Acta Univ. Sapientiae, Mathematica, **1**, 2 (2009) 101–108

On ν-curvature tensor of C3-like conformal Finsler space

S. K. Narasimhamurthy
Department of Mathematics and Computer Science,
Kuvempu University,
Shankaraghatta - 577451,
Shimoga, Karnataka, India.
email: nmurthysk@gmail.com

S. T. Aveesh
Department of Mathematics,
Kuvempu University,
Shankaraghatta - 577451,
Shimoga, Karnataka, India.
email: aveeshst@gmail.com

Pradeep Kumar
Department of Mathematics,
Kuvempu University,
Shankaraghatta - 577451,
Shimoga, Karnataka, India.
email: pradeepget@gmail.com

Abstract. The purpose of the present paper is to study the properties of the ν-curvature tensor in C-reducible Finsler space and conformal ν-curvature tensor in C3-like Finsler space F^n of dimension $(n \geq 4)$, in which the conformal Cartan torsion tensor $\overline{C}_{ijk}$ is said to be a conformal C3-like Finsler space.

1 Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function L(x,y).

AMS 2000 subject classifications: AMS Subject Classification (2000): 53B40, 53C60
Key words and phrases: Finsler space, C-reducible, C3-like, ν-curvature tensor

We use the following notations [2, 6]:

$$
\begin{aligned}
&a) && g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad g^{ij} = (g_{ij})^{-1}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i},\\
&b) && C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^k_{ij} = \frac{1}{2} g^{km}(\dot{\partial}_m g_{ij}),\\
&c) && h_{ij} = g_{ij} - l_i l_j, \quad h^i_k = \delta^i_k - l^i l_k,\\
&d) && m_i = b_i - \beta L^{-1} l^i,\\
&e) && C^h_{ij} l_h = 0,\\
&f) && h^i_k m_i = m_k,\\
&g) && l^i m_i = 0,
\end{aligned} \tag{1}
$$

where l_i, m_i and n_i are the unit vectors, and h_{ij} is a angular metric tensor.

Definition 1 *Let* $F^n = (M^n, L(x,y))$ *and* $\overline{F}^n = (M^n, L(x,y))$ *be two Finsler spaces on the same underlying manifold* M^n*. If the angle in* F^n *is equal to that in* $\overline{F}^n$ *for any tangent vectors, then* F^n *is called conformal to* $\overline{F}^n$ *and the change* $L \to \overline{L} = e^\sigma L$ *of the metric is called a conformal change and* $\sigma(x)$ *is a conformal factor.*

Example 1 *We consider a Finsler space* $F^n = (M^n, L(\alpha,\beta))$*, where* α *is a Riemannian metric,* β *is a 1-form and a conformal change* $L(\alpha,\beta) \to \overline{L} = e^{\sigma(x)} L(\alpha,\beta)$*. Since* $L(\alpha,\beta)$ *is assumed to be (1)p-homogeneous in* α *and* β*, we get* $\overline{L} = L(\overline{\alpha}, \overline{\beta})$*, where* $\overline{\alpha} = e^{\sigma(x)}\alpha$ *and* $\overline{\beta} = e^{\sigma(x)}\beta$*. Thus the conformal change gives rise to the change* $(\alpha,\beta) \longrightarrow (\overline{\alpha},\overline{\beta}) = (e^{\sigma(x)}\alpha, e^{\sigma(x)}\beta)$ *of the pair* (α,β) *independently of the form of the function* $L(\alpha,\beta)$*. Thus we also get the conformal change* $\alpha \to \overline{\alpha} = e^{\sigma(x)}\alpha$ *of the associated Riemannian space* $R^n = (M^n, \alpha)$.

Under the conformal change, we get the following relations [3, 4]:

$$
\begin{aligned}
&a) && \overline{g}_{ij} = e^{2\sigma} g_{ij}, \quad \overline{g}^{ij} = e^{-2\sigma} g^{ij},\\
&b) && \overline{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \overline{C}^i_{jk} = C^i_{jk}, \quad \overline{C}^i_{ik} = \overline{C}_k = C^i_{ik} = C_k,\\
&c) && \overline{l}^i = e^{-\sigma} l^i, \quad \overline{l}_i = e^{\sigma} l_i, \quad \overline{y}_i = e^{2\sigma} y_i,\\
&d) && \overline{h}_{ij} = e^{2\sigma} h_{ij}, \quad \overline{h}^i_j = h^i_j,\\
&e) && \overline{L} = e^{\sigma} L,\\
&f) && \overline{m}_k = e^{\sigma} m_k.
\end{aligned} \tag{2}
$$

