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RICCI ALMOST SOLITON ON (κ, μ) SPACE FORMS

A. Sarkar, P. Bhakta

Abstract. The object of the present paper is to study Ricci almost solitons on (κ, μ) space forms. It is shown that the scalar curvature of a (κ, μ) space form with Ricci almost soliton is invariant by the application of ξ. We have also studied gradient Ricci soliton on (κ, μ) space forms. We have proved that the scalar curvature of a (κ, μ) space form admitting Ricci soliton is constant.

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1. Introduction

The notion of (κ, μ) contact metric manifolds was introduced by Blair[2]. T. Koufogiorgos [9] studied (κ, μ) contact metric manifolds with constant ϕ-sectional curvature. A manifold with constant ϕ-sectional curvature is known as a space-form. A (κ, μ) contact metric manifold with constant ϕ-sectional curvature is called (κ, μ) space form. (κ, μ) space forms have been also studied in the paper [1]. A full classification of (κ, μ) contact metric manifolds has been given in the paper [3] . (κ, μ) contact metric manifolds have been also studied by the first author in the papers [5] and [13].

The notion of Ricci flow has become a popular topic of research due to its application by Perelman [10] to solve the long standing open problem 'Poincare conjecture'. The notion of Ricci flow was introduced by Hamilton [8]. In the same time D. Fridan [6] introduced the concept of Ricci flow to apply it in some relativistic problems in physics. A Ricci soliton is a constant solution of Ricci flow up to diffeomorphism and scaling. A Ricci flow is a heat type parabolic partial differential equation given by

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \quad g_{ij}(0) = g_0,$$

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where g_{ij} are components of Riemannian metric and S_{ij} are components of Ricci curvature. For more details please see [4] . The study of Ricci soliton in contact manifolds was first started by R. Sharma [12]. Again in the paper [7] Ricci soliton on Kenmotsu 3-manifolds has been studied. The notion of Ricci almost soliton has been given by S.Pigola [11]. Ricci solitons have also been studied by the first author in the papers [14] and [15].

The present paper is organized as follows:

Section 2 contains preliminary results. In section 3, we study Ricci almost soliton in (κ, μ)-space forms. Section 4 contains the study of (κ, μ)-space forms with gradient almost Ricci solitons. The last section contains an example.

2. Preliminaries

A $(2n + 1)$ dimensional differential manifold M is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations [2]:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{1}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X). \tag{2}$$

For a contact metric manifold we know

$$\nabla_X \xi = -\phi X - \phi h X, \tag{3}$$

$$h\xi = 0, \quad h\phi = -\phi h \tag{4}$$

for all vector fields X,Y and Z on M. In a contact metric manifold the $(1, 1)$ tensor field h defined by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation is a symmetric operator anti-commutative with ϕ. In [2] Blair et al, introduced a class of contact metric manifold M satisfying

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{5}$$

where κ, μ are real constants. This class of contact metric manifolds are called (κ, μ) contact manifolds.

In a (κ, μ) contact metric manifold, the following relations also hold [2]:

$$g(QX, Y) = S(X, Y), \tag{6}$$

$$h^2 = (\kappa - 1)\phi^2, \kappa \leqslant 1, \tag{7}$$

$$\begin{aligned} S(X,Y) &= \frac{1}{4}\{(c(2n+1)+6n+4\kappa-5)g(X,Y)-(c(2n+1) \\ &+ 6n+4\kappa-5-8n\kappa)\eta(X)\eta(Y)+(8-8n+4\mu)g(Y,hX)\}, \end{aligned} \quad (8)$$

$$r=\frac{n}{2}\{c(2n+1)+6n+4\kappa-5\}+2n\kappa, \quad (9)$$

$$\begin{aligned} 2g((L_V\nabla)(X,Y),Z) &= (\nabla_X L_V g)(Y,X)+(\nabla_Y L_V g)(Z,X) \\ &+ (\nabla_Z L_V g)(X,Y), \end{aligned} \quad (10)$$

where S is the Ricci tensor of type $(0,2)$ and r is the scalar curvature of the manifold. If μ=0, the (κ,μ)-nullity distribution reduces to the κ-nullity distribution, where the κ-nullity distribution $N(\kappa)$ of a Riemannian manifold M is defined by

$$N(\kappa): p \rightarrow N_p(\kappa)=\{W\in T_p(M)/R(X,Y)W=\kappa(g(Y,W)X-g(X,W)Y)\}.$$

If $\xi\in N(\kappa)$, then we call M a $N(\kappa)$-contact metric manifold.

