigning to v is $\begin{bmatrix} s \\ 1 \end{bmatrix}$. Let P be matrix of eigen vectors, $P=\begin{bmatrix} r & s \\ 1 & 1 \end{bmatrix}$ and $P^{-1}=(r-s)^{-1}\begin{bmatrix} 1 & -s \\ -1 & -r \end{bmatrix}$. Now we consider the diagonal matrix D, in which eigen values of L are on the main diagonal, $D=\begin{bmatrix} r(r-s) & 0 \\ 0 & -s(r-s) \end{bmatrix}$. Then by the principle of matrix diagonalization [14, 15], we have

$$L=PDP^{-1}$$

$$\begin{aligned} L^n &= PD^nP^{-1} \\ &= (r-s)^{-1}\begin{bmatrix} r & s \\ 1 & 1 \end{bmatrix}\begin{bmatrix} r^n(r-s)^n & 0 \\ 0 & (-1)^n s^n (r-s)^n \end{bmatrix}\begin{bmatrix} 1 & -s \\ -1 & r \end{bmatrix} \\ &= (r-s)^{n-1}\begin{bmatrix} r^{n+1} & (-1)^n s^{n+1} \\ r^n & (-1)^n s^n \end{bmatrix}\begin{bmatrix} 1 & -s \\ -1 & r \end{bmatrix} \\ &= (r-s)^{n-1}\begin{bmatrix} r^{n+1}-(-1)^n s^{n+1} & -sr^{n+1}+(-1)^n rs^{n+1} \\ r^n-(-1)^n s^n & -sr^n+(-1)^n rs^n \end{bmatrix} \\ &= (r-s)^{n-1}\begin{bmatrix} r^{n+1}-(-1)^n s^{n+1} & r^n-(-1)^n s^n \\ r^n-(-1)^n s^n & r^{n-1}-(-1)^n s^{n-1} \end{bmatrix} \end{aligned}$$

If n is odd then by equation (3.3), we get

$$U^n = \begin{bmatrix} r^{n+1}+s^{n+1} & r^n+s^n \\ r^n+s^n & r^{n-1}+s^{n-1} \end{bmatrix}$$

By equating corresponding terms of the matrices, we have

$$L_{k,n} = r^n + s^n$$

This is the required Binet's formula for k-Lucas sequence.
if n is even then again by equation (3.3), we achieve

$$V^n = (r-s)^{-1}\begin{bmatrix} r^{n+1}-s^{n+1} & r^n-s^n \\ r^n-s^n & r^{n-1}-s^{n-1} \end{bmatrix} \tag{4.5}$$

By using lemma (4), we obtain

$$\begin{aligned} \begin{bmatrix} V_{k,n+1} \\ V_{k,n} \end{bmatrix} &= (r-s)^{-1}\begin{bmatrix} r^{n+1}-s^{n+1} & r^n-s^n \\ r^n-s^n & r^{n-1}-s^{n-1} \end{bmatrix}\begin{bmatrix} p+mk \\ 2m \end{bmatrix} \\ &= (r-s)^{-1}\begin{bmatrix} b_1 & b_2 \\ r^n-s^n & r^{n-1}-s^{n-1} \end{bmatrix}\begin{bmatrix} p+mk \\ 2m \end{bmatrix} \end{aligned}$$

where b_1 and b_2 are the corresponding terms of the matrix. Thus

$$\begin{bmatrix} V_{k,n+1} \\ V_{k,n} \end{bmatrix} = (r-s)^{-1}\begin{bmatrix} (p+mk)\,b_1 + 2mb_2 \\ (p+mk)(r^n-s^n) + 2m\left(r^{n-1}-s^{n-1}\right) \end{bmatrix}$$

$$= (r-s)^{-1}\begin{bmatrix} (p+mk)\,b_1 + 2mb_2 \\ p\,(r^n - s^n) + mkr^n + 2mr^{n-1} - mks^n - 2ms^{n-1} \end{bmatrix}$$
$$= (r-s)^{-1}\begin{bmatrix} (p+mk)\,b_1 + 2mb_2 \\ p\,(r^n - s^n) + mr^{n-1}\,(kr+2) - ms^{n-1}\,(ks+2) \end{bmatrix}$$
$$= (r-s)^{-1}\begin{bmatrix} (p+mk)\,b_1 + 2mb_2 \\ p\,(r^n - s^n) + m\,(r-s)\,r^n + m\,(r-s)\,s^n \end{bmatrix}$$

Equating corresponding terms on both sides, we get

$$V_{k,n} = p\frac{r^n - s^n}{r-s} + m\,(r^n + s^n)$$

This is the Binet's formula for Fibonacci-Like sequence.

Now we present a result which establishes a relation between Fibonacci-Like sequence $\langle V_{k,n}\rangle$ and k-Lucas sequence.

