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# THE POISSON DISTRIBUTION SERIES OF GENERAL SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract. The motivation of this paper is to initiate connections between varied subclasses of univalent functions involving the Poisson distribution series.

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## 1. Introduction

Let $\Delta$ be the unit disk

$$\{z \in C : |z| < 1\},$$

and let $A$ be the class of functions analytic in $\Delta$, satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Then each $f \in A$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_k z^k. \qquad (1)$$

As usual, by $S$ we represent the class of all functions in $A$ which are univalent in $\Delta$.

A function $f \in A$ is said to be starlike of order $\mu$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu \quad (0 \le \mu < 1,\ z \in \Delta),$$

is said to be convex of order $\mu$ if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu \quad (0 \le \mu < 1,\ z \in \Delta).$$

These classes represented by $S^*(\mu)$ and $K(\mu)$, respectively, were first introduced by Robertson [6]. We note that

$$K(\mu) \subset S^*(\mu) \subset A.$$

Let $T$ indicate the subclass of $S$ consisting of functions whose coefficients, from the second on, are non zero given by (see [7])

$$f(z) = z - \sum_{n=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{2}$$

We indicate by $T^*(\mu)$ and $C(\mu)$, respectively, the classes obtained by taking the intersections of $S^*(\mu)$ and $K(\mu)$ $(0 \leq \mu < 1)$ with $T$,

$$T^*(\mu) := S^*(\mu) \cap T \tag{3}$$

$$C(\mu) := K(\mu) \cap T.$$

**Definition 1.** *(See [3]) A function $f \in T$ is said to be in the class $U(\lambda, \alpha, \mu)$, if it satisfies the inequality:*

$$\Re\left(\frac{z\Psi'(z)}{\Psi(z)}\right) > \mu \tag{4}$$

$$(0 \leq \alpha \leq \lambda \leq 1,\ 0 \leq \mu < 1, \quad z \in \Delta),$$

*where*

$$\Psi(z) := \lambda\alpha z^2 f''(z) + (\lambda - \alpha) z f'(z) + (1 - \lambda + \alpha) f(z).$$

The function class $U(\lambda, \alpha, \mu)$ is of notable interest and it comprises many common classes of univalent functions (see [8]). Further we get [cf. equation (3)]

$$U(0, 0, \mu) = T^*(\mu), U(1, 0, \mu) = C(\mu).$$

By choosing $\mu = 0$, we assert the results established by [1], [7].

## 2. Preliminary Results

We employ the tecnique adopted by Porwal [5] to get the Poisson distribution series for univalent functions.

Just recently, in [5], Porwal establish a power series by making use of the Poisson distribution

$$\varphi(\xi, z) = z + \sum_{n=2}^{\infty} \frac{e^{-\xi}\xi^{k-1}}{(k-1)!} z^k \qquad (z \in \Delta).$$

In [5], Porwal also define the series

$$\Omega(\xi, z) = 2z - \varphi(\xi, z) = z - \sum_{n=2}^{\infty} \frac{e^{-\xi}\xi^{k-1}}{(k-1)!} z^k \qquad (z \in \Delta).$$

To demonstrate our first theorem, we express the following Lemma.

**Lemma 1.** *(See [3]) A function $f \in T$ given by (2) is in the class $U(\lambda, \alpha, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} (k-\mu)\left[(k-1)(k\lambda\alpha + \lambda - \alpha) + 1\right] a_k \leq 1 - \mu.$$

Making use of the techniques and methodology used by Porwal [5] (see also [1], [2], [4]), in this present paper, we supply necessary and sufficient conditions for the Poisson distribution series functions belonging to the class $U(\lambda, \alpha, \mu)$. In addition, we establish an integral operator for the series.

## 3. Necessary and sufficient conditions

Our main characterization theorem for the class $U(\lambda, \alpha, \mu)$ is stated as Theorem 2 below.