Definition 2 ([1]) *A Finsler space is said to be C-reducible if it satisfies the equation*

$$C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij})/(n+1). \tag{3}$$

Definition 3 *A conformal Finsler space $\overline{F}^n$ is said to be a semi-C-reducible conformal Finsler space if $\overline{C}_{ijk}$ is of the form,*

$$\overline{C}_{ijk} = e^{2\sigma}\left[\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{q}{C^2}C_iC_jC_k\right].$$

There are three kinds of torsion tensors in Cartan theory of Finsler space F^n. Two of them are $(h)hv$-torsion tensor C_{ijk} and (v)-torsion tensor P_{ijk}, which are symmetric in all its indices. It is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes. For a three dimensional Finsler space F^3, C_{ijk} is always written in the form [5]

$$LC_{ijk} = Hm_im_jm_k - J\mathfrak{U}_{(ijk)}\{m_im_jn_k\} + I\mathfrak{U}_{(ijk)}\{m_in_jn_k\} + Jn_in_jn_k, \tag{4}$$

where $\mathfrak{U}_{(ijk)}\{\}$ denotes the cyclic permutation of the indices i, j, k and addition H, I and J are main scalars; we assume that they are invariant under conformal change and (l_i, m_i, n_i) is Moor's frame. Here $l_i = \dot{\partial}_i L$ is the unit vector along the element of support, m_i is the unit vector along C_i, i.e., $m_i = C_i/C$, where $C^2 = g^{ij}C_iC_j$ and n_i is a unit vector orthogonal to the vector l_i and m_i.

Example 2 *The C-reducible Finsler space (3) it is written for a three dimensional case as*

$$4C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j. \tag{5}$$

The unit vector $m_i = \frac{C_i}{C}$ is orthogonal to l_i, because $C_iy^i = 0$. Therefore equation (5) can be written as

$$4LC_{ijk} = LC\left[3m_im_jm_k + \{m_im_jn_k + m_in_jm_k + n_im_jm_k\}\right], \tag{6}$$

comparing equations (4) and (6) we have, $4H = 3LC$, $LC = (H+I)$ and $J = 0$. So we get $H = 3I$. Conversely, $H = 3I$ and $LC = H + I$ lead to the above. Therefore the necessary and sufficient condition for C-reducible is $H = 3I$, $LC = (H+I)$ and $J = 0$.

Under conformal change, equation (4) can be written as,

$$\overline{LC}_{ijk} = \overline{H}\overline{m}_i\overline{m}_j\overline{m}_k - \overline{J}\mathfrak{U}_{(ijk)}\{\overline{m}_i\overline{m}_j\overline{n}_k\} + \overline{I}\mathfrak{U}_{(ijk)}\{\overline{m}_i\overline{n}_j\overline{n}_k\} + \overline{J}\overline{n}_i\overline{n}_j\overline{n}_k \tag{7}$$

Suppose H, I and J are conformal invariants, then equation (7) reduces to

$$\overline{LC}_{ijk} = e^{3\sigma} LC_{ijk}.$$

The angular metric tensor h_{ij} in F^3 can be written as [5]

$$h_{ij} = m_i m_j + n_i n_j. \tag{8}$$

Under conformal change, equation (8) can be written as

$$\begin{aligned} \overline{h}_{ij} &= \overline{m}_i \overline{m}_j + \overline{n}_i \overline{n}_j, \\ \overline{h}_{ij} &= e^{2\sigma}(m_i m_j + n_i n_j). \end{aligned}$$

After simplification, equation (7) can be written as

$$\overline{C}_{ijk} = \mathfrak{U}_{(ijk)}(\overline{h}_{ij}\overline{a}_k + \overline{C}_i \overline{C}_j \overline{b}_k), \tag{9}$$

where

$$\begin{aligned} \overline{a}_k &= \frac{1}{\overline{L}}\left\{ \overline{I}\overline{m}_k + \frac{\overline{J}}{3}\overline{n}_k \right\}, \\ \overline{b}_k &= \frac{1}{\overline{LC}^2}\left\{ \left(\frac{\overline{H}}{3} - \overline{I}\right)\overline{m}_k - \frac{4\overline{J}}{3}\overline{n}_k \right\}. \end{aligned}$$

Then – by using (2(e),(f)) – the above equation becomes

$$\begin{aligned} \overline{a}_k &= \frac{1}{L}\left\{ I m_k + \frac{J}{3} n_k \right\}, \\ \overline{b}_k &= \frac{e^{2\sigma}}{LC^2}\left[\left(\frac{H}{3} - I\right) m_k - \frac{4J}{3} n_k \right]. \end{aligned}$$

Substitute $\overline{a}_k$ and $\overline{b}_k$ in (9), we get,

$$\begin{aligned} \overline{C}_{ijk} &= \mathfrak{U}_{(ijk)}(e^{2\sigma} h_{ij} a_k + e^{2\sigma} C_i C_j b_k), \\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}(h_{ij} a_k + C_i C_j b_k). \end{aligned} \tag{10}$$

The equation (10) can also be written in the form of

$$\overline{C}^i_{jk} = e^{2\sigma}\mathfrak{U}_{(ijk)}(h_{jk} a^i + C_j C_k b^i). \tag{11}$$

A Finsler space F^n ($n \geq 4$) is called a C3-like conformal Finsler space if there exist two vector fields a_k and b_k, which are positively homogenous of degree -1 and +1, respectively.
The purpose of the present paper is to find the v-curvature tensor of the conformal Finsler space $\overline{F}^n$ when it satisfies (10).

