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SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. In this work, we introduce and investigate a new class $k-\widetilde{U}S_s^m(\alpha,\beta,\mu,\lambda,\gamma,t)$ of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions f belonging to this class.

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1. Introduction

Let A denote the class of analytic functions f defined on the unit disk $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

For $f \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in U). \tag{3}$$

Note that $f * g \in A$.

A function $f \in A$ is said to be in $k - US(\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \quad (k \geq 0), \tag{4}$$

and a function $f \in A$ is said to be in $k - UC(\gamma)$, the class of $k-$uniformly convex functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \quad (k \geq 0). \tag{5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [8] and then studied by various authors. It is known that $f \in k - UC(\gamma)$ or $f \in k - US(\gamma)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{k,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{k,\gamma} = \left\{w = u + iv \in C : u > k\sqrt{(u-1)^2 + v^2} + \gamma, \quad k \geq 0 \text{ and } \gamma \in [0,1)\right\}. \tag{6}$$

Denote by $\mathcal{P}(P_{k,\gamma}), (k \geq 0, 0 \leq \gamma < 1)$ the family of functions p, such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes well-known class of Caratheodory functions. The function $P_{k,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{k,\gamma}$ such that $1 \in \mathcal{R}_{k,\gamma}$ and $\lceil \mathcal{R}_{k,\gamma}$ is a curve defined by the equality

$$\partial\mathcal{R}_{k,\gamma} = \left\{w = u + iv \in C : u^2 = \left(k\sqrt{(u-1)^2 + v^2} + \gamma\right)^2, \quad k \geq 0 \text{ and } \gamma \in [0,1)\right\}. \tag{7}$$

From elementary computations we see that (7) represents conic sections symmetric about the real axis. Thus $\mathcal{R}_{k,\gamma}$ is an elliptic domain for $k > 1$, a parabolic domain for $k = 1$, a hyperbolic domain for $0 < k < 1$ and the right half plane $u > \gamma$, for $k = 0$.

In [11], Sakaguchi defined the class S_s of starlike functions with respect to symmetric points as follows:

Let $f \in A$. Then f is said to be starlike with respect to symmetric points in U if and only if

$$Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \ (z \in U).$$

Recently, Owa et al. [9] defined the class $S_s(\xi, t)$ as follows:

$$Re\left\{\frac{(1-t)zf'(z)}{f(z) - f(tz)}\right\} > \xi, \ (z \in U),$$

where $0 \leq \xi < 1, |t| \leq 1, t \neq 1$. Note that $S_s(0,-1) = S_s$ and $S_s(\xi,-1) = S_s(\xi)$ is called Sakaguchi function of order ξ.

In [6], Darus and Faisal introduced the following differential operator.

For a function $f \in A$,

$$\begin{aligned}
D^0_\lambda(\alpha,\beta,\mu)f(z) &= f(z)\\
D^1_\lambda(\alpha,\beta,\mu)f(z) &= \left(\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}\right)f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)zf'(z)\\
D^2_\lambda(\alpha,\beta,\mu)f(z) &= D\left(D^1_\lambda(\alpha,\beta,\mu)f(z)\right)\\
&\vdots\\
D^m_\lambda(\alpha,\beta,\mu)f(z) &= D_\lambda\left(D^{m-1}_\lambda(\alpha,\beta,\mu)f(z)\right)
\end{aligned}$$

where $\alpha,\beta,\mu,\lambda \geq 0, \alpha+\beta \neq 0$ and $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

If f is given by (1) then from the definition of the operator $D^m_\lambda(\alpha,\beta,\mu)f$ it is easy to see that

$$D^m_\lambda(\alpha,\beta,\mu)f(z) = z + \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)a_nz^n \tag{8}$$

where

$$\phi_n(\alpha,\beta,\mu,\lambda,m) = \left(\frac{\alpha+(\mu+\lambda)(n-1)+\beta}{\alpha+\beta}\right)^m \tag{9}$$

By specializing the parameters of $D^m_\lambda(\alpha,\beta,\mu)f(z)$, we get the following differential operators. If we substitute

