Acta Universitatis Apulensis No. 61/2020
ISSN: 1582-5329 pp. 9-28
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.61.02

GENERALIZED HERMITE-BASED APOSTOL-TYPE FROBENIOUS-GENOCCHI POLYNOMIALS AND ITS APPLICATIONS

Waseem A. Khan and Divesh Srivastava

Abstract. In this article, we introduce a multi-variable hybrid class, namely the Hermite–Apostol-type Frobenius–Genocchi polynomials, and to characterize their properties via different generating function techniques. Several explicit relations involving Hurwitz–Lerch Zeta functions and some summation formulae related to these polynomials are derived. Further, we establish certain symmetry identities involving generalized power sums.

2010 *Mathematics Subject Classification*: 11B68, 05A10, 05A15, 33C45, 16B65.

Keywords: Frobenius-Genocchi polynomials, Apostol-type Genocchi polynomials, 3-variable Hermite polynomials.

1. Introduction

Several investigations have done to introduce and study classical and generalized forms of Apostol type polynomials systematically via various analytic means and generating functions method (see [1]-[3], [7], [8], [11], [12], [14], [15]). Very recently, Araci et. al. [4] introduced and studied a generalized class of 3-variable Hermite-Apostol type Frobenius-Euler polynomials systematically by use of generating method. The following class of polynomials is introduced by convoluting the 3-variable Hermite polynomials $H_n(x, y, z)$ [6] with the Apostol type Frobenius-Euler polynomials $\mathcal{F}_n^{\alpha}(x; u; \lambda)$ (see [9], [17]). The convoluted special polynomials are important as they possess important properties such as recurrence and explicit relations, summation formulae, symmetric and convolution identities, algebraic properties etc. These polynomials are useful and possess potential for applications in certain problems of number theory combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics.

The 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [6], [13]) given by

$$H_n(x,y) = n!\sum_{n=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!\,(n-2r)!}. \tag{1}$$

These polynomials are usually defined by the following generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

The generating function for the three-variable Hermite polynomials (3VHP) $H_n(x,y,z)$ is given by (see [4], [5]):

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!},$$

which for $z = 0$ reduce to the two-variable Hermite–Kampé de Fériet polynomials (2VHKdFP)$H_n(x,y)$ and for $z = 0, x = 2x$ and $y = -1$ become the classical Hermite polynomials $H_n(x)$ [6].

In 2013, Kurt and Simsek (see [9], [10]) introduced the generalized Apostol-type Frobenious-Euler polynomials defined as follows:

$$\left(\frac{a^t-u}{\lambda b^t-u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}. \tag{2}$$

For $u \in \mathbb{C}, u \neq 1$, the generating equation for the Apostol-type Frobenius–Genocchi polynomials (ATFGP) $\mathcal{G}^(a)_n(x;u;\lambda)$ of order α given as:

$$\left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;u:\lambda)\frac{t^n}{n!},$$

which for $x = 0$ gives the Apostol-type Frobenius–Genocchi numbers (ATFGN) $\mathcal{G}_n^{\alpha}(u;\lambda)$, of order α such that:

$$\left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha} = \sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(u:\lambda)\frac{t^n}{n!}.$$

For $u = -1$, the ATFGP reduce to the Apostol–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ [7], which for $\lambda = 1$, become the Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x)$ [7]. Furthermore, the ATFGP for $\lambda = 1$ becomes the Frobenius–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;u)$ (see [11], [12]). The generating equations for the special polynomials are important from different view points and help in finding connection formulas, recursive relations and difference equations and in solving enumeration problems in combinatorics and encoding their solutions.

Recently Araci et al. [5] introduced a new hybrid class, namely the class of three-variable Hermite–based Frobenius–Genocchi polynomials (3VHATFGP) given as

$$\left(\frac{1-u}{\lambda e^t - u}\right)^{\alpha} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_H\mathcal{E}_n^{(\alpha)}(x,y,z;u;\lambda)\frac{t^n}{n!}.$$

Definition 1. *For $u, \lambda \in \mathbb{C}, u \neq 1, x, y, z \in \mathbb{R}$, the three-variable Hermite–based Apostol-type Frobenius–Genocchi polynomials (3VGAFGP) of order α are defined by means of the following generating function:*

$$\left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{F}(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!}, \tag{3}$$

where ${}_HG_n^F(x,y,z;u;\alpha;\lambda)$ denotes the three variable generalized Hermite-based Apostol-type Frobenius-Genocchi polynomials of order α.

For $\lambda = 1$, (3) becomes the three-variable Hermite–Frobenius–Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x,y,z;u)$ of order α, which again for $\alpha = 1$, give the three-variable Hermite-Frobenius–Genocchi polynomials ${}_H\mathcal{G}_n(x,y,z;u)$.

Again, the 3VHATFGP for $u = -1$ give the three-variable Hermite–Apostol–Euler polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x,y,z;\lambda)$ of order α, which for $\lambda = 1$ reduce to the three-variable Hermite–Euler polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x,y,z)$.

The three-variable Hermite–based Apostol-type Frobenius–Genocchi polynomials (3VHATFGP) are also defined as the discrete Apostol-type Frobenious-Genocchi convolution of the 3VHP given as:

$${}_H\mathcal{G}_n^{(\alpha)}(x,y,z;u;\lambda) = n!\sum_{k=0}^{n}\sum_{r=0}^{[k/3]} \frac{\mathcal{G}_{n-k}^{(\alpha)}(u;\lambda)z^r H_{k-3r}(x,y)}{(n-k)!r!(k-3r)!},$$

where $H_n(x,y)$ are the two-variable Hermite–Kampé-de-Férite polynomials.