Corollary 6. *For $n \in \mathbb{Z}^+$, the following result holds*

$$\begin{aligned} &2mV_{k,n+2} - (p+mk)\,V_{k,n+1} + 2mV_{k,n} - (p+mk)\,V_{k,n-1} \\ &= \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n} \end{aligned} \tag{4.6}$$

Proof. If we equate corresponding terms of matrices in the equation (4.5), we get

$$\frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{p^2m^2 + 4m^2 - p^2} = \frac{r^{n+1} - s^{n+1}}{r-s}$$

and

$$\frac{2mV_{k,n} - (p+mk)\,V_{k,n-1}}{p^2m^2 + 4m^2 - p^2} = \frac{r^{n-1} - s^{n-1}}{r-s}$$

Therefore,

$$\begin{aligned} &\frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1} + 2mV_{k,n} - (p+mk)\,V_{k,n-1}}{p^2m^2 + 4m^2 - p^2} \\ &= \frac{r^{n-1}\left(r^2+1\right) - s^{n-1}\left(s^2+1\right)}{r-s} \\ &= \frac{r^n\,(r-s) + s^n\,(r-s)}{r-s} \end{aligned}$$

Thus

$$\begin{aligned} &2mV_{k,n+2} - (p+mk)\,V_{k,n+1} + 2mV_{k,n} - (p+mk)\,V_{k,n-1} \\ &= \left(m^2k^2 + 4m^2 - p^2\right) L_{k,n} \end{aligned}$$

Hence the result.

5. Cassini's Identity for $\langle V_{k,n} \rangle$

In this section we obtain Cassinin's identity for Fibonacci-Like sequence by using matrix V.

Theorem 7. *For $n \geq 1$, we have*

$$V_{k,n}^2 - V_{k,n+1}V_{k,n-1} = (-1)^n \left(m^2k^2 + 4m^2 - p^2\right) \tag{5.1}$$

Proof.

$$det\left(V^n\right) = \begin{vmatrix} \frac{2mV_{k,n+2} - (p+mk)\,V_{k,n+1}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{m^2k^2+4m^2-p^2} \\ \frac{2mV_{k,n+1} - (p+mk)\,V_{k,n}}{m^2k^2+4m^2-p^2} & \frac{2mV_{k,n} - (p+mk)\,V_{k,n-1}}{m^2k^2+4m^2-p^2} \end{vmatrix}$$

$$\begin{aligned} det\left(V^n\right) = {} & \left(m^2k^2+4m^2-p^2\right)^{-2} \Big\{2mV_{k,n+2}\big[2mV_{k,n} - (p+mk)\,V_{k,n-1}\big] \\ & - (p+mk)\,V_{k,n+1}\big[2mV_{k,n} - (p+mk)\,V_{k,n-1}\big] - \big[2mV_{k,n+1} \\ & - (p+mk)\,V_{k,n}\big]^2\Big\} \end{aligned}$$

By using expansion of $V_{k,n+2}$ and $V_{k,n+1}^2$, we have

$$\begin{aligned} det\left(V^n\right) = {} & \left(m^2k^2+4m^2-p^2\right)^{-2} \big[4m^2kV_{k,n+1}V_{k,n} - 2mk\,(p+mk)\,V_{k,n+1} \\ & V_{k,n-1} + 4m^2V_{k,n}^2 - 2m\,(p+mk)\,V_{k,n}V_{k,n-1} + (p+mk)^2\,V_{k,n+1} \\ & V_{k,n-1} - 2m\,(p+mk)\,V_{k,n+1}V_{k,n} - 4m^2k^2V_{k,n}^2 - 4m^2V_{k,n-1}^2 \\ & - 8m^2kV_{k,n}V_{k,n-1} - (p+mk)^2\,V_{k,n}^2 + 4m\,(p+mk)\,V_{k,n+1}V_{k,n}\big] \\ det\left(V^n\right) = {} & \left(m^2k^2+4m^2-p^2\right)^{-2} \big[4m^2kV_{k,n+1}V_{k,n} - 2m\,(p+mk)\,V_{k,n}V_{k,n-1} \\ & + 2m\,(p+mk)\,V_{k,n+1}V_{k,n} - 4m^2V_{k,n-1}^2 - 8m^2kV_{k,n}V_{k,n-1} \\ & - 2mk\,(p+mk)\,V_{k,n+1}V_{k,n-1} + 4m^2V_{k,n}^2 + (p+mk)^2\,V_{k,n+1}V_{k,n-1} \\ & - 4m^2V_{k,n}^2 - (p+mk)^2\,V_{k,n}^2\big] \\ = {} & \left(m^2k^2+4m^2-p^2\right)^{-2} \big[\left(6m^2k+2mp\right)V_{k,n+1}V_{k,n} - \left(2mp+2m^2k\right) \\ & V_{k,n}V_{k,n-1} - 4m^2V_{k,n-1}^2 - 8m^2kV_{k,n}V_{k,n-1} - 2mk\,(p+mk) \\ & V_{k,n+1}V_{k,n-1} + 4m^2V_{k,n}^2 + (p+mk)^2\,V_{k,n+1}V_{k,n-1} - 4m^2V_{k,n}^2 \\ & - (p+mk)^2\,V_{k,n}^2\big] \\ = {} & \left(m^2k^2+4m^2-p^2\right)^{-2} \big(6m^2V_{k,n}^2 + 2mpkV_{k,n}^2 - 4m^2kV_{k,n+1}V_{k,n-1} \\ & - m^2k^2V_{k,n+1}V_{k,n-1} + 4m^2V_{k,n}^2 + p^2V_{k,n+1}V_{k,n-1} - 4m^2k^2V_{k,n}^2 \end{aligned}$$

