Acta Universitatis Apulensis No. 62/2020
ISSN: 1582-5329 pp. 111-118
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.62.09

JANOWSKI STARLIKENESS AND CONVEXITY WITH APPLICATIONS OF POISSON DISTRIBUTION SERIES

A. Çetinkaya

Abstract. In the present paper, we obtain necessary and sufficient conditions for Poisson distribution series belonging to the classes $\mathcal{TS}^*(A,B)$ and $\mathcal{TC}(A,B)$ with negative coefficients. We also establish some new results for an integral operator related to the Poisson distribution series.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Poisson distribution series, integral operator, Janowski starlikeness, Janowski convexity.

1. Introduction

Let $\mathbb{D}=\{z:|z|<1\}$ be the open unit disc and let $\mathcal{A}$ be the class of all functions f that are analytic in $\mathbb{D}$ and normalized by $f(0)=f'(0)-1=0$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f, which are univalent in $\mathbb{D}$. Thus, each $f\in\mathcal{S}$ has the Maclaurin's series expansion of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ \ (z\in\mathbb{D}). \tag{1}$$

A domain $\mathcal{D}\subset\mathbb{C}$ is called starlike with respect to the point $z_0\in\mathcal{D}$ if the closed line segment joining a point $z_0\in\mathcal{D}$ to each point $z\in\mathcal{D}$ lies entirely in $\mathcal{D}$, while the domain $\mathcal{D}\subset\mathbb{C}$ is called convex if the closed line segment between z_1 and z_2 lies entirely in the domain, when $z_1,z_2\in\mathcal{D}$. Denote by $\mathcal{S}^*$ a function $f\in\mathcal{S}$ is called starlike with respect to the origin in the disc $\mathbb{D}$ if the domain $f(\mathbb{D})$ is starlike with respect to the origin. Denote by $\mathcal{C}$ a function $f\in\mathcal{S}$ is called convex in $\mathbb{D}$ if it maps the disc $\mathbb{D}$ onto a convex domain.

Let Ω be the family of Schwarz functions w which are analytic in $\mathbb{D}$ and satisfy the conditions $w(0)=0$, $|w(z)|<1$ for all $z\in\mathbb{D}$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we say that f_1 is subordinate to f_2, denoted by $f_1\prec f_2$, if there exists a Schwarz function $w \in \Omega$ such that $f_1(z) = f_2(w(z))$. We also note that if f_2 is univalent in $\mathbb{D}$, then

$$f_1 \prec f_2 \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}),\ (z \in \mathbb{D}).$$

Also, denote by $\mathcal{P}$ the class of functions p which are analytic and have positive real part in $\mathbb{D}$ with $p(0) = 1$. More details of these definitions can be found in [3].

By using the subordination, Janowski [5] introduced the class $\mathcal{P}(A, B)$. A given analytic function p with $p(0) = 1$ is said to belong to the class $\mathcal{P}(A, B)$ if and only if

$$p(z) \prec \frac{1 + Az}{1 + Bz},\ (-1 \leq B < A \leq 1)$$

for every $z \in \mathbb{D}$.

Geometrically, a function $p \in \mathcal{P}(A, B)$ maps the unit disc $\mathbb{D}$ onto the domain $\Psi(A, B)$ defined by

$$\Psi(A, B) := \left\{ w : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

This domain represents an open circular disc centered on the real axis with diameter end points $\mathcal{D}_1 = \frac{1-A}{1-B}$ and $\mathcal{D}_2 = \frac{1+A}{1+B}$ with $0 < \mathcal{D}_1 < 1 < \mathcal{D}_2$.

For $-1 \leq B < A \leq 1$, let $\mathcal{S}^*(A, B)$ and $\mathcal{C}(A, B)$ be the subclasses of $\mathcal{S}$ consisting of Janowski starlike and Janowski convex functions, respectively, defined analytically by

$$\mathcal{S}^*(A, B) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(A, B) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in \mathbb{D} \right\}.$$

For $A = 1$ and $B = -1$, one easily get the classes $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Comprehensive details of the Janowski starlikeness and Janowski convexity can be found in [4, 5, 6].

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ (z \in \mathbb{D}). \tag{2}$$

With the negative coefficients, the subclasses $\mathcal{S}^*(A, B)$ and $\mathcal{C}(A, B)$ are written as

$$\mathcal{TS}^*(A, B) := \mathcal{T} \cap \mathcal{S}^*(A, B) \quad \text{and} \quad \mathcal{TC}(A, B) := \mathcal{T} \cap \mathcal{C}(A, B).$$

For more details of the Janowski starlikeness and Janowski convexity with negative coefficients, one may refer to [2] and references given therein.

A variable x is said to have Poisson distribution if it takes the values $0, 1, 2, 3, ...$ with probabilities e^{-m}, $me^{-m}/1!$, $m^2e^{-m}/2!, ...$ respectively, where m is called the parameter.

