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On Calderón's conjecture

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1. Introduction

This paper is a successor of [4]. In that paper we considered bilinear operators of the form

$$H_\alpha(f_1, f_2)(x) := \text{p.v.} \int f_1(x-t) f_2(x+\alpha t) \frac{dt}{t}, \tag{1}$$

which are originally defined for f_1, f_2 in the Schwartz class $\mathcal{S}(\mathbb{R})$. The natural question is whether estimates of the form

$$\|H_\alpha(f_1, f_2)\|_p \le C_{\alpha,p_1,p_2} \|f_1\|_{p_1} \|f_2\|_{p_2} \tag{2}$$

with constants C_{α,p_1,p_2} depending only on α, p_1, p_2 and $p := \frac{p_1 p_2}{p_1+p_2}$ hold. The first result of this type is proved in [4], and the purpose of the current paper is to extend the range of exponents p_1 and p_2 for which (2) is known. In particular, the case $p_1 = 2$, $p_2 = \infty$ is solved to the affirmative. This was originally considered to be the most natural case and is known as Calderón's conjecture [3].

We prove the following theorem:

Theorem 1. *Let $\alpha \in \mathbb{R} \setminus \{0, -1\}$ and*

$$1 < p_1, p_2 \le \infty, \tag{3}$$

$$\frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2} < \infty. \tag{4}$$

Then there is a constant C_{α,p_1,p_2} such that estimate (2) *holds for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$.*

If $\alpha = 0, -1, \infty$, then we obtain the bilinear operators

$$H(f_1) \cdot f_2\,,\; H(f_1 \cdot f_2)\,,\; f_1 \cdot H(f_2)\,,$$

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the last one by replacing t with t/α and taking a weak limit as α tends to infinity. Here H is the ordinary linear Hilbert transform, and $\cdot$ is pointwise multiplication. The L^p-bounds of these operators are easy to determine and quite different from those in the theorem. This suggests that the behaviour of the constant C_{α,p_1,p_2} is subtle near the exceptional values of α. It would be of interest to know that the constant is independent of α for some choices of p_1 and p_2.

We do not know that the condition $\frac{2}{3} < p$ is necessary in the theorem. But it is necessary for our proof. An easy counterexample shows that the unconditionality in inequality (6) already requires $\frac{2}{3} \leq p$. The cases of (p_1, p_2) being equal to $(1, \infty)$, $(\infty, 1)$, or (∞, ∞) have to be excluded from the theorem, since the ordinary Hilbert transform is not bounded on L^1 or L^∞.

We assume the reader as somewhat familiar with the results and techniques of [4]. The differences between the current paper and [4] manifest themselves in the overall organization and the extension of the counting function estimates to functions in L^q with $q < 2$.

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2. Preliminary remarks on the exponents

Call a pair (p_1, p_2) good, if for all $\alpha \in \mathbb{R} \setminus \{0, -1\}$ there is a constant C_{α,p_1,p_2} such that estimate (2) holds for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. In this section we discuss interpolation and duality arguments. These, together with the known results from [4], show that instead of Theorem 1 it suffices to prove:

Proposition 1. *If $1 < p_1, p_2 < 2$ and $\frac{2}{3} < \frac{p_1 p_2}{p_1 + p_2}$, then (p_1, p_2) is good.*

In [4] the following is proved:

Proposition 2. *If $2 < p_1, p_2 < \infty$ and $1 < \frac{p_1 p_2}{p_1 + p_2} < 2$, then (p_1, p_2) is good.*

Strictly speaking, this proposition is proved in [4] only in the case $\alpha = 1$, but this restriction is inessential. The necessary modifications to obtain the full result appear in the current paper in Section 3. Therefore we take Proposition 2 for granted.

The next lemma follows by complex interpolation as in [1]. The authors are grateful to E. Stein for pointing out this reference to them.

Lemma 1. *Let $1 < p_1, p_2, q_1, q_2 \leq \infty$ and assume that (p_1, p_2) and (q_1, q_2) are good. Then*

$$\left(\frac{\theta}{p_1} + \frac{1-\theta}{q_1}, \frac{\theta}{p_2} + \frac{1-\theta}{q_2}\right)$$

is good for all $0 < \theta < 1$.

Next we need a duality lemma.

LEMMA 2. *Let $1 < p_1, p_2 < \infty$ such that $\frac{p_1p_2}{p_1+p_2} \geq 1$. If (p_1, p_2) is good, then so are the pairs*

$$\left(p_1, \left(\frac{p_1p_2}{p_1+p_2}\right)'\right) \quad \text{and} \quad \left(\left(\frac{p_1p_2}{p_1+p_2}\right)', p_2\right).$$

Here p' denotes as usual the dual exponent of p. To prove the lemma, fix $\alpha \in \mathbb{R} \setminus \{0, -1\}$ and $f_1 \in \mathcal{S}(\mathbb{R})$ and consider the linear operator $H_\alpha(f_1, \_)$. The formal adjoint of this operator with respect to the natural bilinear pairing is

$$\operatorname{sgn}(1+\alpha) H_{-\frac{\alpha}{1+\alpha}}(f_1, \_),$$

as the following lines show:

$$\begin{aligned}
&\int \left(\text{p.v.} \int f_1(x-t) f_2(x+\alpha t) \frac{1}{t}\, dt\right) f_3(x)\, dx \\
&\qquad = \text{p.v.} \int\int f_1(x-\alpha t - t) f_2(x) f_3(x-\alpha t)\, dx\, \frac{1}{t} dt \\
&\qquad = \operatorname{sgn}(1+\alpha) \int \left(\text{p.v.} \int f_1(x-t) f_3(x - \frac{\alpha}{1+\alpha} t) \frac{1}{t}\, dt\right) f_2(x)\, dx.
\end{aligned}$$

Similarly, we observe that for fixed f_2 the formal adjoint of $H_\alpha(\_, f_2)$ is $-H_{-1-\alpha}(\_, f_2)$. This proves Lemma 2 by duality.

Now we are ready to prove estimate (2) in the remaining cases, i.e., for those pairs (p_1, p_2) for which one of p_1, p_2 is smaller or equal two, and the other one is greater or equal two. In this case the constraint on p is automatically satisfied. By symmetry it suffices to do this for $p_1 \in]1, 2]$ and $p_2 \in [2, \infty]$. First observe that the pairs $(3, 3)$ and $(3/2, 3/2)$ are good by the above propositions. Then the pairs $(2, 2)$ and $(2, \infty)$ are good by interpolation and duality. Let P be the set of all $p_1 \in]1, 2]$ such that the pair (p_1, p_2) is good for all $p_2 \in [2, \infty]$. The previous observations show that $2 \in P$. Define $p := \inf P$ and assume $p > 1$. Pick a small $\varepsilon > 0$ and a $p_1 \in P$ with $p_1 < p + \varepsilon$. If ε is small enough, we can interpolate the good pairs (p_1, ε^{-1}) and $(1+\varepsilon, 2-\varepsilon)$ to obtain a good pair of the form $(q_\varepsilon, q_\varepsilon')$. Since $\lim_{\varepsilon \to 0} q_\varepsilon = \frac{3p-2}{2p-1} < p$ we have $q_\varepsilon < p$ provided ε is small enough. By duality we see that the pair (q, ∞) is good, and by Proposition 1 there is a $p_2 < 2$ such that (q, p_2) is good. By interpolation $q \in P$ follows. This is a contradiction to $p = \inf P$; therefore the assumption $p > 1$ is false and we have $\inf P = 1$. Again by interpolation we observe $P =]1, 2]$, which finishes the prove of estimate (2) for the remaining exponents.

3. Time-frequency decomposition of H_α

In this section we write the bilinear operators H_α approximately as finite sums over rank one operators, each rank one operator being well localized in time and frequency. We mostly follow the corresponding section in [4], adopting the basic notation and definitions from there such as that of a phase plane representation.