Simsek [18] constructed the λ-stirling type number of second kind $S(n,\nu;a,b;\lambda)$ by means of the following generating function:

$$\sum_{n=0}^{\infty}\mathcal{S}(n,\nu;a,b;\lambda)\frac{t^n}{n!}=\frac{(\lambda b^t-a^t)^\nu}{\nu!}$$

and the generalized array type polynomials is defined by Simsek (see [17], [18]) as:

$$\sum_{n=0}^{\infty}\mathcal{S}_\nu^n(x;a,b;\lambda)\frac{t^n}{n!}=\frac{(\lambda b^t-a^t)^\nu}{\nu!}b^{xt}. \tag{4}$$

We give the generating function of the polynomial $Y_n(x;\lambda;a)$ as:

$$\frac{t}{\lambda a^t-1}a^{xt}=\sum_{n=0}^{\infty}Y_n(x;\lambda;a)\frac{t^n}{n!}. \quad (a\geq 1).$$

We also note that for $x=0$, above equation gives a relation as

$$Y_n(0;\lambda;a)=Y_n(\lambda;a). \tag{5}$$

Again if we set $x=0$ & $\alpha=1$, in (5), we get

$$Y_n(\lambda;1)=\frac{1}{\lambda-1}.$$

The main goal of this paper is as follows. In section two, we establish some explicit properties of three-variable Hermite–based Apostol-type Frobenius–Genocchi polynomials (3VHATFGP). In section three, we derive some implicit formulae for three-variable Hermite–based Apostol-type Frobenius–Genocchi polynomials (3VHATFGP) and in last section, we introduce some symmetric identities for three-variable Hermite–based Apostol-type Frobenius–Genocchi polynomials (3VHATFGP) by applying generating functions.

2. Generalized Hermite-based Apostol-type Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^F(x,y,z;u;\lambda)$

Theorem 1. *The following relation holds true for 3-VHATFGP of order α:*

$$\begin{aligned}(n+1){}_H\mathcal{G}_{n+1}^F(x,y,z;u;\lambda)&=x(n+1){}_H\mathcal{G}_n^F(x,y,z;u;\lambda)+2yn(n+1){}_H\mathcal{G}_{n-1}^F(x,y,z;u;\lambda)\\&+3zn(n-1){}_H\mathcal{G}_{n-2}^F(x,y,z;u;\lambda)+{}_H\mathcal{G}_{n+1}^F(x,y,z;u;\lambda)\\&-\sum_{m=0}^{n}\binom{n}{m}\left(\frac{\lambda}{u}\right){}_H\mathcal{G}_{n+1-m}^F(x,y,z;u;\lambda)Y(1;\lambda/u;e).\end{aligned} \tag{6}$$

Proof. Making $\alpha = 1$ and differentiating (4) with respect to t, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_{n+1}^F(x,y,z;u;\lambda)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u}e^{xt+yt^2+zt^3}(x+2yt+3zt^2)$$
$$+e^{xt+yt^2+zt^3}\left[\frac{(\lambda e^t-u)(1-u)-(1-u)t(\lambda e^t)}{(\lambda e^t-u)^2}\right]$$
$$= x\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\lambda)\frac{t^n}{n!} + 2y\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\lambda)\frac{t^{n+1}}{n!}$$
$$+3z\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\lambda)\frac{t^{n+2}}{n!} + \frac{1}{t}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\lambda)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{\lambda}{u}Y_m(1;\lambda/u;e)\frac{t^{m-1}}{m!}. \tag{7}$$

Which, upon comparing the coefficients of t^n on both sides of (7), gives the recurrence relation (6).

Corollary 2. *The following relation holds true for 3-VHATFP of order* α*:*

$$(n+1){}_HG_{n+1}^F(x,y,z;\lambda) = x(n+1){}_HG_n^F(x,y,z;\lambda) + 2yn(n+1){}_HG_{n-1}^F(x,y,z;\lambda)$$
$$+3zn(n-1){}_HG_{n-2}^F(x,y,z;\lambda) + {}_HG_{n+1}^F(x,y,z;\lambda)$$
$$-\sum_{m=0}^{n}\binom{n}{m}(-\lambda){}_HG_{n+1-m}^F(x,y,z;\lambda)Y(1;-\lambda;e).$$

Theorem 3. *The following result holds true for 3-VHATFGP of order* α*:*

$$(1-u)^\gamma {}_H\mathcal{G}_n^F(x,y,z;u;(\alpha-\gamma);\lambda) = \frac{n!}{(n+\gamma)!}\sum_{k=0}^{n+\gamma}\sum_{p=0}^{\gamma}\binom{\gamma}{p}\binom{n+\gamma}{k}$$
$$\times {}_H\mathcal{G}_{n+\gamma-k}^F(x,y,z;u;\alpha;\lambda)\lambda^p p^k(-u)^{\gamma-p}. \tag{8}$$

Proof. We start with

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;(\alpha-\gamma);\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t-u}\right)^{(\alpha)} e^{xt+yt^2+zt^3}(\lambda e^t-u)^\gamma((1-u)t)^{-\gamma}.$$

$$=\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}\sum_{p=0}^{\gamma}\binom{\gamma}{p}\lambda^p p^k(-u)^{\gamma-p}(1-u)^{-\gamma}t^{-\gamma}\frac{t^k}{k!}. \tag{9}$$

Now, applying Cauchy product in above equation (9) and some simplification leads us to our required result (8).