- $\beta = 0$, we get $D^mf(z) = z + \sum\limits_{n=2}^{\infty}\left(\frac{\alpha+(\mu+\lambda)(n-1)}{\alpha}\right)^m a_nz^n$ of differential operator given by Darus and Faisal [5].
- $\beta = 1, \mu = 0$, we get $D^mf(z) = z + \sum\limits_{n=2}^{\infty}\left(\frac{\alpha+\lambda(n-1)+1}{\alpha+1}\right)^m a_nz^n$ of differential operator given by Aouf et al. [2].
- $\alpha = 1, \beta = 0$ and $\mu = 0$, we get $D^mf(z) = z + \sum\limits_{n=2}^{\infty}(1+\lambda(n-1))^m a_nz^n$ of differential operator given by Al-Oboudi [1].
- $\alpha = 1, \beta = 0, \mu = 0$ and $\lambda = 1$, we get $D^mf(z) = z + \sum\limits_{n=2}^{\infty}(n)^m a_nz^n$ of Salageanis differential operator [12].

- $\alpha = 1, \beta = 1, \mu = 0$ and $\lambda = 1$, we get $D^m f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right)^m a_n z^n$ of differential operator given by Uralegaddi and Somanatha [16].
- $\beta = 1, \mu = 0$ and $\lambda = 1$, we get $D^m f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\alpha}{\alpha+1}\right)^m a_n z^n$ of differential operator given by Cho and Srivastava [3, 4].

Now, by making use of the differential operator $D_\lambda^m f$, we define a new subclass of functions belonging to the class A.

Definition 1. *A function $f \in A$ is said to be in the class $k - US_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ if for all $z \in U$*

$$Re\left\{\frac{(1-t)z\,(D_\lambda^m(\alpha,\beta,\mu)f(z))'}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)}\right\} \geq k\left|\frac{(1-t)z\,(D_\lambda^m(\alpha,\beta,\mu)f(z))'}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)} - 1\right| + \gamma,$$

for $\lambda \geq 0,\ m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$.

Furthermore, we say that a function $f \in k - US_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ is in the subclass $k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ if $f(z)$ is of the following form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0, n \in \mathbb{N},\ z \in U. \tag{10}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$.

Firstly, we shall need the following lemmas.

Lemma 1. *Let $w = u + iv$. Then*

$$Re\ w \geq \alpha \ \text{ if and only if } \ |w - (1+\alpha)| \leq |w + (1-\alpha)|.$$

Lemma 2. *Let $w = u + iv$ and α, γ be real numbers. Then*

$$Re\ w > \alpha|w-1| + \gamma \ \text{ if and only if } \ Re\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma$$

2. Coefficient bounds of the function class $k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$

Theorem 3. *The function f defined by (10) is in the class $k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ if and only if*

$$\sum_{n=2}^{\infty} \phi_n(\alpha, \beta, \mu, \lambda, m)|n(k+1) - u_n(k+\gamma)|a_n \leq 1 - \gamma, \tag{11}$$

where $\lambda \geq 0,\ m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ *and* $u_n = 1 + t + \cdots + t^{n-1}$.
The result is sharp for the function $f(z)$ *given by*

$$f(z) = z - \frac{1-\gamma}{\phi_n(\alpha,\beta,\mu,\lambda,m)|n(k+1) - u_n(k+\gamma)|} z^n$$

Proof. By Definition 1, we get

$$Re\left\{ \frac{(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)} \right\} \geq k\left| \frac{(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)} - 1 \right| + \gamma.$$

Then by Lemma 2, we have

$$Re\left\{ \frac{(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)}(1+ke^{i\theta}) - ke^{i\theta} \right\} \geq \gamma, \quad -\pi < \theta \leq \pi$$

or equivalently

$$Re\left\{ \frac{(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'(1+ke^{i\theta})}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)} - \frac{ke^{i\theta}\left[D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)\right]}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)} \right\} \geq \gamma. \tag{12}$$

Let $F(z) = (1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)\right]$
and $E(z) = D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)$.
By Lemma 1, (12) is equivalent to

$$|F(z) + (1-\gamma)E(z)| \geq |F(z) - (1+\gamma)E(z)|, \quad \text{for } 0 \leq \gamma < 1.$$

But

$$\begin{aligned}
|F(z) + (1-\gamma)E(z)| = &\Big|(1-t)\big\{(2-\gamma)z - \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)(n+u_n(1-\gamma))a_n z^n \\
&- ke^{i\theta}\sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)(n-u_n)a_n z^n\big\}\Big| \\
\geq &|1-t|\Big\{(2-\gamma)|z| - \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)|n+u_n(1-\gamma)||a_n||z^n| \\
&- k\sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)|n-u_n||a_n||z^n|\Big\}.
\end{aligned}$$

