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ON EIGENSTATES FOR SOME $\mathbf{SL_2}$ RELATED HAMILTONIAN

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Abstract. In this paper we consider the stationary Schrdinger equation for a self-conjugated Hamiltonian $\boldsymbol{H} = \dfrac{\boldsymbol{e}+\boldsymbol{f}}{\mathrm{i}}$, where $\boldsymbol{e}$ and $\boldsymbol{f}$ is an anti-unitary pair of the canonical Cartan "creating" and "annihilation" operators for the classical Lie algebra sl_2 taken in the representation with "the lowest weight equals to 1". In this paper we prove that this operator has the continuous spectrum. Construction of eigenstates for $\boldsymbol{H}$ is given in details.

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1. Introduction

This paper will deal with the representation theory of the classical Lie algebra [1]. We will consider the Lie algebra sl_2 in a certain infinitely dimensional representation corresponding to the lowest weight 1. The representation module is equivalent to the Fock Space representation of the quantum oscillator [2]. The "creating" and "annihilation" operators $\boldsymbol{e}$ and $\boldsymbol{f}$ are anti-unitary, so that the operator $\boldsymbol{H} = \frac{1}{\mathrm{i}}(\boldsymbol{e}+\boldsymbol{f})$ is Hermitian, and therefore it can be interpreted as a Hamiltonian for a certain Quantum Mechanical system. This Hamiltonian is related to a Hamiltonian considered in [3, 4] in the limit $q = 1$ (Note, the regime $q = 1$ was not considered in [3, 4]).

This paper organised as follows. In section 2 we fix the proper representation of sl_2 and rewrite the stationary Schrödinger equation as a linear recursion with non-constant coefficients. Section 3 is devoted to the analysis of the recursion equations. Its asymptotic is discussed in section 4. Section 5 contains discussion and conclusion.

2. Formulation of the problem

We consider the algebra sl_2 generated by three operators $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{h}$ satisfying the three fundamental commutation relations [1].

$$[\boldsymbol{e}, \boldsymbol{f}] \;=\; \boldsymbol{h}\;, \quad [\boldsymbol{h}, \boldsymbol{e}] \;=\; 2\boldsymbol{e}\;, \quad [\boldsymbol{h}, \boldsymbol{f}] \;=\; -2\boldsymbol{f}\;. \tag{1}$$

Let $\mathfrak{F}$ stands for the Fock Space,

$$\mathfrak{F} \;=\; \mathrm{Span}\left\{\; |\, n\rangle\;, \quad n\in\mathbb{Z}_{n\geq 0}\;\right\}\;. \tag{2}$$

The map

$$\boldsymbol{e} \;\stackrel{\pi}{\to}\; \pi(\boldsymbol{e}) \;\in\; \mathrm{End}\,(\mathfrak{F})\;, \quad \text{etc.}, \tag{3}$$

we define as

$$\boldsymbol{e}\,|n\rangle \;=\; |n{+}1\rangle\; \mathsf{i}(n{+}1)\;, \quad \boldsymbol{f}\,|n\rangle \;=\; |n{-}1\rangle\; \mathsf{i}n\;, \quad \boldsymbol{h}\,|n\rangle \;=\; |n\rangle\;(2n{+}1)\;, \quad n\in\mathbb{Z}_{n\geq 0}\;, \tag{4}$$

where for shortness we use notation $\boldsymbol{e}$ instead of $\pi(\boldsymbol{e})$, etc. Our representation (4) is the representation with the lowest weight 1,

$$\boldsymbol{h}|0\rangle \;=\; |0\rangle\;. \tag{5}$$

(in Physics this is called "spin $= -1/2$ representation"). The Fock co-module is defined by

$$\langle n|n'\rangle \;=\; \delta_{n,n'}\;, \quad n, n'\geq 0\;. \tag{6}$$

An essential feature of our paper is that this representation not unitary:

$$\boldsymbol{e}^{\dagger} \;=\; -\boldsymbol{f}\;, \tag{7}$$

where the "dagger" means the Hermitian conjugation. Subject of our interest is self-conjugated unbounded Hamiltonian

$$\boldsymbol{H} \;=\; \frac{\boldsymbol{e}+\boldsymbol{f}}{\mathsf{i}}\;, \tag{8}$$

and the stationary Schrödinger equation for it,

$$\boldsymbol{H}\,|\psi\rangle \;=\; |\psi\rangle\, E\;. \tag{9}$$

In what follows, we will study the structure of $|\psi\rangle$ for any $E\in\mathbb{R}$ and deduce that our Hamiltonian has continuous spectrum.