The class of (κ,μ)-contact metric manifolds contain both the class of Sasakian ($\kappa=1$ and $h=0$) and non-Sasakian ($\kappa\neq 1$ and $h\neq 0$) manifolds. Through out the paper we denote by M^{2n+1} a $(2n+1)$-dimensional non-Sasakian (κ,μ)-space form. A contact metric manifold is said to be η-Einstein if $Q=aId+b\eta\otimes\eta$, where a, b are smooth functions on M^{2n+1}.

A space form is said to be (κ,μ)-space form if the ϕ-sectional curvature is constant. In this space form the curvature tensor is given by [1]

$$\begin{aligned} 4R(X,Y)Z &= [(c+3))\{g(Y,Z)X-g(x,Z)Y\}+(c+3-4\kappa)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \\ &+ (c-1)\{2g(X,\phi Y)\phi Z+g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X\} \\ &- 2\{g(hX,Z)hY-g(hY,Z)hX)+2g(X,Z)hY-2g(Y,Z)hX \\ &- 2\eta(X)\eta(Z)hY+2\eta(Y)\eta(Z)hX+2g(hX,Z)Y-2g(hY,Z)X \\ &+ 2g(hY,Z)\eta(X)\xi-2g(hX,Z)\eta(Y)\xi-g(\phi hX,Z)\phi hY \\ &+ g(\phi hY,Z)\phi hX\}+4\mu\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY \\ &+ g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi\}]. \end{aligned} \quad (11)$$

3. Ricci almost soliton on (κ,μ) space forms

Definition 3.1. A metric g of a manifold M is called Ricci almost soliton if it satisfies

$$(\mathcal{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0 \quad (12)$$

for a function λ. The notion of Ricci almost soliton was introduced in the paper [11] by S.Pigola. Let us consider a (κ, μ) space form. From (8) we get

$$\begin{aligned} QY &= \frac{1}{4}\{(c(2n+1)+6n+4\kappa-5)Y-(c(2n+1) \\ &+ 6n+4\kappa-5-8n\kappa)\eta(Y)\xi+(8-8n+4\mu)hY\}. \end{aligned} \tag{13}$$

From the property of covariant derivative and Lie derivative we have from (13)

$$\begin{aligned} (\nabla_X Q)Y &= -\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\nabla_X\eta(Y))\xi+\eta(Y)\nabla_X\xi\} \\ &+ \frac{1}{2}(8-8n+4\kappa)\{\nabla_X\mathcal{L}_\xi\phi)Y+\mathcal{L}_\xi\phi\nabla_X Y\} \\ &+ \{c(2n+1)+6n+4\kappa-5-8n\kappa\}\eta(\nabla_X Y)\xi \\ &+ (8-8n+4\mu)h\nabla_X Y. \end{aligned} \tag{14}$$

Now, from (14),

$$\begin{aligned} g((\nabla_X Q)Y,X) &= -\{c(2n+1)+(6n+4\kappa-5-8n\kappa)\} \\ &\quad \{g((\nabla_X\eta(Y))\xi,X)+g(\eta(Y)\nabla_X\xi,X)\} \\ &+ \frac{1}{2}(8-8n+4\kappa)\{g(\nabla_X\mathcal{L}_\xi\phi)Y,X) \\ &+ \{c(2n+1)+6n+4\kappa-5-8n\kappa\}g(\eta(\nabla_X Y)\xi,X). \end{aligned} \tag{15}$$

Let$\{e_1,e_2,\xi\}$ be an orthonormal ϕ-basis of the tangent space of the manifold at any point. Then we know

$$\text{div}QY = g((\nabla_{e_1}Q)Y,e_1)+g((\nabla_{e_2}Q)Y,e_2)+g((\nabla_{e_3}Q)Y,e_3). \tag{16}$$