2 Properties of C3-like conformal Finsler space

Let C_{ijk} be the indicatory tensor and contract equation (10) with g^{jk}, we get

$$\begin{aligned}\overline{C}_{ijk}g^{jk} &= e^{2\sigma}(h_{ij}a_k + C_iC_jb_k + h_{jk}a_i + C_jC_kb_i + h_{ki}a_j + C_kC_ib_j)g^{jk},\\ C_i &= (C_iC_b + (n+1)a_i + C^2b_i + C^iC_b),\\ C_i - 2C_iC_b &= ((n+1)a_i + C^2b_i),\\ C_i(1-2C_b) &= (n+1)a_i + C^2b_i, \qquad (12)\end{aligned}$$

where $C_b = C_ib^i$.

Lemma 1 *The three vectors a_i, b_i, C_i are linearly dependent vectors.*

Contracting (12) with C^i, we get

$$\begin{aligned}(1-2C_b)C_iC^i &= (n+1)a_iC^i + C^2b_iC^i,\\ (1-2C_b)C^2 &= (n+1)C_a + C^2C_b,\\ (1-2C_b)C^2 - C^2C_b &= (n+1)C_a,\\ (1-3C_b)C^2 &= (n+1)C_a.\end{aligned}$$

Lemma 2 *If C_i is perpendicular to b_i, then $C_a = \frac{C^2}{(n+1)}$, and if C_i is perpendicular to a_i, then $C_b = \frac{1}{3}$.*

Now equation (12) can be written as

$$b_i = \frac{(1-2C_b)C_i}{C^2} - \frac{(n+1)a_i}{C^2}. \qquad (13)$$

Substitute (13) in equation (10), we get

$$\begin{aligned}\overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\{h_{ij}a_k + C_iC_jb_k\},\\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\left[h_{ij}a_k + C_iC_j\left\{\frac{(1-2C_b)C_k}{C^2} - \frac{(n+1)a_k}{C^2}\right\}\right],\\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\left\{h_{ij}a_k - \frac{(n+1)}{C^2}C_iC_ja_k\right\} + e^{2\sigma}\frac{3(1-2C_b)}{C^2}C_iC_jC_k.\end{aligned}$$

If a_i is parallel to C_i, i.e. $a_i = \frac{p}{n+1}C_i$, where p is some scalar, then $\overline{C}_{ijk}$ reduces to

$$\begin{aligned}\overline{C}_{ijk} &= e^{2\sigma}\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} - \frac{e^{2\sigma}3P}{C^2}C_iC_jC_k +\\ &+ \frac{e^{2\sigma}3(1-2C_b)}{C^2}C_iC_jC_k,\end{aligned}$$

$$\begin{aligned}\overline{C}_{ijk} &= e^{2\sigma}\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{3e^{2\sigma}C_iC_jC_k}{C^2}(1-2C_b-P),\\ \overline{C}_{ijk} &= e^{2\sigma}\left[\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{q}{C^2}C_iC_jC_k\right],\end{aligned}$$

where $q = 3(1-2C_b-p)$. Hence we state:

Theorem 1 *A* $C3$*-like conformal Finsler space reduce to a semi-C-reducible conformal Finsler space if the vectors* a_i *and* b_i *are parallel to* C_i*.*

3 The ν-curvature tensor of C-redusible Finsler space

The ν-curvature tensor S_{hijk} of F^n is given by

$$S_{hijk} = C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik}. \tag{14}$$

Using (3), the above equation can be written as

$$S_{hijk} = [C_iC_jh_{hk} + C_hC_kh_{ij} - C_iC_kh_{hj} - C_hC_jh_{ik}]/(n+1)^2. \tag{15}$$

Therefore equation (15) reduces to,

$$S_{hijk} = C_i[C_jh_{hk} - C_kh_{hj}]/(n+1)^2 + C_h[C_kh_{ij} - C_jh_{ik}]/(n+1)^2. \tag{16}$$

Contracting (16) with respect to y^i, and after some simplification, we get

$$\begin{aligned} 0 &= C_h[C_ky_j - C_jy_k]/(n+1)^2,\\ C_ky_j &= C_jy_k. \end{aligned}\tag{17}$$