In contrast to [4] we work out how the decomposition and the constants depend on α, and we add an additional assumption (iv) in Proposition 3 which is necessary to prove L^p- estimates for $p < 2$. The reader should think of the functions $\theta_{\xi,\imath}$ in this assumption as being exponentials $\theta_{\xi,\imath}(x) = e^{i\eta_\imath x}$ for certain frequencies $\eta_\imath = \eta_\imath(\xi)$.

PROPOSITION 3. *Assume we are given exponents* $1 < p_1, p_2 < 2$ *such that* $\frac{p_1 p_2}{p_1+p_2} > \frac{2}{3}$, *and we are given a constant* C_m *for each integer* $m \geq 0$. *Then there is a constant* C *depending on these data such that the following holds*:

Let S *be a finite set,* $\phi_1, \phi_2, \phi_3 : S \to \mathcal{S}(\mathbb{R})$ *be injective maps, and* $I, \omega_1, \omega_2, \omega_3 : S \mapsto \mathcal{J}$ *be maps such that* $I(S)$ *is a grid,* $\mathcal{J}_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S)$ *is a grid, and the following properties* (i)–(iv) *hold for all* $\imath \in \{1, 2, 3\}$:

(i) *The map*

$$\rho_\imath : \phi_\imath(S) \to \mathcal{R},\ \phi_\imath(s) \mapsto I(s) \times \omega_\imath(s)$$

is a phase plane representation with constants C_m.

(ii) $\omega_\imath(s) \cap \omega_\jmath(s) = \emptyset$ *for all* $s \in S$ *and* $\jmath \in \{1, 2, 3\}$ *with* $\imath \neq \jmath$.

(iii) *If* $\omega_\imath(s) \subset J$ *and* $\omega_\imath(s) \neq J$ *for some* $s \in S$, $J \in \mathcal{J}_\omega$, *then* $\omega_\jmath(s) \subset J$ *for all* $\jmath \in \{1, 2, 3\}$.

(iv) *To each* $\xi \in \mathbb{R}$ *there is associated a measurable function* $\theta_{\xi,\imath} : \mathbb{R} \to \{z \in \mathbb{C} : |z| = 1\}$ *such that for all* $s \in S$, $\jmath \in \{1, 2, 3\}$ *and* $J \in I(S)$ *the following holds: If* $\xi \in \omega_\jmath(s)$, $|J| \leq |I(s)|$, *then*

$$\inf_{\lambda \in \mathbb{C}} \|\phi_\imath(s) - \lambda\theta_{\xi,\imath}\|_{L^\infty(J)} \leq C_0 |J|\, |I(s)|^{-\frac{3}{2}} \left(1 + \frac{|c(J) - c(I(s))|}{|I(s)|}\right)^{-2}. \tag{5}$$

For all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ *and all maps* $\varepsilon : S \to [-1, 1]$, *we then have*:

$$\left\| \sum_{s \in S} \varepsilon(s) |I(s)|^{-\frac{1}{2}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s) \right\|_{\frac{p_1 p_2}{p_1+p_2}} \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}. \tag{6}$$

In the rest of this section we prove Proposition 2 under the assumption that Proposition 3 above is true. Let $1 < p_1, p_2 < 2$ with $\frac{2}{3} < p := \frac{p_1 p_2}{p_1+p_2}$ and $\alpha \in \mathbb{R} \setminus \{0, -1\}$.

Let L be the smallest integer larger than

$$2^{10} \max\left\{|\alpha|, \frac{1}{|\alpha|}, \frac{1}{|1+\alpha|}\right\}.$$

The dependence on α will enter into our estimate via a polynomial dependence on L.

Define $\varepsilon := L^{-3}$. Pick a function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi}$ is supported in $[L^3-1, L^3+1]$ and

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(2^{\varepsilon k}\xi) = 1 \quad \text{for} \quad \text{all} \quad \xi > 0.$$

Define

$$\psi_k(x) := 2^{-\frac{\varepsilon k}{2}} \psi(2^{-\varepsilon k}x)$$

and

$$\tilde{H}_\alpha(f_1, f_2)(x) := \sum_{k\in\mathbb{Z}} 2^{-\frac{\varepsilon k}{2}} \int_{\mathbb{R}} f_1(x-t) f_2(x+\alpha t)\psi_k(t)dt. \tag{7}$$

It suffices to prove boundedness of $\tilde{H}_\alpha$. Pick a $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\varphi}$ is supported in $[-1,1]$ and

$$\sum_{n,l\in\mathbb{Z}} \left\langle f, \varphi_{k,n,\frac{l}{2}} \right\rangle \varphi_{k,n,\frac{l}{2}} = f \tag{8}$$

for all Schwartz functions f, where

$$\varphi_{\kappa,n,l}(x) := 2^{-\frac{\varepsilon\kappa}{2}}\, \varphi(2^{-\varepsilon\kappa}x - n)\, e^{2\pi i 2^{-\varepsilon\kappa} x l}.$$

We apply this formula three times in (7) to obtain:

$$\tilde{H}_\alpha(f_1, f_2)(x) = \sum_{k,n_1,n_2,n_3,l_1,l_2,l_3\in\mathbb{Z}} C_{k,n_1,n_2,n_3,l_1,l_2,l_3} H_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2)(x) \tag{9}$$

with

$$H_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2)(x) := 2^{-\frac{\varepsilon k}{2}} \left\langle f_1, \varphi_{k,n_1,\frac{l_1}{2}} \right\rangle \left\langle f_2, \varphi_{k,n_2,\frac{l_2}{2}} \right\rangle \varphi_{k,n_3,\frac{l_3}{2}}(x)$$

and

$$C_{k,n_1,n_2,n_3,l_1,l_2,l_3} := \int\int \varphi_{k,n_1,\frac{l_1}{2}}(x-t) \varphi_{k,n_2,\frac{l_2}{2}}(x+\alpha t) \varphi_{k,n_3,\frac{l_3}{2}}(x) \psi_k(t)\, dt\, dx \quad . \tag{10}$$

The proof of the following lemma is a straightforward calculation as in [4].

Lemma 3. *There is a constant C depending on ϕ and ψ such that*

$$|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| \le C\left(1 + \frac{1}{L}\mathrm{diam}\{n_1, n_2, n_3\}\right)^{-100}. \tag{11}$$

Moreover,

$$C_{k,n_1,n_2,n_3,l_1,l_2,l_3} = 0,$$

unless

$$l_1 \in \left[\left(-\frac{\alpha}{1+\alpha}l_3 + \frac{2}{1+\alpha}L^3\right) - L, \left(-\frac{\alpha}{1+\alpha}l_3 + \frac{2}{1+\alpha}L^3\right) + L\right] \tag{12}$$

and

$$l_2 \in \left[\left(-\frac{1}{1+\alpha}l_3 - \frac{2}{1+\alpha}L^3\right) - L, \left(-\frac{1}{1+\alpha}l_3 - \frac{2}{1+\alpha}L^3\right) + L\right]. \tag{13}$$

Now we can reduce Proposition 2 to the following lemma:

LEMMA 4. *There is a constant C depending on p_1, p_2, φ, and ψ such that the following holds*:

Let $\nu > 0$ be an integer and let S be a finite subset of $\mathbb{Z}^3$ such that for $(k,n,l),(k',n',l') \in S$ the following three properties are satisfied:

$$\text{If } k \neq k', \quad \text{then } |k-k'| > L^{10}, \tag{14}$$

$$\text{if } n \neq n', \quad \text{then } |n-n'| > L^{10}\nu, \tag{15}$$

$$\text{if } l \neq l', \quad \text{then } |l-l'| > L^{10}. \tag{16}$$

Let ν_1, ν_2 be integers with $1 + \max\{|\nu_1|,|\nu_2|\} = \nu$ and let $\lambda_1, \lambda_2 : \mathbb{Z} \to \mathbb{Z}$ be functions such that $l_1 := \lambda_1(l_3)$ satisfies (12) *and $l_2 := \lambda_2(l_3)$ satisfies* (13) *for all $l_3 \in \mathbb{Z}$. Then we have for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ and all maps $\varepsilon : S \to [-1,1]$*:

(17)

$$\left\| \sum_{(k,n,l)\in S} \varepsilon(k,n,l) H_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}(f_1,f_2) \right\|_p \leq CL^{30}\nu^{10}\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

Before proving the lemma we show how it implies boundedness of $\tilde{H}_\alpha$ and therefore proves Proposition 2. First observe that the lemma also holds without the finiteness condition on S. We can also remove conditions (14), (15), and (16) on S at the cost of some additional powers of L and ν, so that the conclusion of the lemma without these hypotheses is

(18)

$$\left\| \sum_{(k,n,l)\in S} \varepsilon(k,n,l) H_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}(f_1,f_2) \right\|_p \leq CL^{100}\nu^{20}\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

Here we have used the quasi triangle inequality for L^p which is uniform for $p > \frac{2}{3}$.