Theorem 4. *The following result holds true for 3-VHATFGP of order α:*

$$(2u-1)\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) {}_H\mathcal{G}_r^F(x,y,z;u;\lambda)\ {}_H\mathcal{G}_{n-r}^F(a,y,z;1-u;\lambda)$$

$$=n(u-1){}_H\mathcal{G}_{n-1}^F(x+a,y,z;u;\lambda)\ +nu\ {}_H\mathcal{G}_{n-1}^F(x+a,y,z;1-u;\lambda)$$
$$+{}_H\mathcal{G}_n^F(x+a,y,z;u;\lambda)$$
$$-\ {}_H\mathcal{G}_n^F(x+a,y,z;1-u;\lambda). \tag{10}$$

Proof. In order to proof (10), we set

$$(2u-1)\left(\frac{(1-u)t}{\lambda e^t-u}\right)c^{xt+yt^2+zt^3}\left(\frac{(1-(1-u))t}{\lambda e^t-(1-u)}\right)e^{at+yt^2+zt^3}$$

$$=t^2(1-u)(1-(1-u))e^{(x+a)t+2(yt^2+zt^3)}\left[\frac{1}{\lambda e^t-u}-\frac{1}{\lambda e^t-(1-u)}\right]. \tag{11}$$

Employing the result of (4), equation (11) reduces as

$$(2u-1)\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x,y;u;a,b,c;\lambda)\frac{t^r}{r!}\ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(a,y,z;1-u;\lambda)\frac{t^n}{n!}$$

$$=(1-(1-u)t)\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;u;\lambda)\frac{t^r}{r!}-(1-u)t$$

$$\times\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;1-u;\lambda)\frac{t^r}{r!}.$$

$$(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) {}_H\mathcal{G}_r^F(x,y,z;u;\lambda)\ {}_H\mathcal{G}_{n-r}^F(a,y,z;1-u;\lambda)\frac{t^n}{n!}$$

$$=(1-(1-u)t)\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;u;\lambda)\frac{t^r}{r!}-(1-u)t$$

$$\times \sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;1-u;\lambda)\frac{t^r}{r!}$$

$$= (u-1)\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;u;\lambda)\frac{t^{r+1}}{r!} + u\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;1-u;\lambda)\frac{t^{r+1}}{r!}$$

$$+\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;u;\lambda)\frac{t^n}{n!}$$

$$-\sum_{r=0}^{\infty} {}_H\mathcal{G}_r^F(x+a,y,z;1-u;\lambda)\frac{t^n}{n!}.$$

On comparing the coefficient of t^n form the above equation, we arrive at our desired result.

Theorem 5. *The following result holds true for 3-VHATFGP of order α:*

$$\sum_{m=0}^{n} {}_H\mathcal{G}_n^F(y,x,z;u;\alpha;\lambda)\frac{t^n}{n!} = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)!k!}\mathcal{G}_{n-2k}^F(y,z;u;\alpha;\lambda)x^k. \tag{12}$$

Proof. Interchanging x and y in (4), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(y,x,z;u;\alpha;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{yt+xt^2+zt^3}$$

$$= \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(y,z;u;\alpha;\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(xt)^{2m}}{m!}$$

Replacing n by $n-2m$ in the above equation, we leads to required result (12).

Theorem 6. *The following result holds true for 3-VHATFGP of order α:*

$$\sum_{k=0}^{n} {}_H\mathcal{G}_k^F(-x,-y,-z;u;-\alpha;\lambda)\ {}_H\mathcal{G}_{(n-k)}^F(x,y,z;u;(\alpha-m);\lambda) = \mathcal{G}_n^F(u;\alpha-m;\lambda). \tag{13}$$

Proof. In order to proof (13), replacing x with $-x$, y with $-y$, z with $-z$ and α with $-\alpha$ in (6), we get get

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(-x,-y,-z;u;-\alpha;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{(-\alpha)} e^{-(xt+yt^2+zt^3)}. \tag{14}$$

Making use of the above equation in the left-hand side of (14), we can write

$$\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^F(-x,-y,-z;u;-\alpha;\lambda)\frac{t^k}{k!} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha-m;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{-m}.$$

We can write the above equation as

$$\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^F(-x,-y,-z;u;-\alpha;\lambda)\frac{t^k}{k!} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;-\alpha-m;\lambda)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \mathcal{G}_n^F(u;-m;\lambda)\frac{t^n}{n!}.$$

Using Cauchy product in the above equation and on comparing the coefficients of t^n in the obtained equation, we immediately come to our desired result (13).

In a very similar way, for $\alpha,\beta \in \mathbb{Z}$, we can form some more identities which are given below.

$${}_H\mathcal{G}_n^F(2x,2y,2z;u^2;-\alpha;\lambda^2) = \sum_{k=0}^{n} \binom{n}{k} {}_H\mathcal{G}_k^F(x,y,z;u;-\alpha;\lambda)$$

$$\times {}_H\mathcal{H}_{n-k}^F(x,y,z;-u;-\alpha;\lambda).$$

Proof of this identity can be solved by making use of (4) with some required calculations.

Theorem 7. *The following result holds true for 3-VHATFGP of order α:*

$${}_H\mathcal{G}_n^F(x,y,z;u;-\alpha;\lambda) = \left(\frac{u-1}{u}\right)^{\alpha} \sum_{l=0}^{n-\alpha} \binom{n}{l-\alpha} \Phi_\alpha\left(\frac{\lambda}{u}, l-n-\alpha, x\right) H_l(0,y,z). \tag{15}$$

Proof. From Equation (4), we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!} = ((1-u)t)^{\alpha}\left(\lambda e^t - u\right)^{(-\alpha)} e^{xt+yt^2+zt^3},$$

which on simplification, becomes

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!} = (1-u)^{\alpha}(-u)^{-\alpha}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}$$
$$\times\left(\frac{\lambda}{u}\right)^{\alpha}\frac{(k+x)^n t^{n+\alpha}}{n!}e^{yt^2+zt^3}.$$

Using the definition of Hurwitz-Lerch Zeta function $\Phi_{\mu}(z,s,a) = \sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{n!(n+a)^s}$ and (1), we have result (15).