Also

$$|F(z)-(1+\gamma)E(z)| = \Big|(1-t)\big\{-\gamma z - \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)(n-u_n(1+\gamma))a_n z^n$$

$$- ke^{i\theta}\sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)(n-u_n)a_n z^n\big\}\Big|$$

$$\leq |1-t|\Big\{\gamma|z| + \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)|n-u_n(1+\gamma)||a_n||z^n|$$

$$+ k\sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)|n-u_n|a_n|z^n|\Big\}.$$

So

$$|F(z)+(1-\gamma)E(z)| - |F(z)-(1+\gamma)E(z)|$$

$$\geq |1-t|\Big\{2(1-\gamma)|z| - \sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)\big[|n+u_n(1-\gamma)| + |n-u_n(1+\gamma)| + 2k|n-u_n|\ \big]a_n|z^n|\Big\}$$

$$\geq 2(1-\gamma)|z| - \sum_{n=2}^{\infty}2\phi_n(\alpha,\beta,\mu,\lambda,m)\big|n(k+1)-u_n(k+\gamma)\big|a_n|z^n| \geq 0$$

or

$$\sum_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)\big|n(k+1)-u_n(k+\gamma)\big|a_n \leq 1-\gamma.$$

Conversely, suppose that (11) holds. Then we must show

$$Re\left\{\frac{(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)\right]}{D_\lambda^m(\alpha,\beta,\mu)f(z) - D_\lambda^m(\alpha,\beta,\mu)f(tz)}\right\} \geq \gamma.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)[n(1+ke^{i\theta}) - u_n(\gamma+ke^{i\theta})]a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)u_n a_n z^{n-1}}\right\} \geq 0.$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)[n(1+k) - u_n(\gamma+k]a_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty}\phi_n(\alpha,\beta,\mu,\lambda,m)u_n a_n r^{n-1}}\right\} \geq 0.$$

Letting $r \to 1^-$, we have desired conclusion.

Corollary 4. *If* $f(z) \in k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ *then*

$$a_n \leq \frac{1-\gamma}{\phi_n(\alpha, \beta, \mu, \lambda, m)|n(k+1) - u_n(k+\gamma)|}$$

where $\lambda \geq 0,\ m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ *and* $u_n = 1 + t + \cdots + t^{n-1}$.

3. Neighborhood of the function class $k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$

The concept of neighborhoods of analytic functions was introduced and studied by Goodman [7], Ruscheweyh [10] and Santosh et al. [13].

Definition 2. *Let* $\lambda \geq 0,\ m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, \varsigma \geq 0$ *and* $u_n = 1 + t + \cdots + t^{n-1}$. *We define the* $\varsigma-$*neighborhood of a function* $f \in A$ *and denote by* $N_\varsigma(f)$ *consisting of all functions* $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n \in S(b_n \geq 0, n \in \mathbb{N})$ *satisfying*

$$\sum_{n=2}^{\infty} \frac{\phi_n(\alpha, \beta, \mu, \lambda, m)|n(k+1) - u_n(k+\gamma)|}{1-\gamma} |a_n - b_n| \leq 1 - \varsigma.$$

Theorem 5. *Let* $f(z) \in k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ *and for all real* θ *we have* $\gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0$. *For any complex number* ϵ *with* $|\epsilon| < \varsigma(\varsigma \geq 0)$, *if* f *satisfies the following condition:*

$$\frac{f(z) + \epsilon z}{1+\epsilon} \in k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$$

then $N_\varsigma(f) \subset k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$.

Proof. It is obvious that $f \in k - \widetilde{U}S_s^m(\alpha, \beta, \mu, \lambda, \gamma, t)$ if and only if

$$\left| \frac{(1-t)z\left(D_\lambda^m(\alpha, \beta, \mu)f(z)\right)'(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)\left(D_\lambda^m(\alpha, \beta, \mu)f(z) - D_\lambda^m(\alpha, \beta, \mu)f(tz)\right)}{(1-t)z\left(D_\lambda^m(\alpha, \beta, \mu)f(z)\right)'(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)\left(D_\lambda^m(\alpha, \beta, \mu)f(z) - D_\lambda^m(\alpha, \beta, \mu)f(tz)\right)} \right| < 1,$$