Observe that (18) and (11) imply

$$\Big\| \sum_{(k,n,l)\in S} C_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l} H_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}(f_1,f_2) \Big\|_p \le C L^{200} \nu^{-50} \|f_1\|_{p_1} \|f_2\|_{p_2}. \tag{19}$$

Conditions (12) and (13) give a bound on the number of values the functions λ_1 and λ_2 can take at a fixed l_3 so that the coefficient $C_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}$ does not vanish. Moreover there are of the order ν pairs ν_1,ν_2 such that $1+\max\{|\nu_1|,|\nu_2|\}=\nu$. Hence,

$$\Big\| \sum_{(k,n,l)\in S, n_1,n_2,l_1,l_2\in\mathbb{Z}, 1+\max\{|n-n_1|,|n-n_2|\}=\nu} C_{k,n_1,n_2,n,l_1,l_2,l} H_{k,n_1,n_2,n,l_1,l_2,l}(f_1,f_2) \Big\|_p \le C L^{300} \nu^{-20} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Summing over all ν gives boundedness of $\tilde{H}_\alpha$.

It remains to prove Lemma 4. Clearly we intend to do this by applying Proposition 3. Fix data $S,\nu,\nu_1,\nu_2,\lambda_1,\lambda_2$ as in Lemma 4. Define functions $\phi_\imath : S \mapsto \mathcal{S}(\mathbb{R})$ as follows:

$$\begin{aligned} \phi_1(k,n,l) &:= L^{-10}\nu^{-2}\varphi_{k,n+\nu_1,\frac{\lambda_1(l)}{2}}, \\ \phi_2(k,n,l) &:= L^{-10}\nu^{-2}\varphi_{k,n+\nu_2,\frac{\lambda_2(l)}{2}}, \\ \phi_3(k,n,l) &:= L^{-10}\nu^{-2}\varphi_{k,n,\frac{l}{2}}. \end{aligned}$$

If E is a subset of $\mathbb{R}$ and $x \neq 0$ a real number we use the notation $x \cdot E := \{xy \in \mathbb{R} : y \in E\}$. This is not to be confused with the previously defined xI for positive x and intervals I. Pick three maps $\omega_1,\omega_2,\omega_3 : S \to \mathcal{J}$ such that the following properties (20)–(25) are satisfied for all $s=(k,n,l)\in S$:

$$-\frac{1+\alpha}{\alpha} \cdot \operatorname{supp}(\widehat{\phi_1(s)}) \subset \omega_1(s), \tag{20}$$

$$-(1+\alpha) \cdot \operatorname{supp}(\widehat{\phi_2(s)}) \subset \omega_2(s), \tag{21}$$

$$\operatorname{supp}(\widehat{\phi_3(s)}) \subset \omega_3(s), \tag{22}$$

$$2^{-\varepsilon(k+1)} L \le |\omega_\imath(s)| \le 2^{-\varepsilon k} L \ \text{ for } \ \imath = 1,2,3, \tag{23}$$

$$\mathcal{J}_\omega := \omega_1(S)\cup\omega_2(S)\cup\omega_3(S) \ \text{ is a grid}, \tag{24}$$

and, for all $\imath,\jmath \in \{1,2,3\}$,

$$\text{If } \omega_\imath(s) \subset J \text{ and } \omega_\imath(s) \neq J \text{ for some } J \in \mathcal{J}_\omega, \text{ then } \omega_\jmath(s) \subset J. \tag{25}$$

The existence of such a triple of maps is proved as in [4].

Next pick a map $I : S \to \mathcal{J}$ which satisfies the following three properties (26)–(28) for all $s = (k, n, l) \in S$:

$$|c(I(s)) - 2^{\varepsilon k} n| \leq 2^{\varepsilon k}\nu, \tag{26}$$

$$2^4 2^{\varepsilon k}\nu \leq |I(s)| \leq 2^\varepsilon 2^4 2^{\varepsilon k}\nu, \tag{27}$$

$$I(S) \text{ is a grid.} \tag{28}$$

The existence of such a map is again proved as in [4].

Now Lemma 4 follows immediately from the fact that the data S, ϕ_1, ϕ_2, ϕ_3, I, ω_1, ω_2, and ω_3 satisfy the hypotheses of Proposition 3. The verification of these hypotheses is as in [4] except for hypothesis (iv).

We prove hypothesis (iv) for $\imath = 1$, the other cases being similar. Define for $\xi \in \mathbb{R}$:

$$\theta_{\xi,1}(x) := e^{-2\pi i \frac{\alpha}{\alpha+1} x\xi}.$$

Pick $s = (k, n, l) \in S$. Obviously,

$$\nu^{-2}\varphi_{k,n+\nu_1,0}(x) \leq C|I(s)|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|}\right)^{-2}$$

and

$$\nu^{-2}(\varphi_{k,n+\nu_1,0})'(x) \leq C|I(s)|^{-\frac{3}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|}\right)^{-2}.$$

Now let $\xi \in \omega_\jmath(s)$. By choice of $\theta_{\xi,1}$ we see that the function

$$\varphi_{k,n+\nu_1,\frac{\lambda_1(l)}{2}}\theta_{\xi,1}^{-1}$$

arises from $\varphi_{k,n+\nu_1,0}$ by modulating with a frequency which is contained in $L^{10}[-|I(s)|^{-1}, |I(s)|^{-1}]$. Therefore,

$$(\phi_1(s)\theta_{\xi,1}^{-1})'(x) \leq C|I(s)|^{-\frac{3}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|}\right)^{-2}.$$

Now let $J \in I(S)$ with $|J| \leq |I(s)|$. Then we have

$$\begin{aligned} \inf_\lambda \|\phi_1(s)\theta_{\xi,1}^{-1} - \lambda\|_{L^\infty(J)} &\leq |J| \left\| \left(\phi_1(s)\theta_{\xi,1}^{-1}\right)' \right\|_{L^\infty(J)} \\ &\leq C|J||I(s)|^{-\frac{3}{2}} \left(1 + \frac{|c(J) - c(I(s))|}{|I(s)|}\right)^{-2}. \end{aligned}$$

This proves hypothesis (iv), and therefore finishes the reduction of Proposition 2 to Proposition 3.

4. Reduction to a symmetric statement

The following proposition is a variant of Proposition 3 which is symmetric in the indices 1, 2, and 3.

PROPOSITION 4. *Let $1 < p_1, p_2, p_3 < 2$ be exponents with*

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2$$

and let $C_m > 0$ for $m \geq 0$. Then there are constants $C, \lambda_0 > 0$ such that the following holds: Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3$ be as in Proposition 3, let $f_\imath$, $\imath = 1, 2, 3$ be Schwartz functions with $\|f_\imath\|_{p_\imath} = 1$, and define

$$E := \left\{ x \in \mathbb{R} : \max_\imath \left(M_{p_\imath}(Mf_\imath)(x) \right) \geq \lambda_0 \right\}.$$

Then we have

$$\sum_{s \in S : I(s) \not\subset E} |I(s)|^{-\frac{1}{2}} \left| \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \langle f_3, \phi_3(s) \rangle \right| \leq C.$$

We now prove that Proposition 3 follows from Proposition 4.