Theorem 8. *The following result holds true for 3-VHATFGP of order α:*

$$\alpha!\sum_{m=0}^{n}\binom{n}{m}{}_H\mathcal{G}_{n-m}^F(x,y,z;u;\alpha;\lambda)S\left(m,\alpha,\frac{\lambda}{u}\right) = \frac{n!}{(n-\alpha)!}\left(\frac{1-u}{u}\right)^{\alpha}H_{n-\alpha}(x,y,z). \tag{16}$$

$${}_H\mathcal{G}_n^F(x,y,z;u;\alpha-\gamma;\lambda) = \gamma!\frac{n!}{(n+\gamma)!m!}\left(\frac{1-u}{u}\right)^{\gamma}\sum_{m=0}^{n+\gamma}\binom{n+\gamma}{m}$$
$$\times\,{}_H\mathcal{G}_{n+\gamma-m}^F(x,y,z;u;\alpha;\lambda)S\left(m,\gamma,\frac{\lambda}{u}\right). \tag{17}$$

Proof. We start with generating function (4) to get (16)

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{xt+yt^2+zt^3}.$$

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{(\frac{\lambda}{u}e^t-1)^{\alpha}}{\alpha!}\alpha!\frac{t^n}{n!} = \left(\frac{(1-u)t}{u}\right)^{\alpha} e^{xt+yt^2+zt^3}.$$

By using result (2), Cauchy product application and comparing the coefficients of equal powers of t^n in the obtained equation, yields (16).

Again we consider (4) as

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha-\gamma;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{\alpha} e^{xt+yt^2+zt^3}$$

$$\times\left(\frac{u}{1-u}\right)^{\gamma}\gamma!\frac{\left(\frac{\lambda}{u}e^t-1\right)^{\gamma}}{\gamma!}t^{-\gamma}.$$

Making use of result (2), Cauchy product application and comparison of equal powers of t^n in the obtained equation, yields (17).

Theorem 9. *The following inequality holds true:*

$$ {}_H\mathcal{G}_{n-2\nu}^F(x,y,z;u;-\nu;\lambda) = \frac{(\nu)!}{(-n)_{2\nu}}\sum_{k=0}^{n}\sum_{m=0}^{l}\left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} n \\ m \end{array}\right) S(k,v,1,b;\frac{\lambda}{u})$$

$$\times Y_{m-k}^{(\nu)}\left(\frac{1}{u};a\right)H_{l-m}(x,y). \tag{18}$$

Proof. In order to proof (18), we replace α with $-\nu$ in equation (2), we get

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;-\nu;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^{(-\nu)} e^{xt+yt^2+zt^3}.$$

On arranging the above equation, we arrive at

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;-\nu;\lambda)\frac{t^n}{n!} = (\nu!)\frac{\left(\frac{\lambda}{u}e^t-1\right)^{\nu} e^{xt+yt^2+zt^3}}{(\nu!)\left(\frac{1}{u}-1\right)^{\nu}t^{\nu}}\frac{t^{\nu}}{t^{\nu}}.$$

By assistance of generating function for λ-Stirling number, above equation reduces to

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;-\nu;\lambda)\frac{t^{n+2\nu}}{n!} = (\nu!)\sum_{k=0}^{\infty}S(k,v;\frac{\lambda}{u})\frac{t^k}{k!}$$

$$\times\sum_{m=0}^{\infty}Y_m^{(\nu)}\left(1;\frac{1}{u}\right)\frac{t^m}{m!}\sum_{n=0}^{\infty}H_l(x,y,z)\frac{t^l}{l!}.$$

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y;u;a,b,b;-\nu;\lambda)\frac{t^{n+2\nu}}{n!} = \nu!\sum_{l=0}^{\infty}\sum_{k=0}^{m}\sum_{m=0}^{l}\left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} l \\ m \end{array}\right)S(k,v,1,b;\frac{\lambda}{u})$$

$$\times Y_{m-k}^{(\nu)}\left(\frac{1}{u};a\right)H_{l-m}(x,y)\frac{t^l}{l!}.$$

Using Cauchy product in the above equation, we get our result.

3. Summation Formulae

In order to prove the summation formulae for the 3VHATFGP ${}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)$, we have the following theorems:

Theorem 10. *The following result holds true for 3-VHATFGP of order α:*

$${}_H\mathcal{G}_{k+l}^F(s,y,z;u;\alpha;\lambda)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(s-x)^{m+n}{}_H\mathcal{G}_{k-n+l-m}^F(x,y,z;u;\alpha;\lambda). \tag{19}$$

Proof. Replacing t with $(t+w)$ in (4) and then using required calculations, we get

$$\left(\frac{(1-u)(t+w)}{\lambda e^{t+w}-u}\right)^{\alpha}e^{y(t+w)^2+z(t+w)^3}=e^{-x(t+w)}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(x,y,z;u;\alpha;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}. \tag{20}$$

Replacing x by s and then equating the obtained equation from the above equation (20), we get

$$e^{(s-x)(t+w)}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(x,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}=\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(s,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}.$$

Expanding the exponent part of left-hand side, the above equation converts as

$$\sum_{N=0}^{\infty}\frac{[(s-x)(t+w)]^N}{N!}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(x,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}$$
$$=\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(s,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}.$$

$$\sum_{n,m=0}^{\infty}\frac{(s-x)^{m+n}(t+w)^{m+n}}{m!n!}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}_{k+l}^F(x,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}$$

$$= \sum_{k,l=0}^{\infty} {}_H\mathcal{G}_{k+l}^{F}(s,y,z;u;\alpha;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}. \qquad (21)$$

On comparing the coefficients of equal powers of t and w after taking the reference of Cauchy product to the above equation (21), we attain our required result (19).