$$
\begin{aligned}
&- p^2V_{k,n}^2 - m^2k^2V_{k,n}^2 - 2mpkV_{k,n}^2\big) \\
&= \left(m^2k^2 + 4m^2 - p^2\right)^{-2} \big(m^2V_{k,n}^2 + 4m^2V^2k,n - p^2V_{k,n}^2 - m^2k^2 \\
&\quad V_{k,n+1}V_{k,n-1} - 4m^2V_{k,n+1}V_{k,n-1} + p^2V_{k,n+1}V_{k,n-1}\big) \\
&= \left(m^2k^2 + 4m^2 - p^2\right)^{-2} \left(m^2k^2 + 4m^2 - p^2\right) \left(V_{k,n}^2 - V_{k,n+1}V_{k,n-1}\right) \\
&= \left(m^2k^2 + 4m^2 - p^2\right)^{-1} \left(V_{k,n}^2 - V_{k,n+1}V_{k,n-1}\right)
\end{aligned}
$$

Since $det\,(V^n) = (-1)^n$, we have

$$
V_{k,n}^2 - V_{k,n+1}V_{k,n-1} = (-1)^n \left(m^2k^2 + 4m^2 - p^2\right)
$$

Hence the result.
From the proof of this theorem, we conclude that

$$
\begin{aligned}
&\left[2mV_{k,n+2} - (p + mk)\,V_{k,n+1}\right]\left[2mV_{k,n} - (p + mk)\,V_{k,n-1}\right] \\
&\quad - \left[2mV_{k,n+1} - (p + mk)\,V_{k,n}\right]^2 = (-1)^2 \left(m^2k^2 + 4m^2 - p^2\right)
\end{aligned} \tag{5.2}
$$

6. Characteristic Equation of V^n

In theorem (5) we easily saw the characteristic equation of V. But in this section we obtain the characteristic equation for V^n.

Theorem 8. *For $n \in \mathbb{Z}_0$, the characteristic equation of V^n is given by*

$$
x^2 - L_{k,n}x + (-1)^n = 0 \tag{6.1}
$$

Proof. Since V^n is a square matrix then by Cayley Hamilton theorem, we have

$$
det\,(V^n - xI_2) = 0
$$

Here

$$
\begin{aligned}
&det\,(V^n - xI_2) \\
&= \begin{vmatrix} \dfrac{2mV_{k,n+2} - (p + mk)\,V_{k,n+1}}{m^2k^2 + 4m^2 - p^2} - x & \dfrac{2mV_{k,n+1} - (p + mk)\,V_{k,n}}{m^2k^2 + 4m^2 - p^2} \\ \dfrac{2mV_{k,n+1} - (p + mk)\,V_{k,n}}{m^2k^2 + 4m^2 - p^2} & \dfrac{2mV_{k,n} - (p + mk)\,V_{k,n-1}}{m^2k^2 + 4m^2 - p^2} - x \end{vmatrix} \\
&= \left(m^2k^2 + 4m^2 - p^2\right)^{-2} \Big\{ \left[2mV_{k,n+2} - (p + mk)\,V_{k,n+1}\right]\left[2mV_{k,n} - (p + mk)\right. \\
&\quad \left. V_{k,n-1}\right] - x\left(m^2k^2 + 4m^2 - p^2\right)\left[2mV_{k,n+2} - (p + mk)\,V_{k,n+1}\right] - x
\end{aligned}
$$