Using (15) in (16) we have

$$\begin{aligned} \text{div}QY &= -\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{g(\nabla_{e_1}\eta(Y)\xi,e_1)+g(\eta(Y)\nabla_{e_1}\xi,e_1) \\ &+ g(\nabla_{e_2}\eta(Y)\xi,e_2)+g(\eta(Y)\nabla_{e_2}\xi,e_2)+g(\nabla_{e_3}\eta(Y)\xi,e_3) \\ &+ g(\eta(Y)\nabla_{e_3}\xi,e_3)\}+\frac{1}{2}(8-8n+4\kappa)\{g((\nabla_{e_1}\mathcal{L}_\xi\phi)Y,e_1) \\ &+ g((\nabla_{e_2}\mathcal{L}_\xi\phi)Y,e_2)+g((\nabla_{e_3}\mathcal{L}_\xi\phi)Y,e_3)\} \\ &+ \{c(2n+1)+6n+\kappa-5-8n\kappa\}\{g(\eta(\nabla_{e_1}Y)\xi,e_1) \\ &+ g(\eta(\nabla_{e_2}Y)\xi,e_2)+g(\eta(\nabla_{e_3}Y)\xi,e_3). \end{aligned} \tag{17}$$

But it is well known that

$$\frac{1}{2}dr(Y) = \text{div}QY.$$

So, from (17) we get

$$\begin{aligned}
\frac{1}{2}dr(Y) &= -\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{g(\nabla_{e_1}\eta(Y)\xi, e_1)+g(\eta(Y)\nabla_{e_1}\xi, e_1) \\
&+ g(\nabla_{e_2}\eta(Y)\xi, e_2)+g(\eta(Y)\nabla_{e_2}\xi, e_2)+g(\nabla_{e_3}\eta(Y)\xi, e_3) \\
&+ g(\eta(Y)\nabla_{e_3}\xi, e_3)\}+\frac{1}{2}(8-8n+4\kappa)\{g((\nabla_{e_1}\mathcal{L}_\xi\phi)Y, e_1) \\
&+ g((\nabla_{e_2}\mathcal{L}_\xi\phi)Y, e_2)+g((\nabla_{e_3}\mathcal{L}_\xi\phi)Y, e_3)\} \\
&+ \{c(2n+1)+6n+\kappa-5-8n\kappa\}\{g(\eta(\nabla_{e_1}Y)\xi, e_1) \\
&+ g(\eta(\nabla_{e_2}Y)\xi, e_2)+g(\eta(\nabla_{e_3}Y)\xi, e_3). \qquad (18)
\end{aligned}$$

Putting $Y=\xi$ in (18) and using (3) we have

$$\begin{aligned}
\frac{1}{2}dr(\xi) &= (8-8n+4\mu)\{g(-h(-\phi e_1-\phi he_1), e_1)+g(-h(-(\phi e_2-\phi he_2), e_2) \\
&+ g(-h(-\phi e_3-\phi he_3), e_3)\}. \qquad (19)
\end{aligned}$$

If we consider $\{e_1, e_2, e_3\}$ as a ϕ-basis and $e_3=\xi$ then (18) and (4) we have

$$\xi r = 0.$$

Thus we can state the following:

Theorem 3.1. If a (2n+1)-dimensional (κ, μ)-space form admits Ricci almost soliton, then the scalar curvature is invariant by the application of ξ.

Since the manifold is Ricci almost soliton, we get

$$(\mathcal{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0, \qquad (20)$$

here λ is a function.
Using (8) in (20) we get,

$$\begin{aligned}
(\mathcal{L}_V g)(X,Y) &= 2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\eta(X)\eta(Y) \\
&- \frac{1}{2}(8-8n+4\mu)g(Y,hX) \\
&- \{\frac{1}{2}(c(2n+1)+6n+4\kappa-5)+2\lambda\}g(X,Y). \qquad (21)
\end{aligned}$$

Differentiating covariantly with respect to W we get from (21)

$$\begin{aligned}
(\nabla_W\mathcal{L}_V g)(X,Y) &= 2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\nabla_W\eta(X))\eta(Y) \\
&+ \eta(X)(\nabla_W\eta(Y))\}. \qquad (22)
\end{aligned}$$

Replacing W by Z in (22) we get,

$$\begin{aligned}(\nabla_Z\mathcal{L}_V g)(X,Y) &= 2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\nabla_Z\eta(X))\eta(Y)\\ &+ \eta(X)(\nabla_Z\eta(Y))\}.\end{aligned} \tag{23}$$

Again replacing W by Y and X by Z , Y by X in (22) we get

$$\begin{aligned}(\nabla_Y\mathcal{L}_V g)(Z,X) &= 2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\nabla_Y\eta(Z))\eta(X)\\ &+ \eta(Z)(\nabla_Y\eta(X))\}.\end{aligned} \tag{24}$$