Corollary 11. *For $l=0$ in (19), we have:*

$${}_H\mathcal{G}_{k}^{F}(s,y,z;u;\alpha;\lambda) = \sum_{n=0}^{k}\binom{k}{n}(s-x)^n {}_H\mathcal{G}_{k-n}^{F}(x,y,z;u;\alpha;\lambda).$$

Corollary 12. *For s with $s+x$ in (19) we have:*

$${}_H\mathcal{G}_{k+l}^{F}(s+x,y,z;u;\alpha;\lambda) = \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(s)^{m+n} {}_H\mathcal{G}_{k-n+l-m}^{F}(x,y,z;u;\alpha;\lambda).$$

Corollary 13. *For s with $s+x$ and $y=z=0$ in (19), we have:*

$$\mathcal{G}_{k+l}^{F}(s+x;u;\alpha;\lambda) = \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(s)^{m+n}\mathcal{G}_{k-n+l-m}^{F}(x;u;\alpha;\lambda).$$

Corollary 14. *For $s=0$ in (19), we have:*

$${}_H\mathcal{G}_{k+l}^{F}(y,z;u;\alpha;\lambda) = \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(-x)^{m+n} {}_H\mathcal{G}_{k-n+l-m}^{F}(x,y,z;u;\alpha;\lambda).$$

Theorem 15. *The following result holds true for 3-VHATFGP of order α:*

$${}_H\mathcal{G}_{n}^{F}(x+w,y,z;u;\alpha;\lambda) = \sum_{m=0}^{n}\binom{n}{m} {}_H\mathcal{G}_{n-m}^{F}(x,y,z;u;\alpha;\lambda)w^m. \qquad (22)$$

$${}_H\mathcal{G}_{n}^{F}(x+w,y,z;u;\alpha;\lambda) = \sum_{n=0}^{m}\binom{m}{n} {}_H\mathcal{G}_{n}^{F}(x,y,z;u;\alpha;\lambda)w^{m-n}. \qquad (23)$$

Proof. Substituting x with $x+w$ in (4), then making use of Equation (4) and with the series expansion of e^{wt} in the resultant equation, we have:

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x+w,y,z;u;\alpha;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)w^m\frac{t^{n+m}}{n!m!}.$$

which, upon simplification, gives Assertion (22).
Similarly we can obtain (23).

Theorem 16. *The following result holds true for 3-VHATFGP of order α:*

$$ {}_H\mathcal{G}_n^F(x,y,z;u;\alpha\pm\beta;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(u;\lambda)\,{}_H\mathcal{G}_{n-k}^F(x,y,z;u;\pm\beta;\lambda). \tag{24}$$

Proof. On replacing α with $(\alpha\pm\beta)$ in (4) and some simplifications leads us to result (24).

Corollary 17. *For $u=-1$, the above result reduces to 3-VHATGP of order α as:*

$$ {}_HG_n^F(x,y,z;\alpha\pm\beta;\lambda) = \sum_{k=0}^{n}\binom{n}{k}G_k^{(\alpha)}(\lambda)\,{}_HG_{n-k}^F(x,y,z;\pm\beta;\lambda),$$

where ${}_HG_n^F(x,y,z;\alpha;\lambda)$ is known as 3-VHATGP of order α.

Theorem 18. *The following result holds true for 3-VHATFGP of order α:*

$$ {}_H\mathcal{G}_n^F(x+1,y,z;u;\alpha;\lambda) = \sum_{m=0}^{n}\binom{n}{m}\,{}_H\mathcal{G}_{n-m}^F(x,y,z;u;\alpha;\lambda).$$

Proof. From equation (4), we replace x with $x+1$

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha} e^{xt+yt^2+zt^3}.$$

$$= \sum_{n=0}^{\infty}\mathcal{G}_n^F(x,y,z;u;\alpha;\lambda)\frac{t^n}{n!}\ \sum_{m=0}^{\infty}\frac{t^m}{m!}.$$

On replacing n by $n-m$ and on comparing the coefficient of equal powers of t, we arrive at the required result.

Theorem 19. *The following result holds true for 3-VHATFGP of order α:*

$$ {}_H\mathcal{G}_n^F(x,y,z;u;\alpha+1;\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_H\mathcal{G}_m^F(x,y,z;u;\alpha;\lambda)\ \mathcal{G}_{(n-m)}(u;\lambda). $$

Proof. Replacing α to $(\alpha+1)$ in equation (4), we have

$$ \sum_{n=0}^{\infty}{}_H\mathcal{G}_n^F(x,y,z;u;\alpha+1;\lambda)\frac{t^n}{n!}=\left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha+1}e^{xt+yt^2+zt^3} $$

$$ =\left(\frac{(1-u)t}{\lambda e^t-u}\right)\ \left(\frac{(1-u)t}{\lambda e^t-u}\right)^{\alpha}e^{xt+yt^2+zt^3} $$

$$ =\sum_{n=0}^{\infty}\mathcal{G}_n^F(u;\lambda)\frac{t^n}{n!}\ \sum_{m=0}^{\infty}{}_H\mathcal{G}_m^F(x,y,z;u;\alpha;\lambda)\frac{t^m}{m!}. $$

On using Cauchy product and on comparing coefficient of t^n from the resulting equation, we lead to our required result.