Let $1 < p_1, p_2 < 2$ and assume

$$p := \frac{p_1 p_2}{p_1 + p_2} > \frac{2}{3}.$$

Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3, \varepsilon$ be as in the proposition and define for each $S' \subset S$

$$H_{S'}(f_1, f_2) = \sum_{s \in S'} \varepsilon(s) |I|^{-\frac{1}{2}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s).$$

By Marcinkiewicz interpolation ([2]), it suffices to prove a corresponding weak-type estimate instead of (6). By linearity and scaling invariance it suffices to prove that there is a constant C such that for $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ we have

$$|\{x \in \mathbb{R} : |H_S(f_1, f_2)(x)| \geq 2\}| \leq C.$$

Pick an exponent p_3 such that the triple p_1, p_2, p_3 satisfies the conditions of Proposition 4, and let λ_0 be as in this proposition. Let f_1 and f_2 be Schwartz functions with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$.

Define

$$E_0 := \{x : \max\{M_{p_1}(Mf_1)(x), M_{p_2}(Mf_2)(x)\} \geq \lambda_0\}.$$

and

$$E_{\text{in}} := \left\{ x \in R : \left| H_{\{s \in S : I(s) \subset E_0\}}(f_1, f_2)(x) \right| \geq 1 \right\},$$
$$E_{\text{out}} := \left\{ x \in R : \left| H_{\{s \in S : I(s) \not\subset E_0\}}(f_1, f_2)(x) \right| \geq 1 \right\}.$$

It suffices to bound the measures of E_{in} and E_{out} by constants. We first estimate that of E_{out} using Proposition 4 . Let $\delta > 0$ be a small number and let $\theta : [0,\infty) \to [0,1]$ be a smooth function which vanishes on the interval $[0, 1-\delta]$ and is constant equal to 1 on $[1,\infty)$. Extend this function to the complex plane by defining in polar coordinates $\theta(re^{i\phi}) := \theta(r)e^{-i\phi}$. Assume that δ is chosen sufficiently small to give

$$|E_{\text{out}}|^{\frac{1}{p_3}} \leq \left\| \theta \left(H_{\{s\in S: I(s)\not\subset E_0\}}(f_1, f_2) \right) \right\|_{p_3} \leq 2|E_{\text{out}}|^{\frac{1}{p_3}}.$$

Define

$$f_3 := \frac{\theta \left(H_{\{s\in S: I(s)\not\subset E_0\}}(f_1, f_2) \right)}{\left\| \theta \left(H_{\{s\in S: I(s)\not\subset E_0\}}(f_1, f_2) \right) \right\|_{p_3}}.$$

We can assume that $|E_{\text{out}}| > \lambda_0^{-p_3}$, because otherwise nothing is to prove. This assumption implies $\|M_{p_3}(Mf_3)\|_\infty < \lambda_0$. By applying Proposition 4, we obtain:

$$|E_{\text{out}}|^{1-\frac{1}{p_3}} \leq 2 \left| \int H_{\{s\in S: I(s)\not\subset E_0\}}(f_1, f_2)(x) f_3(x)\, dx \right| \leq C.$$

Therefore $|E_{\text{out}}|$ is bounded by a constant.

It remains to estimate the measure of the set E_{in}, which is an elementary calculation. We need the following lemma:

Lemma 5. *Let J be an interval and define*

$$S_J := \{s \in S : I(s) = J\}.$$

Then for all $m > 0$ there is a C_m such that for all $A > 1$ and $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ we have:

$$\|H_{S_J}(f_1, f_2)\|_{L^1((AJ)^c)} \leq C_m |J| A^{-m} \left(\inf_{x\in J} M_{p_1} f_1(x) \right) \left(\inf_{x\in J} M_{p_2} f_2(x) \right).$$

We prove the lemma for $|J| = 1$, which suffices by homogeneity. For $m \geq 0$ define the weight

$$w_m(x) := (1 + \operatorname{dist}(x, J))^m .$$

Then for $1 \leq r < 2$ we obtain the estimates

$$\| \sum_{s\in S_J} \alpha_s \phi_\imath(s) \|_{L^{r'}(\omega_m)} \leq C_m \left\| (\alpha_s)_{s\in S_J} \right\|_{l^r(S_J)} \tag{29}$$

and

$$\left\| (\langle f, \phi_\imath(s) \rangle)_{s\in S_J} \right\|_{l^{r'}(S_J)} \leq C_m \|f\|_{L^r(\omega_m^{-1})}, \tag{30}$$

which follow easily by interpolation ([6]) from the trivial weighted estimate at $r = 1$ and the nonweighted estimate at $r = 2$.

Now define r by

$$\frac{1}{r} = \frac{1}{p_1{}'} + \frac{1}{p_2{}'};$$

in particular we have $1 < r < 2$. By writing $H_{S_J}(f_1, f_2) = (H_{S_J}(f_1, f_2) w_m^{\frac{1}{r'}}) w_m^{-\frac{1}{r'}}$ and applying Hölder we have for large m:

$$\|H_{S_J}(f_1, f_2)\|_{L^1((AJ)^c)} \leq C_M A^{-M} \|H_{S_J}(f_1, f_2)\|_{L^{r'}(w_m)} .$$

Here M depends on m and r and can be made arbitrarily large by picking m accordingly. By estimates (29) and (30) we can estimate the previously displayed expression further by

$$\begin{aligned}
&\leq C_M A^{-M} \left\| (\langle f_1, \phi_1(s)\rangle \langle f_2, \phi_2(s)\rangle)_{s \in S_J} \right\|_{l^r(S_J)} \\
&\leq C_M A^{-M} \left\| (\langle f_1, \phi_1(s)\rangle)_{s \in S_J} \right\|_{l^{p_1{}'}(S_J)} \left\| (\langle f_2, \phi_2(s)\rangle)_{s \in S_J} \right\|_{l^{p_2{}'}(S_J)} \\
&\leq C_M A^{-M} \|f_1\|_{L^{p_1}(w_{10}^{-1})} \|f_2\|_{L^{p_2}(w_{10}^{-1})} \\
&\leq C_M A^{-M} \left(\inf_{x \in J} M_{p_1} f_1(x)\right) \left(\inf_{x \in J} M_{p_2} f_2(x)\right).
\end{aligned}$$

This finishes the proof of Lemma 5.

We return to the estimate of the set E_{in}. Define

$$E' := E_0 \cup \bigcup_{J \in I(S): J \subset E} 4J.$$

Since $|E'| \leq 5|E_0| \leq C$, it suffices to prove

$$\|H_{\{s \in S: I(s) \subset E_0\}}(f_1, f_2)\|_{L^1(E'^c)} \leq C. \tag{31}$$

Fix $k > 1$ and define

$$\mathcal{I}_k := \{J \in I(S) : J \subset E_0, 2^k J \subset E', 2^{k+1} J \not\subset E'\}.$$

Let $J \in \mathcal{I}_k$. Then for $\imath = 1, 2$ we have:

$$\inf_{x \in J} M_{p_\imath} f_\imath(x) \leq 2^{k+1} \inf_{x \in 2^{k+1} J} M_{p_\imath} f_\imath(x) \leq 2^{k+1},$$

since outside the set E' the maximal function is bounded by 1. Hence, by the previous lemma,

$$\|H_{S_J}(f_1, f_2)\|_{L^1((E')^c)} \leq C_m |J| 2^{-km}.$$

Since $I(S)$ is a grid, it is easy to see that the intervals in $\mathcal{I}_k$ are pairwise disjoint; hence we have

$$\left\| H_{\{s \in S, I(s) \in \mathcal{I}_k\}}(f_1, f_2) \right\|_{L^1((E')^c)} \leq C_m |E_0| 2^{-km}.$$

By summing over all $k > 1$ we prove (31). This finishes the estimate of the set $|E_{\text{in}}|$ and therefore the reduction of Proposition 3 to Proposition 4.