$(-\pi \leq \theta \leq \pi)$,

for any complex number s with $|s| = 1$, we have

$$\frac{(1-t)z\left(D_\lambda^m(\alpha, \beta, \mu)f(z)\right)'(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)\left(D_\lambda^m(\alpha, \beta, \mu)f(z) - D_\lambda^m(\alpha, \beta, \mu)f(tz)\right)}{(1-t)z\left(D_\lambda^m(\alpha, \beta, \mu)f(z)\right)'(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)\left(D_\lambda^m(\alpha, \beta, \mu)f(z) - D_\lambda^m(\alpha, \beta, \mu)f(tz)\right)} \neq s.$$

In other words, we must have

$$(1-s)(1-t)z\left(D_\lambda^m(\alpha,\beta,\mu)f(z)\right)'(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma+s(-1+ke^{i\theta}+\gamma))$$
$$\times\left(D_\lambda^m(\alpha,\beta,\mu)f(z)-D_\lambda^m(\alpha,\beta,\mu)f(tz)\right)\neq 0.$$

which is equivalent to

$$z-\sum_{n=2}^{\infty}\frac{\phi_n(\alpha,\beta,\mu,\lambda,m)\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}z^n\neq 0.$$

However, $f\in k-\widetilde{U}S_s^m(\alpha,\beta,\mu,\lambda,\gamma,t)$ if and only $\frac{(f*h)}{z}\neq 0, z\in U-\{0\}$, where $h(z)=z-\sum\limits_{n=2}^{\infty}c_nz^n$ and

$$c_n=\frac{\phi_n(\alpha,\beta,\mu,\lambda,m)\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}$$

we note that

$$|c_n|\leq\frac{\phi_n(\alpha,\beta,\mu,\lambda,m)\,|n(1+k)-u_n(k+\gamma)|}{1-\gamma}$$

since $\frac{f(z)+\epsilon z}{1+\epsilon}\in k-\widetilde{U}S_s^m(\alpha,\beta,\mu,\lambda,\gamma,t)$, therefore $z^{-1}\left(\frac{f(z)+\epsilon z}{1+\epsilon}*h(z)\right)\neq 0$, which is equivalent to

$$\frac{(f*h)(z)}{(1+\epsilon)z}+\frac{\epsilon}{1+\epsilon}\neq 0. \tag{13}$$

Now suppose that $\left|\frac{(f*h)(z)}{z}\right|<\varsigma$. Then by (13), we must have

$$\left|\frac{(f*h)(z)}{(1+\epsilon)z}+\frac{\epsilon}{1+\epsilon}\right|\geq\frac{|\epsilon|}{|1+\epsilon|}-\frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z}\right|$$
$$>\frac{|\epsilon|-\varsigma}{|1+\epsilon|}\geq 0,$$

this is a contradiction by $|\epsilon|<\varsigma$ and however, we have $\left|\frac{(f*h)(z)}{z}\right|\geq\varsigma$. If $g(z)=z-\sum\limits_{n=2}^{\infty}b_nz^n\in N_\varsigma(f)$, then

$$\varsigma-\left|\frac{(g*h)(z)}{z}\right|\leq\left|\frac{((f-g)*h)(z)}{z}\right|\leq\sum_{n=2}^{\infty}|a_n-b_n||c_n||z^n|$$
$$<\sum_{n=2}^{\infty}\frac{\phi_n(\alpha,\beta,\mu,\lambda,m)|n(1+k)-u_n(k+\gamma)|}{1-\gamma}|a_n-b_n|\leq\varsigma.$$

4. Partial sums of the function class $k-\widetilde{U}S_s^m(\alpha,\beta,\mu,\lambda,\gamma,t)$

In this section, applying methods used by Silverman [14] and Silvia [15], we investigate the ratio of a function of the form (10) to its sequence of partial sums $f_m(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Theorem 6. *If f of the form* (1) *satisfies the condition* (11) *then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{\delta_{m+1}} \tag{14}$$

and

$$\delta_n = \begin{cases} 1, & \text{if } n = 2,3,\cdots,m; \\ \delta_{m+1}, & \text{if } n = m+1, m+2, \cdots. \end{cases} \tag{15}$$

where

$$\delta_n = \frac{\phi_n(\alpha,\beta,\mu,\lambda,m)|n(1+k) - u_n(k+\gamma)|}{1-\gamma}. \tag{16}$$