3. Analysis of the recursion

We will use the Dirac notations for $\langle\mathrm{bra}|$ and $|\mathrm{ket}\rangle$ vectors. In components,

$$\psi_n \;=\; \langle n|\psi\rangle\;, \tag{10}$$

where $\langle n|$ is a state of Fock co-module, cf. (6), and $|\psi\rangle$ is a required wavefunction. The stationary Schrödinger equation (9) in components reads

$$(n+1)\,\psi_{n+1} \;+\; n\,\psi_{n-1} \;=\; E\,\psi_n\;, \tag{11}$$

where we assume

$$\psi_0 \;=\; 1 \qquad \forall\;E\;\in\;\mathbb{R}\,. \tag{12}$$

Our aim now is to understand the asymptotic behaviour of ψ_n when $n\to\infty$. Since E for now is only one free parameter, we assume implicitly

$$|\psi\rangle \;=\; |\psi_E\rangle\;,\quad \psi_n \;=\; \psi_n(E)\,. \tag{13}$$

Recursion (11) can be identically rewritten in matrix form [3, 4]:

$$(\psi_n,\psi_{n+1}) \;=\; (\psi_{n-1},\psi_n)\;\cdot\;L_{n+1}\;, \tag{14}$$

where

$$L_n \;=\; \begin{pmatrix} 0 & -1+\dfrac{1}{n} \\ 1 & \dfrac{E}{n} \end{pmatrix}\;. \tag{15}$$

Thus,

$$(\psi_{n-1},\psi_n) \;=\; (0,1)\;L_1\cdot L_2\cdots L_{n-1}\cdot L_n\;. \tag{16}$$

Since

$$L_\infty \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;,\quad L_\infty^4 \;=\; 1\;, \tag{17}$$

we expect *mod* 4 pattern for ψ_n. Diagonalising matrix L_n ,

$$L_n \;=\; P_n^{-1}\begin{pmatrix} \lambda_n & 0 \\ 0 & \overline{\lambda_n} \end{pmatrix} P_n\;, \tag{18}$$

where

$$\lambda_n \;=\; \mathrm{i}\left(\sqrt{1-\frac{1}{n}-\frac{E^2}{4n^2}}-\mathrm{i}\frac{E}{2n}\right) \;=\; \mathrm{i}\sqrt{1-\frac{1}{n}}\;\exp\left\{-\mathrm{i}\arcsin\frac{E}{2\sqrt{n(n-1)}}\right\}\;, \tag{19}$$

and

$$P_n P_{n+1}^{-1} \;=\; 1 + \frac{1}{2n^2} \begin{pmatrix} 0 & 0 \\ -E & 1 \end{pmatrix} + \mathcal{O}(1/n^3)\;, \tag{20}$$

one can deduce the following asymptotic straightforwardly from (16):

$$\psi_n(E) \;=\; \frac{A_n(E)}{\sqrt{n}}\; \cos\left(\frac{E}{2}\log n - \frac{\pi n}{2} + \varphi_n(E)\right)\;, \quad n \gg 1\;. \tag{21}$$

Intensive numerical computations allow one to conclude that the sequences $A_n(E)$ and $\varphi_n(E)$ smoothly converge to $A(E)$ and $\varphi(E)$ when $n \to \infty$. Therefore, we can postulate the $1/n$ expansion for A_n and φ_n:

$$A_n(E) \;=\; A(E)\,(1 + \frac{\delta_1}{n} + \frac{\delta_2}{n^2} + \cdots)\;, \quad \varphi_n(E) \;=\; \varphi(E) + \frac{\epsilon_1}{n} + \frac{\epsilon_2}{n^2} + \cdots \tag{22}$$

with some n-independent coefficients

$$\delta_j \;=\; \delta_j(E)\;, \quad \epsilon_j = \epsilon_j(E)\;, \quad j \;\geq\; 1\;. \tag{23}$$

Values of $\delta_j,\ \epsilon_j$ must follow from (11). In what follows, let us combine all correction terms in (22) into

$$\delta(n,E) \;=\; \sum_{j=1}^{\infty} \frac{\delta_j(E)}{n^j}\;, \qquad \epsilon(n,E) \;=\; \sum_{j=1}^{\infty} \frac{\epsilon_j(E)}{n^j}\;. \tag{24}$$