5. The combinatorics on the set S

We prove Proposition 4. Let $1 < p_1, p_2, p_3 < 2$ be exponents with

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2.$$

Let $\eta > 0$ be the largest number such that $\frac{1}{\eta}$ is an integer and

$$\eta \leq 2^{-100} \left(2 - \sum_{\imath} \frac{1}{p_\imath}\right) \min_{\jmath} \left(1 - \frac{1}{p_\jmath}\right).$$

Let S, ϕ_1, ϕ_2, ϕ_3, I, ω_1, ω_2, and ω_3 be as in Propositions 3 and 4. Let $f_\imath$, $\imath = 1, 2, 3$ be Schwartz functions with $\|f_\imath\|_{p_\imath} = 1$. Without loss of generality we can assume that for all $s \in S$,

$$I(s) \not\subset \left\{x \in \mathbb{R} : \max_{\imath} \left(M_{p_\imath}(Mf_\imath)(x)\right) \geq \lambda_0\right\}, \tag{32}$$

where λ_0 is a constant which we will specify later.

Define a partial order $\ll$ on the set of rectangles by

$$J_1 \times J_2 \ll J_1' \times J_2' \ , \ \text{if } \ J_1 \subset J_1' \ \text{and } \ J_2' \subset J_2. \tag{33}$$

A subset $T \subset S$ is called a *tree of type* $\imath$, if the set $\rho_\imath(T)$ has exactly one maximal element with respect to $\ll$. This maximal element is called the *base* of the tree T and is denoted by s_T. Define $J_T := I(s_T)$.

Define $S_{-1} := S$. Let $k \geq 0$ be an integer and assume by recursion that we have already defined S_{k-1}. Define

$$S_k := S_{k-1} \setminus \bigcup_{\imath,\jmath=1}^{3} \left(\bigcup_{l=0}^{\infty} T_{k,\imath,\jmath,l}\right),$$

where the sets $T_{k,\imath,\jmath,l}$ are defined as follows. Let $k \geq 0$ and $\imath, \jmath \in \{1, 2, 3\}$ be fixed. Let $l \geq 0$ be an integer and assume by recursion that we have already defined $T_{k,\imath,\jmath,\lambda}$ for all integers λ with $0 \leq \lambda < l$. If one of the sets $T_{k,\imath,\jmath,\lambda}$ with $\lambda < l$ is empty, then define $T_{k,\imath,\jmath,l} := \emptyset$. Otherwise let $\mathcal{F}$ denote the set of all trees T of type $\imath$ which satisfy the following conditions (34)–(36):

$$T \subset S_{k-1} \setminus \bigcup_{\lambda < l} T_{k,\imath,\jmath,\lambda}, \tag{34}$$

$$\text{if } \imath = \jmath, \quad \text{then} \quad |\langle f_\jmath, \phi_\jmath(s)\rangle| \geq 2^{-\eta k} 2^{-\frac{k}{p_\jmath'}} |I(s)|^{\frac{1}{2}} \quad \text{for all } s \in T, \tag{35}$$

$$\text{if } \imath \neq \jmath, \quad \text{then} \quad \left\| \left(\sum_{s \in T} \frac{|\langle f_\jmath, \phi_\jmath(s)\rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \geq 2^4 2^{-\frac{k}{p_\jmath'}} |J_T|. \tag{36}$$

If $\mathcal{F}$ is empty, then we define $T_{k,\imath,\jmath,l} := \emptyset$. Otherwise define $\mathcal{F}_{\max}$ to be the set of all $T_{\max} \in \mathcal{F}$ which satisfy:

$$\text{if } T \in \mathcal{F} \ , \ T_{\max} \subset T \ , \ \text{then } T = T_{\max}. \tag{37}$$

Choose $T_{k,\imath,\jmath,l} \in \mathcal{F}_{\max}$ such that for all $T \in \mathcal{F}_{\max}$,

$$\text{(38)} \qquad \text{if } \imath < \jmath, \qquad \text{then } \omega_\imath(s_{T_{k,\imath,\jmath,l}}) \not< \omega_\imath(s_T),$$

$$\text{(39)} \qquad \text{if } \imath > \jmath, \qquad \text{then } \omega_\imath(s_T) \not< \omega(s_{T_{k,\imath,\jmath,l}}).$$

Here $[a,b[\not< [a',b'[$ means $b > a'$. Observe that $T_{k,\imath,\jmath,l}$ actually satisfies (38) and (39) for all $T \in \mathcal{F}$. This finishes the definition of the sets $T_{k,\imath,\jmath,l}$ and S_k.

Since S is finite, $T_{k,\imath,\jmath,l} = \emptyset$ for sufficiently large l. In particular, each $s \in S_k$ satisfies

$$\text{(40)} \qquad |\langle f_\imath, \phi_\imath(s)\rangle| \le 2^{-\eta k} 2^{-\frac{k}{p_\imath{}'}} |I(s)|^{\frac{1}{2}}$$

for all $\imath$, since the set $\{s\}$ is a tree of type $\imath$ which by construction of S_k does not satisfy (35) for $\jmath = \imath$. Similarly for $j \neq i$ each tree $T \subset S_k$ of type $\imath$ satisfies

$$\text{(41)} \qquad \left\| \left(\sum_{s\in T} \frac{|\langle f_\jmath, \phi_\jmath(s)\rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \le 2^4 2^{-\frac{k}{p_\jmath{}'}} |J_T|.$$

Moreover, (40) implies that the intersection of all S_k contains only elements s with $\prod_j \langle f_\jmath, \phi_\jmath(s)\rangle = 0$.

Let $k \le \eta^{-2}$ and assume $T_{k,\imath,\jmath,l}$ is a tree. Observe that (35) and (36) together with Lemma 6 in Section 7 provide a lower bound on the maximal function $M_{p_\jmath}(Mf_\jmath)(x)$ for $x \in J_{T_{k,\imath,\jmath,l}}$. This lower bound depends only on η, $p_\jmath$ and the constants C_m of the phase plane representation. Therefore if we choose the constant λ_0 in (32) small enough depending on η, $p_\jmath$, and C_m, it then is clear that $T_{k,\imath,\jmath,l} = \emptyset$ for $k \le \eta^{-2}$.

Now we have

$$\sum_{s\in S} |I(s)|^{-\frac{1}{2}} \prod_j |\langle f_\jmath, \phi_\jmath(s)\rangle| \quad \le \quad \sum_{k>\eta^{-2}} \sum_{\imath,\jmath} \sum_{l=0}^{\infty} \left(\sup_{s\in T_{k,\imath,\jmath,l}} |I(s)|^{-\frac{1}{2}} |\langle f_\imath, \phi_\imath(s)\rangle| \right)$$
$$\times \prod_{\kappa\neq\imath} \left(\sum_{s\in T_{k,\imath,\jmath,l}} |\langle f_\kappa, \phi_\kappa(s)\rangle|^2 \right)^{\frac{1}{2}}.$$

Using (40), (41) and Lemma 7 of Section 7 we can bound this by

$$\le C \sum_{k>\eta^{-2}} 2^{-\sum_\jmath \frac{k}{p_\jmath{}'}} \sum_{\imath,\jmath} \sum_{l=0}^{\infty} |J_{T_{k,\imath,\jmath,l}}|.$$

Now we apply the estimate

$$\text{(42)} \qquad \sum_{l=0}^{\infty} |J_{T_{k,\imath,\jmath,l}}| \le C 2^{10\eta p_\jmath{}' k} 2^k,$$

for each $k > \eta^{-2}, \imath\jmath$, which is proved in Sections 6 and 8. This bounds the previously displayed expression by

$$\leq C \sum_{k>\eta^{-2}} 2^{-\sum \frac{k}{p_\jmath{}'}} 2^{10\eta p_\jmath{}' k} 2^k. \tag{43}$$

This is less than a constant since

$$\sum_\jmath \frac{1}{p_\jmath{}'} \geq 1 + 10\eta \max_\jmath p_\jmath{}'$$

by the choice of η. This finishes the proof of Proposition 4 up to the proof of estimate (42) and Lemmata 6 and 7.

