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A secondary Chern-Euler class

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Introduction

Let ξ be a smooth oriented vector bundle, with n-dimensional fibre, over a smooth manifold M. Denote by $\hat{\xi}$ the fibrewise one-point compactification of ξ. The main purpose of this paper is to define geometrically a canonical element $\Upsilon(\xi)$ in $H^n(\hat{\xi}, \mathbb{Q})$ ($H^n(\hat{\xi}, \mathbb{Z}) \otimes \frac{1}{2}$, to be more precise). The element $\Upsilon(\xi)$ is a secondary characteristic class to the Euler class in the fashion of Chern-Simons. Two properties of this element are described as follows.

The first one is in a very classical setting. Suppose ξ is the tangent bundle TM of M (hence M is oriented). In this case we denote $\hat{\xi}$ by ΣM and simply write Υ for $\Upsilon(\hat{\xi})$.

Suppose M is the boundary of a compact $(n+1)$-dimensional smooth manifold X. Let V be a nowhere zero smooth vector field given on M which is tangent to X, but not necessarily tangent or transversal to M. The vector field V naturally defines a cross section $\alpha : M \to \Sigma M$. One can extend V to a smooth tangent vector field $\overline{V}$ on X with only isolated (hence only a finite number of) zeros. Since such extensions are generic we shall, for convenience, call any such extension a generic extension. At an isolated zero point p of $\overline{V}$, let $\mathrm{ind}_p(\overline{V})$ be the index of $\overline{V}$ at p defined as usual. We then have the following:

Theorem 0.1. *For any generic extension $\overline{V}$ of V, if $p_1, \dots, p_k$ are the zero points of $\overline{V}$ then*

$$\sum_{j=1}^{k} \mathrm{ind}_{p_j}(\overline{V}) = \begin{cases} \chi(X) + \alpha^*(\Upsilon)([M]) & \text{if } n \text{ is odd} \\ \alpha^*(\Upsilon)([M]) & \text{if } n \text{ is even} \end{cases}$$

where $\chi(X)$ is the Euler characteristic of X.

Notice that, in case M is empty, if we establish as a convention that $\alpha^*(\Upsilon)([M]) = 0$, then the theorem above is a generalization of a well-known theorem of Poincaré-Hopf (cf. [M]). In general if M is not empty, it is easy to see from the Poincaré-Hopf theorem that the sum $\sum_{j=1}^{k} \mathrm{ind}_{p_j}(\overline{V})$ does not depend on the extension $\overline{V}$; and in case n is even, it does not depend on X.

Our theorem above relates the sum to a specific topological invariant of the boundary.

Note. Generalizing the Poincaré-Hopf index theorem for vector fields to manifolds with boundary has been studied by C. Pugh and D. Gottlieb (cf. [G], [P]). The formulae obtained in [G] and [P] however do not seem to link directly to the global topological invariant of the boundary in general.

The second property of $\Upsilon(\xi)$ is that it is closely related to the Thom class. Let ξ_∞ be the ∞-section of $\hat{\xi}$, and let $\gamma(\xi) \in H^n(\hat{\xi}, \xi_\infty)$, with integer coefficients, be the Thom class of ξ. We shall show the following:

THEOREM 0.2. *The natural homomorphism* $\jmath^* : H^n(\hat{\xi}, \xi_\infty) \to H^n(\hat{\xi})$ *is injective, and*

$$\jmath^*(\gamma(\xi)) = \Upsilon(\xi) + \frac{1}{2}\sigma^*(e(\xi))$$

where $e(\xi)$ *is the Euler class of* ξ, *and* $\sigma : \hat{\xi} \to M$ *is the projection.*

The construction of $\Upsilon(\xi)$ is explicit, and is inspired by Chern′s well-known proof of the Gauss-Bonnet theorem. While $\Upsilon(\xi)$ can be defined formally in a pretty straightforward way, in order to see its nature as a secondary characteristic class and prove Theorem 0.1 above, we shall first construct it as an element in $H^n(\hat{\xi}, \mathbb{R})$ in Section 1; the construction depends on choice of a connection on ξ. A proof of Theorem 0.1 is given in Section 2, while the proof of the topological invariance of the $\Upsilon(\xi)$ constructed in Section 1 is postponed to Section 3. There we shall see that $\Upsilon(\xi)$ is defined in $H^n(\hat{\xi}, \mathbb{Z}) \otimes \frac{1}{2}$, and prove Theorem 0.2.

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Section 1

In this section we first construct, in a natural way, a closed differential n-form Ψ on $\hat{\xi}$ (note that $\hat{\xi}$ has a canonical smooth structure). The form Ψ then represents an element in the de Rham cohomology $H^n(\hat{\xi}, \mathbb{R})$. It will be seen in subsequent sections that this element is in fact half integral, and does not depend on various choices involved in the construction.