4. Symmetric identities for Apostol-type Hermite-based-Frobenius-Genocchi polynomials

The identities for the generalized special functions are useful in electromagnetic processes, combinatorics, numerical analysis, etc. Several types of identities and relations related to Apostol-type polynomials and related polynomials are considered in (see [4],[5],[7],[11]-[17]). This provides the motivation to explore symmetry identities for the 3VHATFGP. We recall the following:

For any $\gamma\in\mathbb{R}$ or $\mathbb{C}$, the generalized sum of integer powers $S_k(p;\gamma)$ is given by:

$$ \frac{\gamma^{p+1}e^{(p+1)t}-1}{\gamma e^t-1}=\sum_{k=0}^{\infty}S_k(p;\gamma)\frac{t^k}{k!}. $$

For any $\gamma\in\mathbb{R}$ or $\mathbb{C}$, the multiple power sums $S_k^l(m;\gamma)$ is given by:

$$ \left(\frac{1-\gamma^m e^{mt}}{1-\gamma e^t}\right)^{\gamma}=\frac{1}{\gamma^l\sum_{n=0}^{\infty}}\left[\sum_{p=0}^{n}\binom{n}{p}(-l)^{n-p}S_k^l()m;\gamma\right]\frac{t^m}{m!}. $$

To prove the symmetry identities for the 3VHATFGP, we have the following theorems:

Theorem 20. *For all integers $c,d>0$ and $n\geq 0, a\geq 1, \lambda,\mu\in\mathbb{C}$, the following symmetry relation holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{k}u^{c-1}{}_{H}\mathcal{G}_{n-k}^{F}(dx,d^2y,d^3z;u;\alpha;\lambda)\sum_{l=0}^{k}\binom{k}{l}S_l(c-1;\frac{\lambda}{u})$$
$$\times{}_{H}\mathcal{G}_{k-1}^{F}(cX,C^2Y,c^3Z;u;\alpha-1;\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{k}u^{d-1}{}_{H}\mathcal{G}_{n-k}^{F}(cx,c^2y,c^3z;u;\alpha;\lambda)\sum_{l=0}^{k}\binom{k}{l}S_l(d-1;\frac{\lambda}{u})$$
$$\times\,{}_{H}\mathcal{G}_{k-1}^{F}(dX,d^2Y,d^3Z;u;\alpha-1;\lambda). \quad (25)$$

Proof. Let

$$G(t)=\frac{((1-u)t)^{2\alpha-1}e^{cdxt+y(cdt)^2+z(cdt)^3}(\lambda^c e^{cdt}-u^c)e^{cdXt+Y(cdt)^2+Z(cdt)^3}}{(\lambda e^{ct}-u)^{\alpha}(\lambda e^{dt}-u)^{\alpha}},$$

which on rearranging the powers, we have

$$G(t)=\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{F}(dx,d^2y,d^3z;u;\alpha;\lambda)\frac{(ct)^n}{n!}u^{c-1}\sum_{l=0}^{\infty}S_l(c-1;\frac{\lambda}{u})\frac{(dt)^l}{l!}$$
$$\times\sum_{k=0}^{\infty}{}_{H}\mathcal{G}_{n}^{F}(cX,C^2Y,c^3Z;u;\alpha-1;\lambda)\frac{(dt)^k}{k!}.$$

$$G(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}c^{n-k}d^{k}u^{c-1}{}_{H}\mathcal{G}_{n-k}^{F}(dx,d^2y,d^3z;u;\alpha;\lambda)\sum_{l=0}^{k}\binom{k}{l}S_l(c-1;\frac{\lambda}{u})$$
$$\times\,{}_{H}\mathcal{G}_{k-1}^{F}(cX,C^2Y,c^3Z;u;\alpha-1;\lambda)\frac{t^n}{n!}. \quad (26)$$

In a similar way, we have

$$G(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}d^{n-k}c^{k}u^{d-1}{}_{H}\mathcal{G}_{n-k}^{F}(cx,c^2y,c^3z;u;\alpha;\lambda)\sum_{l=0}^{k}\binom{k}{l}S_l(d-1;\frac{\lambda}{u})$$
$$\times\,{}_{H}\mathcal{G}_{k-1}^{F}(dX,d^2Y,d^3Z;u;\alpha-1;\lambda)\frac{t^n}{n!}. \quad (27)$$

Equating the coefficients of the like powers of t in the R.H.S of expansions (26) and (27), we lead to identity (25).

Theorem 21. *The following inequality holds true*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}u^{d+c-2}\left(\frac{\lambda}{u}\right)^{i+j}c^{n-k}d^{k}{}_{H}\mathcal{G}_{n}^{F}(cX+\frac{c}{d}j,c^{2}Y,c^{3}Z;u;\alpha;\lambda)$$
$$\times{}_{H}\mathcal{G}_{n-k}^{F}(dx+\frac{d}{c}j,d^{2}y,d^{3}z;u;\alpha;\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}u^{d+c-2}\left(\frac{\lambda}{u}\right)^{i+j}d^{n-k}c^{k}{}_{H}\mathcal{G}_{n}^{F}(dX+\frac{d}{c}j,d^{2}Y,d^{3}Z;u;\alpha;\lambda)$$
$$\times{}_{H}\mathcal{G}_{n-k}^{F}(cx+\frac{c}{d}j,c^{2}y,c^{3}z;u;\alpha;\lambda). \quad (28)$$