Theorem 2 *The C-reducible Finsler space and* ν*-curvature tensor* S_{hijk} *satisfies the symmetric property and it holds (17).*

Corollary 1 *Under conformal change, the C-reducible condition and* υ*-curvature tensor also satisfy property (17).*

Example 3 *Let* T_{ij} *be a tensor of (0,2)-type of a two dimensional Finsler space and* $T_{\alpha\beta}$ *be scalar components of* T_{ij} *with respect to the Berwald frame:*

$$T_{ij} = T_{11}l_il_j + T_{12}l_im_j + T_{21}m_il_j + T_{22}m_im_j.$$

If T_{ij} *is symmetric, we have* $T_{12} = T_{21}$*, and if* T_{ij} *is skew-symmetric, then* $T_{0j} = 0$*,* $T_{ij} = 0$*; therefore, by this condition, the* υ*-curvature tensor* S_{hijk} *of* $C\Gamma$ *of any two dimensional Finsler space vanishes identically.*

4 The v-curvature tensor of C3-like conformal Finsler space

Under conformal change, equation (14) can be written as,

$$\overline{S}_{hijk} = e^{2\sigma}[C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik}],$$

using (10), (11), the above equation becomes

$$\begin{aligned}\overline{S}_{hijk} = \; & e^{2\sigma}[\{h_{hk}a_r + C_hC_kb_r + h_{kr}a_h + C_kC_rb_h + h_{rh}a_k + C_rC_hb_k\} \times \\ & \{h_{ij}a^r + C_iC_jb^r + h^r_ja_i + C_jC^rb_i + h^r_ia_j + C^rC_ib_j\} - \\ & \{h_{hj}a_r + C_hC_jb_r + h_{jr}a_h + C_jC_rb_h + h_{rh}a_j + C_rC_hb_j\} \times \\ & \{h_{ik}a^r + C_iC_kb^r + h^r_ka_i + C_kC^rb_i + h^r_ia_k + C^rC_ib_k\}].\end{aligned}$$

After some simplification and rearrangement, we get the following equation:

$$\begin{aligned}\overline{S}_{hijk} = \; & e^{2\sigma}\Bigg[\left\{h_{hk}\left(\frac{a^2}{2}h_{ij} + a_ia_j + C_iC_ja_rb^r + (C_ib_j + C_jb_i)C_a\right)\right\} \times \\ & \left\{h_{ij}\left(\frac{a^2}{2}h_{hk} + a_ha_k + C_hC_ka_rb^r + (C_kb_h + C_hb_k)C_a\right)\right\} - \\ & \left\{h_{hj}\left(\frac{a^2}{2}h_{ik} + a_ia_k + C_iC_ka_rb^r + (C_kb_i + C_ib_k)C_a\right)\right\} \times \\ & \left\{h_{ik}\left(\frac{a^2}{2}h_{hj} + a_ha_j + C_hC_ja_rb^r + (C_jb_h + C_hb_j)C_a\right)\right\} + \\ & C^2(C_ib_h - C_hb_i)(C_kb_j - C_jb_k)\Bigg].\end{aligned}$$

The above equation can be rewritten in the form:

$$\begin{aligned}\overline{S}_{hijk} = \; & e^{2\sigma}[h_{hk}B_{ij} + h_{ij}B_{hk}l - h_{hj}B_{ik} - h_{ik}B_{hj} \\ & + C^2(C_ib_h - C_hb_i)(C_kb_j - C_jb_k)], \qquad (18)\end{aligned}$$

where $B_{ij} = \frac{a^2}{2}h_{ij} + a_ia_j + C_iC_ja_rb^r + (C_ib_j + C_jb_i)C_a$.

Theorem 3 *The conformal v-curvature tensor $\overline{S}_{hijk}$ on a C3-like conformal Finsler space reduces to equation (18).*

References

[1] Kitayama M., Finsler spaces admitting a parallel vector field, *Balkan J. Geometry and its Appl.*, **3** (1998), 29–36.

[2] Matsumoto M., *Foundations of Finsler geometry and special Finsler spaces,* Kaiseisha press, Otsu, Saikawa, 1986.

[3] Narasimhamurthy S.K., Bagewadi C.S. and Nagaraja H.G., Infinitesimal C-conformal motions of special Finsler spaces, *Tensor,* N.S., **64** (2003), 241–247.

[4] Narasimhamurthy S.K. and Bagewadi C.S., C-conformal special Finsler spaces admitting a parallel vector field, *Tensor,* N.S., **65** (2004), 162–169.

[5] Prasad C.M., and Dube O.P., On T-Tensor and ν-curvature tensor of C3-like Finsler space, *Indian J. Pure Appl. Math.,* **23** (1992), 791–795.

[6] Rund H., *The differential Geometry of Finsler spaces,* Springer-Verlag, Berlin, 1959.

Received: May 15, 2009