The construction of Ψ follows the well-known work of Chern in [C], with some modifications particularly in the case when the dimension n of the fibre of ξ is even. For completeness we shall show the construction in detail, while leaving some needed fundamental background in differential geometry to the references (e.g. [KN]).

To start with, we fix an $\mathrm{SO}(n)$-connection ω on ξ, and let Ω be the curvature. Let us first explain some notational conventions that we are going to use, most of them standard.

We denote by $\langle\ ,\ \rangle$ and $\|\ \|$ the underlying metric and the induced norm, respectively, on ξ. The same notation will be used for the induced metric and norm on any other vector bundle associated to ξ.

Let ν be the *canonical* trivial oriented real line bundle over M with the trivial connection. Let $E = \nu \oplus \xi$. We then have an obvious (orientation-preserving) diffeomorphism

$$\hat{\xi} \approx \{v \in E : \|v\| = 1\}$$

in which the 0-section of $\hat{\xi}$ is identified with $1\oplus 0$, the ∞-section of $\hat{\xi}$ is identified with $-1 \oplus 0$, and the unit sphere bundle of ξ is in $0 \oplus \xi$. We shall always use this diffeomorphism without further notice.

The obviously induced $\mathrm{SO}(n+1)$-connection and curvature on E will still be denoted by ω and Ω respectively. Throughout our calculation, we shall choose an oriented local orthonormal frame field for ξ on M. Together with the canonical (positive) unit vector of ν in the first position, this forms the oriented local orthonormal frame field we shall choose for E on M. To simplify the notation without causing any ambiguity, we shall view ω (Ω, resp.) as an $\mathrm{so}(n+1)$-valued 1-form (2-form, resp.) on M, with respect to the chosen frame field. Recall $\Omega = d\omega + \omega \wedge \omega$, where matrix multiplication is understood. Also notice that the first row and column of ω and Ω are always 0.

As in the introduction, let $\sigma : \hat{\xi} \to M$ be the projection. For any differential form A on M, for the sake of simplicity, we shall write A for $\sigma^*(A)$ on $\hat{\xi}$ wherever it can be easily understood from the context.

Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix}$ be the $\mathbb{R}^{n+1}$-valued function on $\hat{\xi}$, associated to a chosen local frame field $e = (\, e_1, \ldots, e_{n+1}\,)$ for E described above, defined by

$$v = \sum_{i=1}^{n+1} u_i(v) e_i, \quad \forall v \in \hat{\xi},$$

and let $\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{n+1} \end{pmatrix}$ be the $\mathbb{R}^{n+1}$-valued 1-form defined by

$$\theta = du + \omega u.$$

The definition of u and θ depends on the choice of the local frame field of course. However, if the local frame field e is replaced by any other frame field eg for some $\mathrm{SO}(n+1)$-valued local function g, then it is easily seen that u and θ are replaced by $g^{-1}u$ and $g^{-1}\theta$ correspondingly.

We are now ready to define the form Ψ. Suppose $n = 2m$ or $2m+1$. Set

$$\Psi_j = \sum_\tau (-1)^\tau u_{\tau(1)}\theta_{\tau(2)}\wedge\cdots\wedge\theta_{\tau(n-2j+1)}\wedge\Omega_{\tau(n-2j+2)\tau(n-2j+3)}\wedge\cdots\wedge\Omega_{\tau(n)\tau(n+1)}$$

for $j = 0, 1, \ldots, m$, where the summation is over all the permutations τ of $\{1, \ldots, n+1\}$, and Ω_{st} denotes the (s,t)-entry of the matrix Ω as usual.

It is easy to see that the definition of each of the Ψ_j above does not depend on the choice of local frame, and hence is a globally well defined n-form on $\hat{\xi}$. We now define

$$\Psi = \frac{1}{(n-1)!!c_n}\sum_{j=0}^{m}\frac{1}{2^j j!(n-2j)!!}\Psi_j$$

where

$$c_n = \begin{cases} \frac{2(2\pi)^m}{(n-1)!!} & \text{if } n = 2m \\ \frac{(2\pi)^{m+1}}{(n-1)!!} & \text{if } n = 2m+1 \end{cases}$$

is the volume of the Euclidean n-dimensional sphere S^n.

We summarize some basic properties of Ψ in the following proposition. Its proof follows from the computations in [C], and hence is omitted. We state this proposition in the more general setting where E is an arbitrary oriented vector bundle over M, with $(n+1)$-dimensional fiber, and ω is an arbitrary $\mathrm{SO}(n+1)$-connection on E.