Thus

$$P(x = k) = \frac{m^k e^{-m}}{k!}, \quad (k \geq 0). \tag{3}$$

The power series whose coefficients are probabilities of the Poisson distribution is given by Porwal [7] as below:

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \tag{4}$$

and

$$F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{5}$$

We note that by ratio test, the radius of convergence of the above series is infinity.

In this paper, we obtain necessary and sufficient conditions for the function $F(m, z)$ belonging to the classes $\mathcal{TS}^*(A, B)$ and $\mathcal{TC}(A, B)$. Finally, we give necessary and sufficient conditions for an integral operator $G(m, z)$ belonging to the classes $\mathcal{TS}^*(A, B)$ and $\mathcal{TC}(A, B)$, respectively.

2. Main Results

In this section, we prove necessary and sufficient conditions for the function $F(m, z)$ belonging to the classes $\mathcal{TS}^*(A, B)$ and $\mathcal{TC}(A, B)$. For our main theorems, we need the following lemma.

Lemma 1. [2] *Let $-1 \leq B < A \leq 1$. A function f defined by (2) is in the class $\mathcal{TS}^*(A, B)$ if and only if*

$$\sum_{n=2}^{\infty} \big[(n-1)(1-B) + (A-B)\big] |a_n| \leq A - B, \tag{6}$$

and a function f defined by (2) is in the class $\mathcal{TC}(A, B)$ if and only if

$$\sum_{n=2}^{\infty} n\big[(n-1)(1-B) + (A-B)\big] |a_n| \leq A - B. \tag{7}$$

In the following theorem, we get necessary and sufficient condition for the function $F(m,z)$ belonging to the class $\mathcal{TS}^*(A,B)$.

Theorem 2. *If $m>0$, then $F(m,z)$ is in the class $\mathcal{TS}^*(A,B)$ if and only if*

$$(1-B)me^m \leq A-B. \tag{8}$$

Proof. Since

$$F(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,$$

taking into account inequality in (6), we must show that

$$\sum_{n=2}^{\infty}\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{(n-1)!}e^{-m}\leq A-B. \tag{9}$$

In view of Poisson distribution series, from the left hand side of (9) we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&\quad=e^{-m}\left[(1-B)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+(A-B)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]\\
&\quad=e^{-m}\left[(1-B)\sum_{n=0}^{\infty}\frac{m^{n+1}}{n!}+(A-B)\sum_{n=1}^{\infty}\frac{m^{n}}{n!}\right]\\
&\quad=e^{-m}\left[(1-B)m\sum_{n=0}^{\infty}\frac{m^{n}}{n!}+(A-B)\sum_{n=1}^{\infty}\frac{m^{n}}{n!}\right]\\
&\quad=e^{-m}\big[(1-B)me^m+(A-B)(e^m-1)\big]\\
&\quad=(1-B)m+(A-B)(1-e^{-m}).
\end{aligned} \tag{10}$$

Since the last expression given in (10) is bounded by $A-B$, then (8) is obtained. Thus the proof is completed.

The next theorem gives necessary and sufficient condition for the function $F(m,z)$ belonging to the class $\mathcal{TC}(A,B)$.

Theorem 3. *If $m>0$, then $F(m,z)$ is in the class $\mathcal{TC}(A,B)$ if and only if*

$$e^m\big[(1-B)m^2+(2-3B+A)m\big]\leq A-B. \tag{11}$$

Proof. Since

$$F(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,$$

according to Lemma 1 inequality (7), we must show that

$$\sum_{n=2}^{\infty} n\big[(n-1)(1-B)+(A-B)\big] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq A-B. \tag{12}$$

Applying Poisson distribution series, the left side of (12) gives

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[(1-B)(n-1)(n-2)+(2-3B+A)(n-1)+(A-B)\right] \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&\quad = e^{-m}\left[(1-B)\sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} + (2-3B+A)\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + (A-B)\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}\right] \\
&\quad = e^{-m}\left[(1-B)\sum_{n=0}^{\infty} \frac{m^{n+2}}{n!} + (2-3B+A)\sum_{n=0}^{\infty} \frac{m^{n+1}}{n!} + (A-B)\sum_{n=1}^{\infty} \frac{m^{n}}{n!}\right] \\
&\quad = e^{-m}\left[(1-B)m^2\sum_{n=0}^{\infty} \frac{m^{n}}{n!} + (2-3B+A)m\sum_{n=0}^{\infty} \frac{m^{n}}{n!} + (A-B)\sum_{n=1}^{\infty} \frac{m^{n}}{n!}\right] \\
&\quad = e^{-m}\big[(1-B)m^2 e^m + (2-3B+A)me^m + (A-B)(e^m-1)\big] \\
&\quad = (1-B)m^2 + (2-3B+A)m + (A-B)(1-e^{-m}).
\end{aligned} \tag{13}$$

The last expression in (13) is bounded by $A-B$, therefore (11) is satisfied. This completes the proof of Theorem 3.