The result in (14) *is sharp for every m, with the extremal function*

$$f(z) = z + \frac{z^{m+1}}{\delta_{m+1}} \tag{17}$$

Proof. Define the function w, we may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} =& \delta_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{\delta_{m+1}}\right)\right\} \\ =& \left\{\frac{1 + \sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}}\right\}. \end{aligned} \tag{18}$$

Then, from (18), we can obtain

$$w(z) = \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\delta_{m+1}\sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{m} a_n - \delta_{m+1}\sum_{n=m+1}^{\infty} a_n}.$$

Now $|w(z)| \leq 1$ if

$$2\delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 2 - 2\sum_{n=2}^{m} a_n,$$

which is equivalent to

$$\sum_{n=2}^{m} a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 1. \tag{19}$$

It is suffices to show that the left hand side of (19) is bounded above by $\sum_{n=2}^{\infty} \delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n - 1)a_n + \sum_{n=m+1}^{\infty} (\delta_n - \delta_{m+1})a_n \geq 0.$$

To see that the function given by (17) gives the sharp result, we observe that for $z = re^{i\pi/n}$,

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\delta_{m+1}}. \tag{20}$$

Taking $z \to 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{\delta_{m+1}}.$$

This completes the proof of Theorem 6.

We next determine bounds for $\frac{f_m(z)}{f(z)}$.

Theorem 7. *If f of the form* (1) *satisfies the condition* (11) *then*

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{\delta_{m+1}}{1+\delta_{m+1}}. \tag{21}$$

The result is sharp with the function given by (17).

Proof. We may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} =& (1+\delta_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{\delta_{m+1}}{1+\delta_{m+1}}\right\} \\ =& \left\{\frac{1+\sum_{n=2}^{m} a_n z^{n-1} - \delta_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}}\right\}, \end{aligned} \tag{22}$$

where

$$w(z) = \frac{(1+\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{-\left(2+2\sum\limits_{n=2}^{m} a_n z^{n-1} - (1-\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}\right)}$$

and

$$|w(z)| \leq \frac{(1+\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n}{2-2\sum\limits_{n=2}^{m} a_n + (1-\delta_{m+1})\sum\limits_{n=m+1}^{\infty} a_n} \leq 1. \tag{23}$$

This last inequality is equivalent to

$$\sum_{n=2}^{m} a_n + \delta_{m+1}\sum_{n=m+1}^{\infty} a_n \leq 1. \tag{24}$$

It is suffices to show that the left hand side of (24) is bounded above by $\sum\limits_{n=2}^{\infty} \delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{m}(\delta_n - 1)a_n + \sum_{n=m+1}^{\infty}(\delta_n - \delta_{m+1})a_n \geq 0.$$

This completes the proof of Theorem 7

We next turn to ratios involving derivatives.

Theorem 8. *If f of the form* (1) *satisfies the condition* (11) *then*

$$Re\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq 1 - \frac{m+1}{\delta_{m+1}}, \tag{25}$$

$$Re\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{\delta_{m+1}}{1+m+\delta_{m+1}} \tag{26}$$

where

$$\delta_n \geq \begin{cases} 1, & \text{if } n = 1,2,3,\cdots,m; \\ n\frac{\delta_{m+1}}{m+1}, & \text{if } n = m+1, m+2, \cdots. \end{cases}$$

and δ_n is defined by (16). *The estimates in* (25) *and* (26) *are sharp with the extremal function given by* (17).

Proof. Firstly, we will give proof of (25). We write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \delta_{m+1}\left\{\frac{f'(z)}{f'_m(z)}-\left(1-\frac{m+1}{\delta_{m+1}}\right)\right\}\\ &= \left\{\frac{1+\sum\limits_{n=2}^{m} na_n z^{n-1}+\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{m} a_n z^{n-1}}\right\},\end{aligned}$$

where

$$w(z)=\frac{\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{2+2\sum\limits_{n=2}^{m} na_n z^{n-1}+\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)|\leq\frac{\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n}{2-2\sum\limits_{n=2}^{m} na_n+\frac{\delta_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n}.$$

Now $|w(z)|\leq 1$ if and only if

$$\sum_{n=2}^{m} na_n+\frac{\delta_{m+1}}{m+1}\sum_{n=m+1}^{\infty} na_n\leq 1, \tag{27}$$

since the left hand side of (27) is bounded above by $\sum\limits_{n=2}^{\infty}\delta_n a_n$. The proof of (26) follows the pattern of that in Theorem (15).
This completes the proof of Theorem 8.

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