Proof. Let

$$H(t)=\frac{((1-u)t)^{2\alpha}e^{cdxt+y(cdt)^{2}+z(cdt)^{3}}(\lambda^{c}e^{cdt}-u^{c})(\lambda^{d}e^{cdt}-u^{d})e^{cdXt+Y(cdt)^{2}+Z(cdt)^{3}}}{(\lambda e^{ct}-u)^{\alpha+1}(\lambda e^{dt}-u)^{\alpha+1}},$$

which on rearranging the powers, we have

$$H(t)=\left(\frac{(1-u)t}{\lambda e^{ct}}\right)^{\alpha}e^{dx(ct)+d^{2}y(ct)^{2}+d^{3}z(ct)^{3}}u^{c-1}\sum_{i=0}^{c-1}\left(\frac{\lambda}{u}\right)^{i}e^{dti}$$
$$\times\left(\frac{(1-u)t}{\lambda e^{dt}}\right)^{\alpha}e^{cX(dt)+c^{2}Y(dt)^{2}+c^{3}Z(dt)^{3}}u^{d-1}\sum_{i=0}^{d-1}\left(\frac{\lambda}{u}\right)^{i}e^{cti}.$$

$$H(t)=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}u^{d+c-2}\left(\frac{\lambda}{u}\right)^{i+j}c^{n-k}d^{k}{}_{H}\mathcal{G}_{n}^{F}(cX+\frac{c}{d}j,c^{2}Y,c^{3}Z;u;\alpha;\lambda)$$
$$\times{}_{H}\mathcal{G}_{n-k}^{F}(dx+\frac{d}{c}j,d^{2}y,d^{3}z;u;\alpha;\lambda). \quad (29)$$

In a similar way, we have

$$H(t)=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}u^{d+c-2}\left(\frac{\lambda}{u}\right)^{i+j}d^{n-k}c^{k}{}_{H}\mathcal{G}_{n}^{F}(dX+\frac{d}{c}j,d^{2}Y,d^{3}Z;u;\alpha;\lambda)$$
$$\times{}_{H}\mathcal{G}_{n-k}^{F}(cx+\frac{c}{d}j,c^{2}y,c^{3}z;u;\alpha;\lambda). \quad (30)$$

On comparing the coefficients of the like powers of t in the R.H.S. of expansions (29) and (30), we obtain the required identity (28).

Theorem 22. *The following inequality holds true*

$$\sum_{m=0}^{d-1} u^{d-1}\left(\frac{\lambda}{u}\right)^m \sum_{l=0}^{m}\binom{n}{l}{}_H\mathcal{G}_{n-l}\left(cx, c^2y, c^3z; u; \lambda\right) d^{n-l}(cm)^l$$

$$\sum_{m=0}^{c-1} u^{c-1}\left(\frac{\lambda}{u}\right)^m \sum_{l=0}^{m}\binom{n}{l}{}_H\mathcal{G}_{n-l}\left(dx, d^2y, d^3z; u; \lambda\right) c^{n-l}(dm)^l. \tag{31}$$

Proof. Let

$$N(t) = \frac{((1-u)t)e^{cdxt+y(cdt)^2+z(cdt)^3}(\lambda^d e^{cdt} - u^d)}{(\lambda e^{dt} - u)(\lambda e^{ct} - u)}.$$

Proceeding on the same lines of proof as in Theorem 4.2, we get identity (31). Thus, we omit the proof.

Theorem 23. *The following inequality holds true*

$$\sum_{l=0}^{n}\binom{n}{l}{}_H\mathcal{G}^F_{n-l}(dx, d^2y, d^3z; u; \lambda)c^{n-l}u^{d\alpha}\lambda^{-\alpha}\sum_{m=0}^{l}\binom{l}{m}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d; \frac{\lambda}{u}\right){}_H\mathcal{G}^F_{l-m}(cx, c^2y, c^3z; u; \alpha+1; \lambda)c^m d^{l-m}$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}{}_H\mathcal{G}^F_{n-l}(dx, d^2y, d^3z; u; \lambda)c^{n-l}u^{d\alpha}\lambda^{-\alpha}\sum_{m=0}^{l}\binom{l}{m}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d; \frac{\lambda}{u}\right){}_H\mathcal{G}^F_{l-m}(cx, c^2y, c^3z; u; \alpha+1; \lambda)c^m d^{l-m}. \tag{32}$$

Proof. Let

$$F(t) = \frac{((1-u)t)^{\alpha+2}e^{cdxt+y(cdt)^2+z(cdt)^3}(\lambda^d e^{cdt} - u^d)^{\alpha}e^{cdXt+Y(cdt)^2+Z(cdt)^3}}{(\lambda e^{dt} - u)^{\alpha+1}(\lambda e^{ct} - u)^{\alpha+1}},$$

which on rearranging the powers, we have

$$F(t) = \sum_{n=0}^{\infty}{}_H\mathcal{G}_n(dx, d^2y, d^3z; u; \lambda)c^n\frac{t^n}{n!}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\binom{m}{r}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d; \frac{\lambda}{u}\right)c^m\frac{t^m}{m!}\sum_{l=0}^{\infty}{}_H\mathcal{G}_l^{(\alpha+1)}(cX, c^2Y, c^3Z; u; \lambda)d^l\frac{t^l}{l!}.$$

$$F(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}{}_{H}\mathcal{G}_{n-l}^{F}(dx,d^2y,d^3z;u;\lambda)c^{n-l}u^{d\alpha}\lambda^{a}-\alpha\sum_{m=0}^{l}\binom{l}{m}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d;\frac{\lambda}{u}\right){}_{H}\mathcal{G}_{l-m}^{F}(cx,c^2y,c^3z;u;\alpha+1;\lambda)c^m d^{l-m}\frac{t^n}{n!}. \tag{33}$$

Again we can write

$$F(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}{}_{H}\mathcal{G}_{n-l}^{F}(dx,d^2y,d^3z;u;\lambda)c^{n-l}u^{d\alpha}\lambda^{a}-\alpha\sum_{m=0}^{l}\binom{l}{m}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d;\frac{\lambda}{u}\right){}_{H}\mathcal{G}_{l-m}^{F}(cx,c^2y,c^3z;u;\alpha+1;\lambda)c^m d^{l-m}\frac{t^n}{n!}. \tag{34}$$

On comparing the coefficients of equal powers of t in the r.h.s. of Expansions (33) and (34), yields identity (32).