6. Counting the trees for $\imath = \jmath$

We prove estimate (42) in the case $\imath = \jmath$. Thus fix $k > \eta^{-2}, \imath, \jmath$ with $\imath = \jmath$. Let $\mathcal{F}$ denote the set of all trees $T_{k,\imath,\jmath,l}$. Observe that for $T, T' \in \mathcal{F}, T \neq T'$ we have, by (37), that $T \cup T'$ is not a tree; therefore

$$\rho_\imath(s_T) \cap \rho_\imath(s_{T'}) = \emptyset.$$

Define $b := 2^{-\eta k} 2^{-\frac{k}{p_\imath{}'}}$. Then by (35) for all $T \in \mathcal{F}$

$$|\langle f_\imath, \phi_\imath(s_T)\rangle| \geq b|J_T|^{\frac{1}{2}}. \tag{44}$$

Finally recall that for all $s \in S$:

$$I(s) \not\subset \{x : M_{p_\imath}(Mf_\imath)(x) \geq \lambda_0\}. \tag{45}$$

Our proof goes in the following four steps:

Step 1. Define the counting function

$$N_\mathcal{F}(x) := \sum_{T\in\mathcal{F}} 1_{J_T}(x). \tag{46}$$

We have to estimate the L^1-norm of the counting function. Since the counting function is integer-valued, it suffices to show a weak-type $1 + \varepsilon$ estimate for small ε. More precisely it suffices to show for all integers $\lambda \geq 1$ and sufficiently small $\delta, \varepsilon > 0$, $\delta = \delta(\eta, p_\imath), \varepsilon = \varepsilon(\eta, p_\imath)$:

$$|\{x \in \mathbb{R} : N_\mathcal{F}(x) \geq \lambda\}| \leq b^{-p_\imath{}' - \delta} \lambda^{-1-\varepsilon}.$$

Fix such a λ. As in [4] there is a subset $\mathcal{F}' \subset \mathcal{F}$ such that, if we define $N_{\mathcal{F}'}$ analogously to $N_\mathcal{F}$,

$$\{x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq \lambda\} = \{x \in \mathbb{R} : N_\mathcal{F}(x) \geq \lambda\}$$

and $\|N_{\mathcal{F}'}\|_\infty \leq \lambda$. This is due to the grid structure of $I(S)$.

Step 2. Let $A > 1$ be a number whose value will be specified later. We can write

$$\mathcal{F}' = \left(\bigcup_{m=1}^{A^{10}} \mathcal{F}_m \right) \cup \mathcal{F}'' \tag{47}$$

such that if $T, T' \in \mathcal{F}_m$ for some m and $T \neq T'$, then

$$(AJ_T \times \omega(s_T)) \cap (AJ_{T'} \times \omega(s_{T'})) = \emptyset,$$

and

$$\sum_{T \in \mathcal{F}''} |J_T| \leq Ce^{-A} \sum_{T \in \mathcal{F}_1} |J_T|. \tag{48}$$

For a proof of this fact see the proof of the separation lemma in [4].

Step 3. Let $1 \leq m \leq A^{10}$. The following lines hold for all sufficiently small $\delta, \varepsilon > 0$. The arguments may require δ, ε to change from line to line. For a tempered distribution f, $x \in \mathbb{R}$, and $T \in \mathcal{F}_m$ define

$$Bf(x)(T) := \frac{\langle f, \phi_\imath(s_T) \rangle}{|J_T|^{\frac{1}{2}}} 1_{J_T}(x).$$

Let $L^2(\mathbb{R}, l^2(\mathcal{F}))$ be the Banach space of square-integrable functions on $\mathbb{R}$ with values in $l^2(\mathcal{F})$, and analogously for other exponents. Then we have the following estimate by Lemma 4.3 in [4]

$$\|Bf\|_{L^2(\mathbb{R}, l^2(\mathcal{F}_m))} = \left(\sum_{T \in \mathcal{F}_m} |\langle f_\imath, \phi_\imath(s_T) \rangle|^2 \right)^{\frac{1}{2}} \leq C(1 + A^{-\frac{1}{\varepsilon}} \lambda) \|f\|_2.$$

We also trivially have

$$\begin{aligned} \|Bf\|_{L^{1+\delta}(\mathbb{R}, l^\infty(\mathcal{F}_m))} &= \left(\int \left(\sup_{T \in \mathcal{F}_m : x \in J_T} \frac{|< f_\imath, \phi_\imath(s_T) >|}{|J_T|^{\frac{1}{2}}} \right)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \\ &\leq C \|Mf\|_{1+\delta} \leq C \|f\|_{1+\delta}. \end{aligned}$$

By interpolation we have for $1 < p < 2$:

$$\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_m))} \leq C(1 + A^{-\frac{1}{\varepsilon}} \lambda) \|f\|_p.$$

Let $J \in I(S)$, and let $\mathcal{F}_{m,J}$ be the set of $T \in \mathcal{F}_m$ such that $J_T \subset J$. By a localization argument, as in [4], we see that

$$\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_{m,J}))} \leq C\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) |J|^{\frac{1}{p}} \inf_{x \in J} M_p(Mf)(x).$$

In the following, $g^\sharp$ denotes the sharp maximal function of g with respect to the given grid, as in [4]. We define $N_{\mathcal{F}_{m,J}}$ in analogy to (46) to be the counting function of the trees $T \in \mathcal{F}_m$ for which $I_T \subset J$. We apply the previous estimate for $f_\imath$ and use (44) to obtain

$$\begin{aligned}
\left(N_{\mathcal{F}_m}^{\frac{p}{p'+\delta}}\right)^\sharp(x) &\leq \sup_{J:x\in J}\left(\frac{1}{|J|}\int_J N_{\mathcal{F}_{m,J}}(x)^{\frac{p}{p'+\delta}}\,dx\right)\\
&\leq b^{-p}\sup_{J:x\in J}\frac{1}{|J|}\left\|\left(\sum_{T\in\mathcal{F}_{m,J}}\frac{|\langle f_\imath,\phi_\imath(s_T)\rangle|^{p'+\delta}}{|J_T|^{\frac{p'+\delta}{2}}}1_{J_T)}\right)^{\frac{1}{p'+\delta}}\right\|_p^p\\
&\leq b^{-p}C\left(\lambda^\varepsilon(1+A^{-\frac{1}{\varepsilon}}\lambda)M_p(Mf_\imath)(x)\right)^p.
\end{aligned}$$

Using (45) we can sharpen this argument in the case $p = p_\imath$ to

$$\left(N_{\mathcal{F}_m}^{\frac{p_\imath}{p_\imath'+\delta}}\right)^\sharp(x) \leq Cb^{-p_\imath}\left(\lambda^\varepsilon(1+A^{-\frac{1}{\varepsilon}}\lambda)\min\{M_{p_\imath}(Mf_\imath)(x),\lambda_0\}\right)^{p_\imath}.$$

Taking the $\frac{p_\imath'+2\delta}{p_\imath}$- norm on both sides and raising to the $\frac{p_\imath'+\delta}{p_\imath}$-th power gives

$$\|N_{\mathcal{F}_m}\|_{\frac{p_\imath'+2\delta}{p_\imath'+\delta}} \leq Cb^{-p_\imath'-\delta}\left(\lambda^\varepsilon(1+A^{-\frac{1}{\varepsilon}}\lambda)\right)^{p_\imath'+\delta}. \tag{49}$$

Step 4. We split the counting function $N_{\mathcal{F}'}$ according to (47) and use the weak-type estimate following from (49) on the first part and estimate (48) together with (49) and the fact that the counting function is integer-valued on the second part. This gives

$$\begin{aligned}
\left\{x\in\mathbb{R} : N_{\mathcal{F}'}(x)\geq A^{10}\lambda\right\} &\leq CA^{10}\lambda^{-\frac{p_\imath'+2\delta}{p_\imath'+\delta}}b^{-p_\imath'-2\delta}\left(\lambda^\varepsilon(1+A^{-\frac{1}{\varepsilon}}\lambda)\right)^{p_\imath'+2\delta}\\
&\quad+e^{-A}Cb^{-p_\imath'-2\delta}\left(\lambda^\varepsilon(1+A^{-\frac{1}{\varepsilon}}\lambda)\right)^{p_\imath'+2\delta}.
\end{aligned}$$

Choosing A of the order λ^ε and $\varepsilon \ll \delta$ gives

$$\{x\in\mathbb{R} : N_{\mathcal{F}'}(x)\geq\lambda\} \leq C\lambda^{-1-\varepsilon}b^{-p_\imath'-\delta}.$$

According to Step 1 this finishes the proof of estimate (42) in the case $\imath = \jmath$.