**Theorem 2.** *If $\xi > 0$, then $\Omega(\xi, z)$ is in $U(\lambda, \alpha, \mu)$, if and only if*

$$\begin{aligned} &\lambda\alpha\xi^3 + (5\lambda\alpha + \lambda - \alpha - \mu\lambda\alpha)\xi^2 + (4\lambda\alpha + 2\lambda - 2\alpha - 2\mu\lambda\alpha - \mu\lambda + \mu\alpha + 1)\xi \\ &+(\mu - 1)e^{-\xi} \leq 0. \end{aligned} \tag{5}$$

*Proof.* By using the fact that

$$\Omega(\xi, z) = z - \sum_{n=2}^{\infty} \frac{e^{-\xi}\xi^{k-1}}{(k-1)!} z^k \tag{6}$$

and applying Lemma 1, it is adequate to show that

$$\sum_{n=2}^{\infty} (k-\mu)\left[(k-1)(k\lambda\alpha + \lambda - \alpha) + 1\right] \frac{e^{-\xi}\xi^{k-1}}{(k-1)!} \leq 1 - \mu. \tag{7}$$

It follows from (7) that

$$\sum_{n=2}^{\infty}(k-\mu)\left[(k-1)(k\lambda\alpha+\lambda-\alpha)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$=\sum_{n=2}^{\infty}\left\{k^3\lambda\alpha+k^2(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)+k(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)\right.$$

$$\left.+\mu(\lambda-\alpha-1)\right\}\frac{\xi^{k-1}}{(k-1)!}.$$

By writing

$$k^3=(k-1)(k-2)(k-3)+6(k-1)(k-2)+7(k-1)+1,$$

$$k^2=(k-1)(k-2)+3(k-1)+1$$

and

$$k=(k-1)+1,$$

we obtain

$$\sum_{n=2}^{\infty}(k-\mu)\left[(k-1)(k\lambda\alpha+\lambda-\alpha)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$=\lambda\alpha\sum_{n=2}^{\infty}\left[(k-1)(k-2)(k-3)+6(k-1)(k-2)+7(k-1)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$+(\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)\sum_{n=2}^{\infty}\left[(k-1)(k-2)+3(k-1)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$+(\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)\sum_{n=2}^{\infty}\left[(k-1)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$+\mu(\lambda-\alpha-1)\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$=e^{-\xi}\left\{\left[e^{\xi}\xi^3+6e^{\xi}\xi^2+7e^{\xi}\xi+e^{\xi}-1\right]\lambda\alpha+\left[e^{\xi}\xi^2+3e^{\xi}\xi+e^{\xi}-1\right](\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)\right.$$

$$\left.+\left[e^{\xi}\xi+e^{\xi}-1\right](\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)+(e^{\xi}-1)\mu(\lambda-\alpha-1)\right\}$$

$$=\left[\xi^3+6\xi^2+7\xi+1-e^{-\xi}\right]\lambda\alpha+\left[\xi^2+3\xi+1-e^{-\xi}\right](\lambda-\alpha-\lambda\alpha-\mu\lambda\alpha)$$

$$+\left[\xi+1-e^{-\xi}\right](\mu\lambda\alpha-\mu\lambda+\mu\alpha-\lambda+\alpha+1)+(1-e^{-\xi})\mu(\lambda-\alpha-1).$$

But, this last expression is less than or equal to $1-\mu$ if and only if (5) is satisfied. Hence the proof is completed.

By taking $\lambda = \alpha = 0$ in Theorem 2, we state the following Corollary.

**Corollary 3.** *If* $\xi > 0$, *then* $\Omega(\xi, z)$ *is in* $T^*(\mu)$, *if and only if*

$$\xi - (1-\mu)e^{-\xi} \leq 0.$$

By taking $\lambda = 1$ and $\alpha = 0$ in Theorem 2, we state the following Corollary.

**Corollary 4.** *If* $\xi > 0$, *then* $\Omega(\xi, z)$ *is in* $C(\mu)$, *if and only if*

$$\xi^2 + (3-\mu)\xi - (1-\mu)e^{-\xi} \leq 0.$$

## 4. Inclusion Properties

We next explore a particular integral operator $\Lambda(\xi, z)$ as follows:

$$\Lambda(\xi, z) = \int_0^z \frac{\Omega(\xi, t)}{t} dt. \tag{8}$$

**Theorem 5.** *If* $\xi > 0$, *then* $\Lambda(\xi, z)$ *defined by (8) is in* $U(\lambda, \alpha, \mu)$, *if and only if*

$$\begin{aligned} &\lambda\alpha\xi^2 + (2\lambda\alpha + \lambda - \alpha - \mu\lambda\alpha)\xi + [1 - \mu(\lambda - \alpha)](1 - e^{-\xi}) \\ &+ \tfrac{\mu(\lambda-\alpha-1)}{\xi}(1 - e^{-\xi} - \xi e^{-\xi}) \leq 1 - \mu. \end{aligned} \tag{9}$$