$$
\begin{aligned}
&\left(m^2k^2+4m^2-p^2\right)\left[2mV_{k,n}-(p+mk)\,V_{k,n-1}\right]+x^2\left(m^2k^2+4m^2-p^2\right)^2\\
&\quad-\left[2mV_{k,n+1}-(p+mk)\,V_{k,n}\right]^2\Big\}\\
&=\left(m^2k^2+4m^2-p^2\right)^{-2}\Big\{x^2\left(m^2k^2+4m^2-p^2\right)^2-x\left(m^2k^2+4m^2-p^2\right)\\
&\quad\left[2mV_{k,n+2}-(1+mk)\,V_{k,n+1}+2mV_{k,n}-(p+mk)\,V_{k,n-1}\right]\\
&\quad+\left[2mV_{k,n+2}-(p+mk)\,V_{k,n+1}\right]\left[2mV_{k,n}-(p+mk)\,V_{k,n-1}\right]\\
&\quad-\left[2mV_{k,n+1}-(p+mk)\,V_{k,n}\right]^2\Big\}
\end{aligned}
$$

If we consider corollary (6) and equation (5.2), we get

$$
\begin{aligned}
det\left(V^n-xI_2\right)&=\left(m^2k^2+4m^2-p^2\right)^{-2}\Big[x^2\left(m^2k^2+4m^2-p^2\right)^2\\
&\quad-L_{k,n}x\left(m^2k^2+4m^2-p^2\right)^2+(-1)^n\left(m^2k^2+4m^2-p^2\right)^2\Big]\\
&=x^2-L_{k,n}x+(-1)^n
\end{aligned}
$$

Hence the characteristic equation of V^n is

$$x^2-L_{k,n}x+(-1)^n=0$$

Hence the result.

References

[1] S. Falcon and A. Plaza, *On the Fibonacci k-Numbers*, Chaos Solitons and Fractals, 32 (2007), 1615-1624.

[2] S. Falcon, *On the k-Lucas Numbers*, Int. J. Contemp. Math. Sciences, 6(21) (2011), 1615-1624.

[3] J. R. Silvester, *Fibonacci Numbers by Matrix Methods*, The Mathematical Gazette, 63(425) (1979), 181-191.

[4] E. Kilic, *The Binet Formula, Sums and Representations of Generalized Fibonacci p-Numbers*, Chaos Solitons and Fractals, 32 (2007), 1615-1624.

[5] S. P. Jun and K. H. Choi, *Some Properties of the Generalized Fibonacci Sequence $\langle q_n \rangle$ by Matrix Methods*, Korean J. Math., 24(4) (2016), 681-691.

[6] S. J. Ying, H. C. Kit , Ibrahim H. and Ahmad N., *Algebraic Properties of Generalized Fibonacci Sequence Via Matrix Methods*, Journal of Engineering and Applied Sciences, 11(11) (2016), 2396-2401.

[7] Z. Akyuz and S. Halici, *Some Identities Deriving from the nth Power of a Special Matrix*, Advances in Difference Equations, 2012.

[8] F. Koken and D. Bozkurt, *On the Jacobsthal Numbers by Matrix Methods*, Int. J. Contemp. Math. Sciences, 3(13) (2008), 605-614.

[9] A. Yalciner, *A Matrix Approach for Divisibility Properties of the Generalized Fibonacci Sequence*, Discrete Dynamics in Nature and Society, 2013.

[10] A. Borges, P. Catarino, A. P. Aires, P. Vasco, and H. Campos, *Two-by-Two Matrices Involving k-Fibonacci and k-Lucas Sequences*, Applied Mathematical Sciences, 8(34) (2014), 1659-1666.

[11] P. Catarino and P. Vasco, *Two-by-Two Matrices Involving k-Fibonacci and k-Lucas Sequences*, Int. Journal of Math. Analysis, 7(45) (2013), 2209-2215.

[12] P. Catarino and P. Vasco, *A Note Involving Two-by-Two Matrices of the k-Pell and k-Pell-Lucas Sequences*, International Mathematical Forum, 8(32) (2013), 1561-1568.

[13] S. Srisawat and W. Sripad, *On the (s,t)-Pell and (s,t)-Pell-Lucas Numbers by Matrix Methods*, Annales Mathematicae et Informaticae, 8(32) (2013), 1561-1568.

[14] http://mathworld.wolfram.com/MatrixDiagonalization.html

[15] http://mathworld.wolfram.com/EigenDecomposition.html

A. A. Wani
School of Studies in Mathematics,
Vikram University Ujjain,
Ujjain, India
email: *arfatahmadwani@gmail.com*

V. H. Badshah
School of Studies in Mathematics,
Vikram University Ujjain,
Ujjain, India
email: *vhbadshah@gmail.com*

S. Halici
Department of Mathematics, Faculty of Arts and Sciences,
Sakarya University,
54187 Sakarya, Turkey
email: *shalici@pau.edu.tr*

Paula Catarino
Department of Mathematics, School of Science and Technology,
University of Trás-os-Montes e Alto Douro (Vila Real Portugal)
email: *pcatarino23@gmail.com*