Finally replacing W by X and X by Y, Y by Z in (22) we get

$$\begin{aligned}(\nabla_X\mathcal{L}_V g)(Y,Z) &= 2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\nabla_X\eta(Y))\eta(Z)\\ &+ \eta(Y)(\nabla_X\eta(Z)\}.\end{aligned} \tag{25}$$

From (10) we have,

$$\begin{aligned}2g((\mathcal{L}_V\nabla)(X,Y),Z) &= 2\{c(2n+1)+6n+4\kappa-8n\kappa\}\{(\nabla_X\eta Y)\eta(Z)\\ &+ \eta(Y)(\nabla_X\eta(Z))+(\nabla_Y\eta(Z))\eta(X)+\eta(Z)(\nabla_Y\eta(X))\\ &+ (\nabla_Z\eta(X))\eta(Y)+\eta(X)(\nabla_Z\eta(Y)\}.\end{aligned} \tag{26}$$

Replacing Z by ϕZ in (26) we get

$$\begin{aligned}2g((\mathcal{L}_V\nabla)(X,Y),\phi Z) &= 2\{c(2n+1)+6n+4\kappa-8n\kappa\}\{(\nabla_X\eta Y)\eta(\phi Z)\\ &+ \eta(Y)(\nabla_X\eta(\phi Z))+(\nabla_Y\eta(\phi Z))\eta(X)+\eta(Z)(\nabla_Y\eta(X))\\ &+ (\phi\nabla_Z\eta(X))\eta(Y)+\eta(X)(\phi\nabla_Z\eta(Y)\}.\end{aligned} \tag{27}$$

Putting $Z=\xi$ in (27) and using (2) we get,

$$2\{c(2n+1)+6n+4\kappa-5-8n\kappa\}\{(\phi\nabla_Z\eta(X))\eta(Y)+(\phi\nabla_Z\eta(Y))\eta(X)\}=0. \tag{28}$$

Assume

$$(\phi\nabla_Z\eta(X))\eta(Y)+\phi(\nabla_Z\eta(Y))\eta(X)\}\neq 0.$$

Then from (28) we get,

$$\kappa=\frac{5-6n-c(2n+1)}{4(1-2n)}.$$

Putting the value of κ in(9) we get,

$$r=\frac{c(1.5n+4n^2+2n^3)+4n^2+12n^3-5n}{2(2n-1)}.$$

Hence we can state the following:

Theorem 3.2. If a $(2n+1)$ dimensional (κ,μ) space admits Ricci almost soliton then, the scalar curvature is constant.

4. (κ, μ) space forms admitting gradient almost Ricci soliton

Definition 4.1. A Ricci almost soliton on a (κ, μ) space form will be called gradient Ricci almost soliton if the vector field V is equal to the gradient of a potential function $-f$.

For the gradient Ricci almost soliton we get the following:

$$\nabla\nabla f = S + \lambda g, \tag{29}$$

where λ is a function. From $(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0$, we have

$$\nabla_Y Df = QY + \lambda Y, \tag{30}$$

where D is the gradient operator of g and Q is the Ricci operator. Putting $X{=}\xi$ and $Z{=}Df$ in (11) we get

$$\begin{aligned} R(\xi,Y)Df &= -\kappa\eta(Df)Y + \kappa g(Y,Df)\xi \\ &- \mu\eta(Df)hY + \mu g(hY,Df)\xi. \end{aligned} \tag{31}$$

Now,

$$g(R(\xi,Y)Df,\xi) = -\kappa\eta(Df)\eta(Y) + \kappa g(Y,Df) + \mu g(hY,Df). \tag{32}$$

From (30) we have

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{33}$$

Using (14) in (33) we get

$$\begin{aligned} R(\xi,Y)Df &= -\{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} \\ &\quad \{(\nabla_\xi\eta(Y))\xi + (QY + \phi hY)\} + (4 - 4n + 2\kappa) \\ &\quad \{(\nabla_\xi 2h)Y + 2h\nabla_\xi Y - (\nabla_Y 2h)\xi + 2h(\phi + \phi hY)\} \\ &+ \{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} \\ &\quad \{\eta(\nabla_\xi Y)\xi - \eta(\nabla_Y\xi)\xi\} \\ &+ (8 - 8n + 4\mu)(h\nabla_\xi Y + h\phi Y + h\phi hY). \end{aligned} \tag{34}$$