7. Estimates on a single tree

This section collects some standard facts from Calderón-Zygmund theory, adapted to the setup of trees.

LEMMA 6. *Fix $k, \imath, \jmath, l$ such that $T := T_{k,\imath,\jmath,l}$ is a tree, assume $\imath \neq \jmath$, and let $1 < p \leq 2$. We then have*

$$\left\|\left(\sum_{s\in T}\frac{|\langle f,\phi_\jmath(s)\rangle|^2}{|I(s)|}1_{I(s)}\right)^{\frac{1}{2}}\right\|_p \leq C\|f\|_p. \tag{50}$$

For each interval $J \in I(S)$ define $T_J := \{s \in T : I(s) \subset J\}$. Then we obtain

$$\left\| \left(\sum_{s \in T_J} \frac{|\langle f, \phi_J(s)\rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_p \leq C|J|^{\frac{1}{p}} \inf_{x \in J} M_p(Mf)(x). \tag{51}$$

For each $s \in T$, let h_s be a measurable function supported in $I(s)$ with $\|h_I(x)\|_\infty = |I(s)|^{-\frac{1}{2}}$, $\|h\|_2 = 1$, and $\langle h_s, h_{s'}\rangle = 0$ for $s \neq s'$. Then for all maps $\varepsilon : T \to \{-1, 1\}$, we have

$$\left\| \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s)\rangle h_s \right\|_p \leq C\|f\|_p. \tag{52}$$

First we prove estimate (52). The estimate is true in the case $p = 2$, as is proved in [4]. By interpolation it suffices to prove the weak-type estimate

$$\left| \left\{ x \in \mathbb{R} : \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s)\rangle h_s(x) \geq C\lambda \right\} \right| \leq C' \frac{\|f\|_1}{\lambda}. \tag{53}$$

Let $f \in L^1(\mathbb{R})$. We write f as the sum of a good function g and a bad function b as follows. Let $\{I_n\}_n$ be the set of maximal intervals of the grid $I(S)$ for which

$$\int_{I_n} |f(x)|\, dx \geq \lambda |I_n|.$$

Let $\xi \in \omega_\imath(s_T)$, and pick a function $\theta_{\xi,\imath}$ as in hypothesis (iv) of Proposition 3. For each of the intervals I_n, define

$$b_n(x) := 1_{I_n}(x) \left(f(x) - \lambda_n \theta_{\xi,\imath}(x) \right),$$

where λ_n is chosen such that b_n is orthogonal to $\theta_{\xi,\imath}$. Obviously λ_n is bounded by $C\|f(x)\|_{L^1(I_n)}$. Define $b := \sum_n b_n$ and $g := f - b$. It suffices to prove estimate (53) for the good and bad function separately. The estimate for the good function follows immediately from estimate (52) for $p = 2$. For the bad function we proceed as follows. Since the set

$$E := \bigcup_n 2I_n$$

is bounded in measure by $C\lambda^{-1}$, it suffices to prove the strong-type estimate

$$\left\| \sum_n \left(\sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s)\rangle h_s \right) \right\|_{L^1(E^c)} \leq C\|f\|_1. \tag{54}$$

We estimate each summand separately. Obviously, we have

$$\left\| \sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s)\rangle h_s \right\|_{L^1(E^c)} \leq \sum_{s \in T : I(s) \not\subset 2I_n} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_J(s)\rangle|.$$

For each integer k let T_k be the set of those $s \in T$, for which $|I(s)| \leq 2^k |I_n| < 2|I(s)|$ and $I(s) \not\subset 2I_n$. For $k < 2$ we use the estimate

$$
\begin{aligned}
(55) \qquad & \sum_{s \in T_k} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_\jmath(s)\rangle| \\
&\leq C\|b_n\|_1 \sum_{s\in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|}\right)^{-2} \\
&\leq C\|b_n\|_1 \int_{(2I_n)^c} \sum_{s \in T_k} \frac{1}{2^k|I_n|} \left(1 + \frac{x - c(I_n)|}{2^k|I_n|}\right)^{-2} 1_{I(s)}(x)\, dx \\
&\leq C\|b_n\|_1 2^k.
\end{aligned}
$$

For the last inequality we have seen that the intervals $I(s)$ with $s \in T_k$ are pairwise disjoint.

For $k > 2$ we use the orthogonality of b_n and $\theta_{\xi,\imath}$ as well as hypothesis (iv) of Proposition 3 to obtain

(56)

$$
\begin{aligned}
\sum_{s\in T_k} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_\jmath(s)\rangle| &\leq \sum_{s\in T_k} |I(s)|^{\frac{1}{2}} \|b_n\|_1 \inf_\lambda \|\phi_\jmath(s) - \lambda\theta_{\xi,\imath}\|_{L^\infty(I_n)} \\
&\leq C\|b_n\|_1 \sum_{s \in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|}\right)^{-2} \frac{|I_n|}{|I(s)|} \\
&\leq C\|b_n\|_1 2^{-k}.
\end{aligned}
$$

The last inequality follows by a similar argument as in the case $k \leq 2$. Summing (55) and (56) over k and n gives (54) and finishes the proof of (52).

We prove estimate (50). Observe that (52) is not void, since functions h_s clearly exist. Therefore we can average (52) over all choices of ε to obtain:

$$
\begin{aligned}
2^{-|T|} \sum_\varepsilon \left\| \sum_{s\in T} \varepsilon(s) \langle f, \phi_\jmath(s)\rangle h_s \right\|_p^p &= \int_{\mathbb{R}} 2^{-n} \sum_\varepsilon \left(\sum_{s \in T} \varepsilon(s) \langle f, \phi_\jmath(s) \rangle h_s(x) \right)^p dx \\
&\leq C\|f\|_p^p.
\end{aligned}
$$

Now Khinchine's inequality gives

$$
\int_{\mathbb{R}} \left(2^{-n} \sum_\varepsilon \left(\sum_{s\in T} \varepsilon(s) \langle f, \phi_\jmath(s)\rangle h_s(x)\right)^2 \right)^{\frac{p}{2}} dx \leq C\|f\|_p^p,
$$

which immediately implies estimate (50).

To prove (51) fix a J and write $f = f1_{2J} + f1_{(2J)^c}$. It suffices to prove the estimate separately for both summands. For the first summand we simply apply (50). For the second summand we write

$$\begin{aligned}\left(\sum_{s\in T_J}\frac{|\left\langle f1_{(2J)^c},\phi_\jmath(s)\right\rangle|^2}{|I(s)|}1_{I(s)}(x)\right)^{\frac12} &\le C\sum_{s\in T_J:x\in I(s)}Mf(x)|I(s)||J|^{-1}\\ &\le CMf(x)1_J(x).\end{aligned}$$

The last inequality follows by summing a geometric series. This proves the estimate for the second summand and finishes the proof of Lemma 6.

LEMMA 7. *Fix $k\ge\eta^{-2},\imath,\jmath,l$ such that $T:=T_{k,\imath,\jmath,l}$ is a tree and assume $\imath\ne\jmath$. Then we have*

$$\left(\sum_{s\in T}|\langle f_\jmath,\phi_\jmath(s)\rangle|^2\right)^{\frac12}\le C\left\|\left(\sum_{s\in T}\frac{|\langle f_\jmath,\phi_\jmath(s)\rangle|^2}{|I(s)|}1_{I(s)}\right)^{\frac12}\right\|_1|J_T|^{-\frac12}. \tag{57}$$

Proof. For each $J\in I(S)$,

$$\frac{1}{|J|}\int_J\left(\left(\sum_{s\in T:I(s)\subset J}\frac{|\langle f_\jmath,\phi_\jmath(s)\rangle|^2}{|I(s)|}1_{I(s)}(x)\right)^{\frac12}\right)dx\le C2^{-\frac{k}{p_\jmath{}'}}, \tag{58}$$

since the set $\{s\in T:I(s)\subset J\}$ is a union of trees $\{T_n\}_n$ which satisfy (41) for $k-1$ and

$$\sum_n|J_{T_n}|\le|J_T|.$$

Define for $x\in\mathbb{R}$ and $s\in T$:

$$F(x)(s):=\sum_{s\in T}\frac{|\langle f_\jmath,\phi_\jmath(s)\rangle|^2}{|I(s)|}1_{I(s)}(x).$$

Since F is supported on J_T, we have

$$||F||_{L^2(\mathbb{R},l^2(T))}\le|J_T|^{\frac12}||F||_{\mathrm{BMO}(\mathbb{R},l^2(T))}.$$

Here BMO is understood with respect to the grid $I(S)$ as in [4]. We prove Lemma 7 by estimating this BMO-norm with (58) and (36).