PROPOSITION 1.1.
(1)

$$d\Psi = \begin{cases} 0 & \textit{if } n = 2m \\ -E(\Omega) & \textit{if } n = 2m+1 \end{cases}$$

where, for $n = 2m+1$,

$$E(\Omega) = \frac{1}{(4\pi)^{m+1}(m+1)!}\sum_\tau (-1)^\tau \Omega_{\tau(1)\tau(2)}\cdots\Omega_{\tau(n)\tau(n+1)}$$

is the Euler curvature form of E.

(2) *If* $\imath : S^n \to \hat{\xi}$ *is any (orientation-preserving) isometry from the euclidean sphere* S^n *to a fibre of* $\sigma : \hat{\xi} \to M$, *then* $\imath^*(\Psi) = \frac{1}{c_n}\mathrm{vol}$, *where* vol *denotes the volume form on* S^n.

Returning to the special case when $E = \nu \oplus \xi$ and ω is induced from a connection on ξ, we have that Ψ is a closed n-form on $\hat{\xi}$, since the first row and column of Ω are 0.

Finally we note that the construction of Ψ is obviously natural (in the category of oriented vector bundles with Riemannian connection).

Section 2

In this section we assume ξ is the tangent bundle TM of M. Let Υ be the cohomology class in $H^n(\Sigma M, \mathbb{R})$ represented by the n-form Ψ constructed in last section. We now prove Theorem 0.1 stated in the introduction. First we note the following:

Remark 2.1. The vector bundle $\nu \oplus TM$ can naturally be viewed as one over $\mathbb{R} \times M$, and identified with the tangent bundle $T(\mathbb{R} \times M)$. The $\mathrm{SO}(n+1)$-connection ω in Section 1 is then associated with the Riemannian product metric on $\mathbb{R} \times M$.

Suppose M is the boundary of a compact $(n+1)$-dimensional manifold X. Assume X is orientable. We orient X consistently with the orientation of M. By Remark 2.1, on a tubular neighborhood of M in X, the tangent bundle TX can be identified with E over $(-1, 0] \times M$.

It is well-known that the connection ω (with curvature Ω) in Section 1 can be extended to an $\mathrm{SO}(n+1)$-connection, which is still denoted by ω (with curvature Ω), on TX. Also notice that the restriction of the tangent unit sphere bundle of X, denoted by STX, to M is ΣM. Let $\bar{\sigma} : STX \to X$ be the projection, which extends σ.

Now let V be a nowhere zero smooth vector field on M which is tangent to X, and let $\overline{V}$ be a generic extension of V on X. Without loss of generality, we may assume $\overline{V}$ has only one zero point p.

For $r > 0$, let $B_r(p)$ be the geodesic ball of radius r around p. Then for small r (when $B_r(p)$ is in the interior of X), $\overline{V}$ naturally defines a cross section $\bar{\alpha} : X \setminus B_r(p) \to STX$, which restricts to α on M.

Assume first that n is odd; it follows from Proposition 1.1:

$$
\begin{aligned}
-\chi(X) &= -\int_X E(\Omega) = -\lim_{r\to 0^+} \int_{X\setminus B_r(p)} \bar{\alpha}^*\bar{\sigma}^*(E(\Omega)) = \lim_{r\to 0^+} \int_{X\setminus B_r(p)} d\bar{\alpha}^*(\Psi) \\
&= \int_M \alpha^*(\Psi) - \lim_{r\to 0^+} \int_{\partial B_r(p)} \bar{\alpha}^*(\Psi) = \int_M \alpha^*(\Psi) - \mathrm{ind}_p(\overline{V})
\end{aligned}
$$

where the first equality follows from the Gauss-Bonnet theorem, the second follows from the fact that $\bar{\sigma}\bar{\alpha} = \mathrm{id}$, and the fourth is by Stokes' theorem.

Theorem 0.1 then clearly follows when n is odd. The case when n is even is similar. If X is not orientable, from the proof above, the theorem easily follows by passing to the orientable double covering of X. The proof is therefore complete.

Some special cases worth mentioning are:

• When V is transversal to M, it is easy to see $\alpha^*(\Psi) = 0$ if n is odd, while $\alpha^*(\Psi) = \frac{1}{2}$ times the Euler curvature form of TM if n is even (and if V is pointing out of X).

• When V is tangent to M, it is easy to see $\alpha^*(\Psi) = 0$ for both odd and even n.

The corresponding formulae for $\sum \mathrm{ind}_{p_j}(\overline{V})$ in these cases can also be seen easily from the Poincaré-Hopf theorem, except maybe one—when n is even and V is tranversal to M, which is the relative Poincaré-Hopf theorem (cf. [P]).

It is interesting to compare our formula with the one in [G] or [P]. This yields

$$\alpha^*(\Upsilon)([M]) = \begin{cases} -\mathrm{Ind}(\partial_- V) & \text{if } n \text{ is odd} \\ \chi(X) - \mathrm{Ind}(\partial_- V) & \text{if } n \text{ is even} \end{cases}.$$

We refer to [G] and [P] for the definition of $\mathrm{Ind}(\partial_- V)$.