Theorem 24. *The following inequality holds true*

$$\sum_{m=0}^{n}\binom{n}{m}{}_{H}\mathcal{G}_{n-m}^{F}(dx,d^2y,d^3z;u;\alpha;\lambda)c^{n-m}u^{d\alpha}\lambda^{-\alpha}\sum_{r=0}^{m}\binom{m}{r}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(d;\frac{\lambda}{u}\right)c^m$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_{H}\mathcal{G}_{n-m}^{F}(dx,d^2y,d^3z;u;\lambda)c^{n-m}u^{d\alpha}\lambda^{-\alpha}\sum_{r=0}^{m}\binom{m}{r}(-\alpha)^{m-r}$$

$$S_k^{(\alpha)}\left(c;\frac{\lambda}{u}\right)d^m. \tag{35}$$

Proof. Let

$$N(t)=\frac{((1-u)t)^{\alpha}e^{cdxt+y(cdt)^2+z(cdt)^3}(\lambda^c e^{cdt}-u^c)^{\alpha}}{(\lambda e^{dt}-u)^{\alpha}(\lambda e^{ct}-u)^{\alpha}}.$$

Proceeding on the same lines of proof as in Theorem 4.4, we get identity (35). Thus, we omit the proof.

5. Conclusion

In this paper, a multi-variable hybrid class of the Hermite–Apostol-type Frobenius–Genocchi polynomials is introduced and their properties are explored using various generating function methods. Several explicit and recurrence relations, summation formulae and symmetry identities are established for these hybrid polynomials. A brief view of the operational approach is also given for these polynomials. The operational representations combined with integral transforms may lead to other interesting results, which may be helpful to the theory of fractional calculus. These aspects will be undertaken in further investigation.

Acknowledgements. All authors would like to thank Integral University, Lucknow, India, for providing the manuscript number "IU/R&D/2019-MCN-000682" for the present research work.

References

[1] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company: New York, NY, USA, 1985.

[2] S. Araci and M. Acikgoz, *A note on the FrobeniusEuler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math., (2012), 399-406.

[3] S. Araci and M. Acikgoz, *On the von Staudt-Clausens theorem related to q-Frobenius-Euler numbers*, J. Number Theory, 159, (2016), 329-339.

[4] S. Araci, M. Riyasat, S.A. Wani and S. Khan, *Several characterizations of the 3-variable Hermite-Apostol type Frobenius-Euler and related polynomials*, Adv. Difference Equ. (To appear)

[5] S. Araci, M. Riyasat, S. A. Wani and S. Khan, *A new class of Hermite-Apostol type Frobenius-Euler polynomials and its applications*, Symmetry, 2018, 10,652;DOI: 10.3390/sym10110652.

[6] G. Dattoli, *Generalized polynomials operational identities and their applications*, J. Comput. Appl. Math., 118, (2000), 111–123.

[7] Y. He, S. Araci, H.M. Srivastava and M. Acikgoz, *Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials*, Appl. Math. Comput., 262, (2015), 31-41.

[8] P. Hernández-Llanos, Y. Quintana and A. Urieles, *About Extensions of Generalized Apostol-Type Polynomials*, Results. Math., 68, (2015), 203-225.

[9] B. Kurt and Y. Simsek, *Frobenius-Euler type polynomials related to Hermite-Bernoulli polyomials*, Numerical Analysis and Appl. Math., ICNAAM (2011) Conf. Proc., 1389, (2011), 385-388.

[10] B. Kurt and Y. Symsek, *On the generalized Apostol type Frobenius-Euler polynomials*, Advances in difference equation, 2013, 1, (2013), 1-9.

[11] S. Khan, M. Riyasat and G. Yasmin, *Finding symmetry identities for the 2-variable Apostol type polynomials*, Tbilisi Math. J., 10, 2, (2017), 65-81.

[12] S. Khan, G. Yasmin and M. Riyasat, *Certain results for the 2-variable Apostol type and related polynomials*, Comput. Math. Appl., 69, (2015), 1367-1382.

[13] W. A. Khan, *Some properties of the Generalized APostol type Hermite-based polynomials*, Kyungpook math. J., 55, (2015), 597-614.

[14] W. A. Khan and Divesh Srivastava, *Certain properties of Apostol-type Hermite-based Frobenius-Genocchi polynomials*, Kragujevac Journal of Mathematics, 45(6), (2021), 856-872.

[15] W. A. Khan and Divesh Srivastava, *On generalized Apostol-type Frobenius Genocchi polynomials*, Filomat, 33(7), (2019), 1969-1977.

[16] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Limited. Co. New York, 1984.

[17] Y. Simsek, *Generating functions for q-Apostol type Frobenius-Euler numbers and polynomials*, Axioms 1 (2012), 395-403.

[18] Y. Symsek, *Generating functions for generalized stirling type polynomials, Eulerian type polynomials and their applications*, Fixed point Th. Appl., DOI:1186/1687-1812-2013-87,(2013).

Waseem A. Khan
Department of Mathematics and Natural Science
Prince Mohammad Bin Fahd University,
P. O. Box 1664, Al Khobar-31952 KSA.
email: *waseem08_khan@rediffmail.com*

Divesh Srivastava
Department of Mathematics, Faculty of Science,
Integral University,
Lucknow-226026, India
email: *divesh2712@gmail.com*