To get these values, let us substitute (21) into (11). To do this in convenient way, let us introduce

$$\Phi_n \;=\; \frac{E}{n}\log_n - \frac{\pi n}{2} + \varphi_n;\,, \quad \Phi_{n+1} \;=\; \Phi_n - \frac{\pi}{2} + \alpha_n;\,, \quad \Phi_{n-1} \;=\; \Phi_n + \frac{\pi}{2} - \alpha'_n\;. \tag{25}$$

The values of α_n and α'_n are then given by

$$\begin{aligned} \alpha_n \;&=\; \Phi_{n+1} - \Phi_n + \frac{\pi}{2} \;=\; \frac{E}{2}\log_{(n+1)} + \varphi_{n+1} - \frac{E}{2}\log_n - \varphi_n \\ &=\; \frac{E}{2}\log(1+\frac{1}{n}) + \epsilon_1(\frac{1}{n+1} - \frac{1}{n}) + \epsilon_2(\frac{1}{(n+1)^2} - \frac{1}{n^2}) + \cdots \end{aligned} \tag{26}$$

and similarly for α'_n. Let further

$$\frac{1}{n} = x \quad \Rightarrow \quad \frac{1}{n+1} = \frac{x}{1+x} = \sum_{j=1}^{\infty}(-)^{j+1}x^j \quad \text{etc.,} \tag{27}$$

so that $1/n$-expansion becomes x-expansion. Then,

$$\begin{aligned}\alpha_n &= \frac{E}{2}\log(1+x)+\epsilon_1(\frac{x}{1+x}-x)+\epsilon_2(\frac{x^2}{(1+x)^2}-x^2)+\cdots \\ &= \frac{E}{2}x-(\frac{E}{4}+\epsilon_1)x^2+(\frac{E}{6}+\epsilon_1-2\epsilon_2)x^3+\mathcal{O}(x^4). \end{aligned} \tag{28}$$

Value of α_n' have similar structure.

Now we can use (25,26 and 28) in (21 and 11):

$$\begin{aligned}\psi_n &= \frac{A_n}{\sqrt{n}}\cos(\Phi_n)\ , \\ \psi_{n+1} &= \frac{A_{n+1}}{\sqrt{n+1}}\cos(\Phi_n-\frac{\pi}{2}+\alpha_n) \ = \ \frac{A_{n+1}}{\sqrt{n+1}}(\sin\Phi_n\cos\alpha_n+\cos\Phi_n\sin\alpha_n) \\ \psi_{n-1} &= \frac{A_{n-1}}{\sqrt{n-1}}(-\sin\Phi_n\cos\alpha_n'+\cos\Phi_n\sin\alpha_n')\ . \end{aligned} \tag{29}$$

Equation (11) can be written as

$$\begin{aligned}&\cos\Phi_n\left[(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\sin\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\sin\alpha_n'-E\frac{A_n}{\sqrt{n}}\right] \\ &+\sin\Phi_n\left[(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\cos\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\cos\alpha_n'-E\frac{A_n}{\sqrt{n}}\right] \ = \ 0. \end{aligned} \tag{30}$$

Expressions in the square brackets are the series in $1/n$. Coefficients $\cos\Phi_n$ and $\sin\Phi_n$ are irregular. Therefore, (30) can be satisfied if and only if:

$$\begin{aligned}(n+1)\frac{A_{n+1}}{\sqrt{n+1}}\sin\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\sin\alpha_n'-E\frac{A_n}{\sqrt{n}} &= 0; \\ (n+1)\frac{A_{n+1}}{\sqrt{n+1}}\cos\alpha_n+n\frac{A_{n-1}}{\sqrt{n-1}}\cos\alpha_n'-E\frac{A_n}{\sqrt{n}} &= 0. \end{aligned} \tag{31}$$

Each LHS of (31) is well defined series in $x=1/n$. They must be zero, so that each coefficient in $x=1/n$ expansion must be zero. Thus (31) provides a set of algebraic equations for δ_j,ϵ_j.