Now,

$$\begin{aligned} g(R(\xi,Y)Df,\xi) &= \{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} \\ &\quad \{\eta(\nabla_\xi Y) - \eta(\nabla_\xi\eta(Y))\xi\} \\ &+ (4 - 4n + 2\kappa)[\eta\{(\nabla_\xi 2h)Y\} - \eta\{(\nabla_Y 2h)\xi\}]. \end{aligned} \tag{35}$$

From (32) and (35) we get

$$\begin{aligned} -\kappa\eta(Df)\eta(Y) + \kappa g(Y, Df) &+ \mu g(hY, Df) \\ &= \{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} \\ &\quad \{\eta(\nabla_\xi Y) - \eta(\nabla_\xi \eta(Y)\xi)\} \\ &+ (4 - 4n + 2\kappa)[\eta\{(\nabla_\xi 2h)Y\} \\ &- \eta\{(\nabla_Y 2h)\xi\}]. \end{aligned} \tag{36}$$

Putting $Y{=}\phi Df$ in (36) we obtain

$$\mu = \frac{\{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} + (4 - 4n + 2\kappa)\eta\{(\nabla_\xi 2h)\phi Df\}}{g(h\phi Df, Df)}. \tag{37}$$

So we state the following:

Theorem 4.1. In (2n+1) dimensional (κ, μ)-space forms admitting gradient Ricci almost soliton, the potential function $-f$ is related with μ by the formula

$$\mu = \frac{\{c(2n+1) + 6n + 4\kappa - 5 - 8n\kappa\} + (4 - 4n + 2\kappa)\eta\{(\nabla_\xi 2h)\phi Df\}}{g(h\phi Df, Df)}.$$

5. Example

In this section, we give an example of a (κ, μ)-space form which admits a Ricci almost soliton.

We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 / x \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$e_1 = \tfrac{2}{x}\tfrac{\partial}{\partial y}\ , \ e_2 = 2\tfrac{\partial}{\partial x} - \tfrac{4z}{x}\tfrac{\partial}{\partial y} + xy\tfrac{\partial}{\partial z},\ e_3 = \tfrac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by

$\eta(U) = g(U, e_3)$ for any $U \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0.$$

Then using the linearity of ϕ and g we have $\eta(e_3) = 1$,
$\phi^2(U) = -U + \eta(U)e_3$

and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$for any U,W$\in \chi(M)$.

Moreover $he_1 = -e_1$, $he_2 = e_2$ and $he_3 = 0$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 2e_3 + \tfrac{2}{x}e_1\ ,\ [e_1, e_3] = 0,\ [e_2, e_3] = 2e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_3 = \xi$ and using the above formula for the Riemannian metric g, we can easily calculate that

$$\nabla_{e_1} e_1 = -2e_3,\ \nabla_{e_1} e_2 = \tfrac{2}{x}e_1,\ \nabla_{e_1} e_3 = 0,$$

$$\nabla_{e_2} e_1 = -\tfrac{2}{x}e_2,\ \nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_3 = 2e_1,$$

$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a (κ, μ)-contact metric structure on M.

Using the above relations, we can easily deduce the following:

$$R(e_1, e_2)e_2 = \tfrac{4}{x^2}e_2,\ R(e_2, e_1)e_1 = (-4 + \tfrac{4}{x^2})e_1 + \tfrac{4}{x}e_3,\ R(e_3, e_2)e_2 = \tfrac{4}{x}e_1.$$

Now

$$S(e_1, e_1) = 0,\ S(e_2, e_2) = 0,\ S(e_3, e_3) = 0, \text{ and } S(e_1, e_2) = \tfrac{4}{x^2}.$$

Thus S is not indetically zero.
Again $r = 0$, a constant. Which varifies Theorem (3.2).

We see that from (3.9)

$$(\mathcal{L}_{e_1} g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{38}$$

$$(\mathcal{L}_{e_2} g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{39}$$

$$(\mathcal{L}_{e_3} g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{40}$$

where $\lambda = -\frac{2}{x}$.

Hence the manifold M is a Ricci almost soliton.

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Avijit Sarkar
Department of Mathematics,
University of Kalyani,
Kalyani 741235
West Bengal
India
email: *avjaj@yahoo.co.in*

Pradip Bhakta
Department of Mathematics,
University of Kalyani,
Kalyani 741235
West Bengal
India
email: *pradip020791@gmail.com*