8. Counting the trees for $\imath\ne\jmath$

We prove estimate (42) in the case $\imath\ne\jmath$. Thus fix $k\ge\eta^{-2},\imath,\jmath$ with $\imath\ne\jmath$. Let $\mathcal{F}$ denote the set of all trees $T_{k,\imath,\jmath,l}$.

As in [4] we define for $T \in \mathcal{F}$:

$$\begin{aligned} T^{\min} &:= \{s \in T : \rho_\imath(s) \text{ is minimal in } \rho_\imath(T)\}, \\ T^{\text{fat}} &:= \{s \in T : 2^5 2^{\eta k}|I(s)| \geq |J_T|\}, \\ T^{\partial} &:= \{s \in T : I(s) \cap (1-2^{-4})J_T = \emptyset\}, \\ T^{\partial \max} &:= \{s \in T^{\partial} : \rho_\imath(s) \text{ is maximal in } \rho_\imath\,(T^{\partial})\}, \\ T^{\text{nice}} &:= T \setminus \left(T^{\min} \cup T^{\text{fat}} \cup T^{\partial}\right). \end{aligned}$$

Define $b := 2^{-\frac{k}{p_\jmath{}'}}$. By similar arguments as in [4] we have the estimate

$$\text{(59)} \qquad \text{if } \imath \neq \jmath, \text{ then } \left\| \left(\sum_{s \in T^{\text{nice}}} \frac{|\langle f_\jmath, \phi_\jmath(s)\rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \geq b|J_T|.$$

Define the counting function

$$N_{\mathcal{F}}(x) := \sum_{T \in \mathcal{F}} 1_{J_T}(x).$$

As in Section 6 it suffices to show

$$\text{(60)} \qquad |\{x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq \lambda\}| \leq b^{-p_\jmath{}' - \delta}\lambda^{-1-\varepsilon}$$

for all integers $\lambda \geq 1$ and small $\varepsilon, \delta > 0$. In addition, we can assume that $\|N_{\mathcal{F}}\|_\infty \leq \lambda$.

Let $y \in \mathbb{R}$, $T \in \mathcal{F}$, $x \in J_T$, and $s \in T$. For $f \in \mathcal{S}(\mathbb{R})$ define

$$Sf(y)(T)(x)(s) := \frac{\langle f, \phi_\jmath(s)\rangle}{|I(s)|^{\frac{1}{2}}} 1_{I(s)}(x) 1_{J_T}(y).$$

Consider J_T as a measure space with Lebesgue measure normalized to 1. Then the operator is bounded from L^2 to $L^2(\mathbb{R}, l^2(\mathcal{F}, (L^2(J_T, l^2(T)))))$, as we see below. We have used a sloppy notation for the second Banach space: The range space $L^2(J_T, l^2(T))$ depends on the variable $T \in \mathcal{F}$. To make this space independent of T, we take the direct sum of these Banach spaces as T varies over $\mathcal{F}$, and we let $Sf(y)(T)$ be nonzero only on the component corresponding to T. This is how we interpret the above notation. To see the claimed estimate we calculate:

$$\begin{aligned} \int \sum_{T \in \mathcal{F}} \frac{1}{|J_T|} \int \sum_{s \in T} \frac{|\langle f, \phi_\jmath(s)\rangle|^2}{|I(s)|} 1_{I(s)}(x) 1_{J_T}(y)\, dx\, dy &= \sum_{s \in \bigcup_{T \in \mathcal{F}} T} |\langle f, \phi_\jmath(s)\rangle|^2 \\ &\leq C(1 + \lambda A^{-\frac{1}{\varepsilon}})\|f\|_2^2, \end{aligned}$$

the last inequality being taken from [4]. The operator is also bounded from $L^{1+2\delta}$ into

$$L^{1+2\delta}(\mathbb{R}, l^\infty(\mathcal{F}, L^{1+\delta}(J_T, l^2(T))))$$

since by Lemma 6:

$$\int \left(\sup_{T\in\mathcal{F}} \left(\frac{1}{|J_T|} \int \left(\sum_{s\in T} \left(\frac{|\langle f, \phi_J(s)\rangle|}{|I(s)|^{\frac{1}{2}}} 1_{I(s)}(x) 1_{J_T}(y) \right)^2 \right)^{\frac{1+\delta}{2}} dx \right)^{\frac{1}{1+\delta}} \right)^{1+2\delta} dy$$

$$\leq \int \left(\sup_{T\in\mathcal{F}: y\in J_T} \frac{1}{|J_T|} \left\| \left(\sum_{s\in T} \left(\frac{|\langle f, \phi_J(s)\rangle|}{|I(s)|^{\frac{1}{2}}} 1_{I(s)} \right)^2 \right)^{\frac{1}{2}} \right\|_{1+\delta}^{1+\delta} \right)^{\frac{1+2\delta}{1+\delta}} dy$$

$$\leq C \int (M_{1+\delta}(Mf)(y))^{1+2\delta}\, dy$$

$$\leq C\|f\|_{1+2\delta}^{1+2\delta}.$$

By complex interpolation and the fact that $L^q(J_T) \subset L^1(J_T)$ for $q \geq 1$ we obtain that S maps L^p into $L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}, L^1(J_T, l^2(T))))$ with norm less than $C(1 + \lambda A^{-\frac{1}{\varepsilon}})$.

Let $J \in I(S)$ and define $\mathcal{F}_J$ to be the set of $T \in \mathcal{F}$ such that $J_T \subset J$. Then we can localize as before to get

$$\|Sf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))} \leq C\lambda^{\varepsilon}(1 + \lambda A^{-\frac{1}{\varepsilon}})|J|^{\frac{1}{p}} \inf_{x\in J} M^p(Mf)(x).$$

Using the estimate (59) on nice trees gives, for $f = f_J$ and $p = p_J$,

$$\begin{aligned} \left(N_{\mathcal{F}}^{\frac{p_J}{p_J'+\varepsilon}} \right)^{\sharp}(x) &\leq \sup_{J: x\in J} \left(\frac{1}{|J|} \int_J N_{\mathcal{F}_J}(x)^{\frac{p_J}{p_J'+\varepsilon}}\, dx \right) \\ &\leq b^{-p_J} \sup_{J: x\in J} \left(\frac{1}{|J|} \|Sf_J\|^{p_J}_{L^{p_J}(\mathbb{R}, l^{p_J'+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))} \right) \\ &\leq b^{-p_J} C\lambda^{\varepsilon}(1 + A^{-\frac{1}{\varepsilon}}\lambda)^{p_J}(M_{p_J}(Mf_J)(x))^{p_J}. \end{aligned}$$

Again we can sharpen this argument to obtain

$$\left(N_{\mathcal{F}}^{\frac{p_J}{p_J'+\delta}} \right)^{\sharp}(x) \leq Cb^{-p_J}\lambda^{\varepsilon}(1 + A^{-\frac{1}{\varepsilon}}\lambda)^{p_J} \max\{M_{p_J}(Mf_J)(x)^{p_J}, \lambda_0\}.$$

Taking the $\frac{p_J'+\delta}{p_J}$-norm on both sides proves estimate (60) and therefore also (42).

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