Precise form of the asymptotic corrections is the following:

$$\begin{aligned}\delta(n,E) &= -\frac{1}{4\,n}+\frac{2E^2+1}{32\,n^2}-\frac{5(2E^2-1)}{128\,n^3}+\frac{20E^4-60E^2-21}{2048\,n^4} \\ &-\frac{180E^4-1380E^2+399}{8192\,n^5}+\frac{120E^6-2540E^4+2518E^2+869}{65536\,n^6}+\mathcal{O}(n^{-7}) \end{aligned} \tag{32}$$

and

$$\epsilon(n,E) = \frac{E}{4\,n} - \frac{E(E^2-5)}{96\,n^2} + \frac{E(E^2-9)}{96\,n^3} - \frac{E(9E^4-490E^2+341)}{15360\,n^4}$$
$$+\frac{E(3E^4-190E^2+375)}{2560\,n^5} - \frac{E(15E^6-2793E^4+22169E^2-7615)}{258048\,n^6} + \mathcal{O}(n^{-7})\ . \tag{33}$$

The correction terms δ_j and ϵ_j can by produced from the recursion by a bootstrap up to any order of $1/n$.

4. Orthogonality

There is a remarkable way to derive the inner product for two states in our model. Consider a truncated state,

$$|\psi_E^{(N)}\rangle = \sum_{n=0}^{N} |n\rangle \psi_n(E)\ , \tag{34}$$

where $\psi_n(E)$ are defined by (11) with the initial condition $\psi_0 = 1$. Straightforward computation gives

$$\boldsymbol{H}\,|\psi_E^{(N)}\rangle = |\psi_E^{(N-1)}\rangle\, E + |N\rangle\, N\psi_{N-1}(E) + |N+1\rangle\,(N+1)\psi_N(E)\ . \tag{35}$$

Considering then

$$\langle\psi_{E'}^{(N)}|\,\boldsymbol{H}\,|\psi_E^{(N)}\rangle\ , \tag{36}$$

one deduces

$$\langle\psi_{E'}^{(N-1)}|\psi_E^{(N-1)}\rangle = \frac{N}{E-E'}\,\big(\psi_N(E)\psi_{N-1}(E') - \psi_N(E')\psi_{N-1}(E)\big)\ . \tag{37}$$

Assuming our asymptotic for ψ_N for $N\to\infty$, one obtains

$$\langle\psi_{E'}^{(N)}|\psi_E^{(N)}\rangle = A(E')A(E)\frac{\sin\left(\dfrac{E'-E}{2}\log N + \varphi(E') - \varphi(E)\right)}{E'-E}\ ,\quad N\to\infty\ . \tag{38}$$

The limit $N\to\infty$ is well defined here. In general, this is the Fresnel integral limit [5],

$$\lim_{K\to\infty}\frac{\sin(Kx)}{x} = \pi\delta(x)\ . \tag{39}$$

Therefore, at $N \to \infty$ one obtains

$$\langle \psi_{E'} | \psi_E \rangle \;=\; \pi A(E)^2 \delta(E - E') \;. \tag{40}$$

In fact, this is the main result of our paper. Numerical analysis also shows that the spectrum is unbounded since

$$A(E) \;=\; A(-E) \;. \tag{41}$$

5. Conclusion and Discussion

In this paper we have considered the stationary Schrödinger equation for the self-conjugated Hamiltonian $\boldsymbol{H} \;=\; \frac{1}{\mathsf{i}}(\boldsymbol{e} + \boldsymbol{f})$, where $\boldsymbol{e}$ and $\boldsymbol{f}$ are creatinig and annihilation operators for the algebra sl_2 considered for the infinite-dimensional representation with lowest weight equals 1, equivalent to the usual Fock Space.

The eigenvector equation for operator $\boldsymbol{H}$ is the the second order recursion equation. In this paper we have given detailed analysis for a solution of the recursion. General expression of $\psi_n(E)$ involves four functions: $A(E)$, $\psi(E)$, $\delta_n(E)$, $\varepsilon_n(E)$, see equation (22). We give the rigorous way to define $\delta(n, E)$ and $\epsilon(n, E)$ analytically in the forms of series expansion with respect to $1/n$ and E, however the functions $A(E)$ and $\psi(E)$ are defined only numerically for real E.

The further development of the problem implies two ways: the first way is the further analysis of equation (11) in order to find analytical expressions for the asymptotic analytical functions $A(E)$ and $\varphi(E)$. The second way could be $q \neq 1$ generalisation of the problem. A preliminary analysis shows that $q \neq 1$ case leads to several unexpected mathematical phenomena.

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