3. Integral Operator

Operators play an important role in geometric function theory. The first mathematician who introduced an integral operator on a class of univalent functions was J. W. Alexander. In 1915, Alexander in [1] defined the operator $I : \mathcal{A} \to \mathcal{A}$ given as

$$I(f)(z) = \int_0^z \frac{f(t)}{t} dt,$$

which is called the Alexander integral operator. Porwal in [7] showed the connection between the Alexander integral operator and the function $F(m,z)$ with the following new operator:

$$G(m,z) = \int_0^z \frac{F(m,t)}{t} dt.$$

In view of this operator, the power series of $G(m,z)$ can be shown by

$$G(m,z) = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{(n-1)!}\frac{z^n}{n} = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{n!}z^n. \tag{14}$$

We now give necessary and sufficient conditions for the integral operator $G(m,z)$ belonging to the classes $\mathcal{TS}^*(A,B)$ and $\mathcal{TC}(A,B)$, respectively.

Theorem 4. *If $m>0$, then $G(m,z)$ is in the class $\mathcal{TC}(A,B)$ if and only if*

$$(1-B)me^m \le A-B. \tag{15}$$

Proof. Since

$$G(m,z) = z - \sum_{n=2}^{\infty} \frac{e^{-m}m^{n-1}}{n!}z^n,$$

by Lemma 1 inequality (7), we must show that

$$\sum_{n=2}^{\infty} n\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{n!}e^{-m} \le A-B.$$

Thus calculations give

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{n!}e^{-m}\\
&\quad= \sum_{n=2}^{\infty}\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&\quad= e^{-m}\left[(1-B)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+(A-B)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]\\
&\quad= e^{-m}\left[(1-B)m\sum_{n=0}^{\infty}\frac{m^{n}}{n!}+(A-B)\sum_{n=1}^{\infty}\frac{m^{n}}{n!}\right]\\
&\quad= e^{-m}\big[(1-B)me^m+(A-B)(e^m-1)\big]\\
&\quad= (1-B)m+(A-B)(1-e^{-m}).
\end{aligned} \tag{16}$$

Here, (16) is bounded by $A-B$ if and only if (15) holds.

Theorem 5. *If $m>0$, then $G(m,z)$ is in the class $\mathcal{TS}^*(A,B)$ if and only if*

$$\left[(1-B)-\frac{1-A}{m}\right](1-e^{-m})+(1-A)e^{-m} \le A-B. \tag{17}$$

Proof. Since

$$G(m,z)=z-\sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{n!}z^n,$$

by inequality in (6), we must show that

$$\sum_{n=2}^{\infty}\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{n!}e^{-m}\le A-B.$$

In view of Poisson distribution series, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\big[(n-1)(1-B)+(A-B)\big]\frac{m^{n-1}}{n!}e^{-m}\\
&=\sum_{n=2}^{\infty}\big[n(1-B)-(1-A)\big]\frac{m^{n-1}}{n!}e^{-m}\\
&=\sum_{n=2}^{\infty}\left[(1-B)-\frac{1-A}{n}\right]\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&=e^{-m}\left[(1-B)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}-(1-A)\sum_{n=2}^{\infty}\frac{m^{n-1}}{n!}\right]\\
&=e^{-m}\left[(1-B)\sum_{n=1}^{\infty}\frac{m^{n}}{n!}-\frac{1-A}{m}\sum_{n=2}^{\infty}\frac{m^{n}}{n!}\right]\\
&=e^{-m}\left[(1-B)(e^m-1)-\frac{1-A}{m}(e^m-1-m)\right]\\
&=\left[(1-B)(1-e^{-m})-\frac{1-A}{m}(1-e^{-m}-me^{-m})\right],
\end{aligned}$$

which is bounded by $A-B$ if and only if (17) holds.

References

[1] J. W. Alexander, *Functions which map the interior of the circle upon simple regions*, Ann. Math. 17 (1915/1916), 12-22.

[2] M. K. Aouf, *Linear combinations of regular functions of order alpha with negative coefficients*, Publ. Inst. Math. (Beograd) (N.S.) 47(61) (1990), 61-67.

[3] A. W. Goodman, *Univalent functions, Volume I and Volume II,* Mariner Pub. Co. Inc. Tampa Florida, 1984.

[4] W. Janowski, *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math. (1971), 159-177.

[5] W. Janowski, *Some extremal problems for certain families of analytic Functions I*, Ann. Polon. Math. 28 (1973), 297-326.

[6] W. Janowski, *Some extremal problems for certain families of analytic functions,* Bulletin of the Polish Academy of Sciences, 21 (1973), 17-25.

[7] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, Journal of Complex Analysis, Volume 2014, Article ID 984135, 3 pages.

Asena Çetinkaya
Department of Mathematics and Computer Science,
Istanbul Kültür University
Istanbul, Turkey
email: *asnfigen@hotmail.com*