*Proof.* From (8), we find

$$\Lambda(\xi, z) = z - \sum_{n=2}^{\infty} \frac{e^{-\xi}\xi^{k-1}}{k!} z^k.$$

By using Lemma 1, it is adequate to show that

$$\sum_{n=2}^{\infty} (k-\mu)[(k-1)(k\lambda\alpha + \lambda - \alpha) + 1] \frac{e^{-\xi}\xi^{k-1}}{k!} \leq 1 - \mu. \tag{10}$$

By virtue of the equation (10), we establish

$$\sum_{n=2}^{\infty}(k-\mu)\left[(k-1)(k\lambda\alpha+\lambda-\alpha)+1\right]\frac{e^{-\xi}\xi^{k-1}}{k!}$$

$$=\sum_{n=2}^{\infty}\left[(k-1)(k\lambda\alpha+\lambda-\alpha)+1\right]\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$-\mu\sum_{n=2}^{\infty}\left[(k-1)(k\lambda\alpha+\lambda-\alpha)+1\right]\frac{e^{-\xi}\xi^{k-1}}{k!}$$

$$=\sum_{n=2}^{\infty}(k\lambda\alpha+\lambda-\alpha)\frac{e^{-\xi}\xi^{k-1}}{(k-2)!}+\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$-\mu\sum_{n=2}^{\infty}(k-1)(k\lambda\alpha+\lambda-\alpha)\frac{e^{-\xi}\xi^{k-1}}{k!}-\mu\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{k!}$$

$$=\lambda\alpha\sum_{n=2}^{\infty}\left[(k-2)+2\right]\frac{e^{-\xi}\xi^{k-1}}{(k-2)!}+(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-2)!}+\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}$$

$$-\mu\lambda\alpha\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-2)!}-\mu(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{(k-1)!}+\mu(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{k!}-\mu\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k-1}}{k!}$$

$$=\lambda\alpha\xi^2\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}+2\lambda\alpha\xi\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}+(\lambda-\alpha)\xi\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}+\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}$$

$$-\mu\lambda\alpha\xi\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}-\mu(\lambda-\alpha)\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}+\frac{\mu(\lambda-\alpha)}{\xi}\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}-\frac{\mu}{\xi}\sum_{n=2}^{\infty}\frac{e^{-\xi}\xi^{k}}{k!}.$$

Hence

$$\begin{aligned}
&= e^{-\xi}\Big[\lambda\alpha\xi^2e^{\xi}+(2\lambda\alpha+\lambda-\alpha-\mu\lambda\alpha)\xi e^{\xi}+\left[1-\mu(\lambda-\alpha)\right](e^{\xi}-1)\\
&\quad+\frac{\mu(\lambda-\alpha-1)}{\xi}(e^{\xi}-1-\xi)\Big]\\
&= \lambda\alpha\xi^2+(2\lambda\alpha+\lambda-\alpha-\mu\lambda\alpha)\xi+\left[1-\mu(\lambda-\alpha)\right](1-e^{-\xi})\\
&\quad+\frac{\mu(\lambda-\alpha+1)}{\xi}(1-e^{-\xi}-\xi e^{-\xi}).
\end{aligned}$$

But, this last expression is not greater than $1-\mu$ if and only if (9) is satisfied.

By taking $\lambda=\alpha=0$ in Theorem 5, we state the following Corollary.

**Corollary 6.** *If $\xi>0$, then $\Lambda(\xi,z)$ is in $T^*(\mu)$, if and only if*

$$1-e^{-\xi}-\frac{\mu}{\xi}(1-e^{-\xi}-\xi e^{-\xi})\le 1-\mu.$$

By taking $\lambda = 1$ and $\alpha = 0$ in Theorem 5, we state the following Corollary.

**Corollary 7.** *If* $\xi > 0$, *then* $\Lambda(\xi, z)$ *is in* $C(\mu)$, *if and only if*

$$\xi - (1-\mu)e^{-\xi} \leq 0.$$

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