Section 3

We now turn to the general oriented vector bundle ξ. Let $\alpha_0 : M \to \hat{\xi}$ be the canonical ∞-cross section, and as before $\imath : S^n \to \hat{\xi}$ be any (orientation-preserving) diffeomorphism from S^n into a fibre of σ.

By Proposition 1.1 and a special case mentioned at the end of Section 2, the element $\Upsilon(\xi) \in H^n(\hat{\xi}, \mathbb{R})$ represented by Ψ constructed in Section 1 has the following properties:

(1) $\imath^*(\Upsilon(\xi)) = s^n$, where s^n denotes the canonical generator of $H^q(S^n, \mathbb{R})$.

(2) $\alpha_0^*(\Upsilon(\xi)) = -\frac{1}{2}e(\xi)$, where $e(\xi) \in H^n(M, \mathbb{R})$ is the real coefficient Euler class of ξ.

Example. Let $M = S^2$, and let $\xi = TS^2$ and $\eta = M \times \mathbb{R}^2$ be the trivial (oriented) plane bundle over S^2. Then topologically $\hat{\xi} = \hat{\eta} = S^2 \times S^2$. Let $i_k : S^2 \times S^2 \to S^2$, $k = 1, 2$, be the projections onto the two factors respectively. It is seen immediately from the construction in Section 1 that $\Upsilon(\xi) = i_1^*(s^2) + i_2^*(s^2)$ and $\Upsilon(\eta) = i_2^*(s^2)$.

Guided by (1), (2) above, we now define $\Upsilon(\xi)$ without using the connections.

PROPOSITION 3.1. *The following sequence*

$$0 \longrightarrow H^n(M, \mathbb{Z}) \xrightarrow{\sigma^*} H^n(\hat{\xi}, \mathbb{Z}) \xrightarrow{\imath^*} H^n(S^n, \mathbb{Z}) \longrightarrow 0$$

is exact.

Proof. The proposition comes easily from the following commutative diagram of the Gysin sequence (cf. [MS])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(M) & \xrightarrow{\sigma^*} & H^n(\hat{\xi}) & \longrightarrow & H^0(M) & \longrightarrow & 0 \\ & & & & \downarrow \imath^* & & \downarrow \approx & & \\ & & & & H^n(S^n) & \xrightarrow{\approx} & H^0(\text{point}) & & \end{array}$$

where the integer coefficients are understood. The first horizontal line, which is exact, is from the Gysin sequence of the vector bundle $\nu \oplus \xi$. As before ν is the canonical trivial oriented line bundle, and we have used the fact that $e(\nu \oplus \xi) = 0$ to conclude that the homomorphism $H^0(M) \to H^{n+1}(M)$ in the Gysin sequence vanishes. □

Proposition 3.1 easily implies that there is a canonical decomposition

$$H^n(\hat{\xi}, \mathbb{Z}) = \sigma^*(H^n(M, \mathbb{Z})) \oplus \alpha_0^{*-1}(0)$$

and $\imath^*|_{\alpha_0^{*-1}(0)} : \alpha_0^{*-1}(0) \to H^n(S^n, \mathbb{Z})$ is an isomorphism. Needless to say $\alpha_0^*|_{\sigma^*(H^n(M,\mathbb{Z}))} : \sigma^*(H^n(M, \mathbb{Z})) \to H^n(M, \mathbb{Z})$ is also an isomorphism.

We can now define $\Upsilon(\xi) \in H^n(\hat{\xi}, \mathbb{Z}) \otimes \frac{1}{2}$, where $H^n(\hat{\xi}, \mathbb{Z}) \otimes \frac{1}{2}$ denotes the tensor product, as $\mathbb{Z}$-module, of $H^n(\hat{\xi}, \mathbb{Z})$ and the subgroup of $\mathbb{Q}$ generated by $\frac{1}{2}$, as follows:

$$\Upsilon(\xi) = -\frac{1}{2}\sigma^*(e(\xi)) + \imath^*|_{\alpha_0^{*-1}(0)}{}^{-1}(s^n).$$

Since the sequence in Proposition 3.1 is clearly also exact with real coefficient, properties (1) and (2) above characterize $\Upsilon(\xi)$, defined in Section 1, in $H^n(\hat{\xi}, \mathbb{R})$. Obviously, this agrees with the $\Upsilon(\xi)$ just defined in this section, after tensoring with $\mathbb{R}$. This shows that the element $\Upsilon(\xi) \in H^n(\hat{\xi}, \mathbb{R})$ constructed as in Section 1 does not depend on the choice of connections.

It is well-known that if an oriented M is the boundary of a compact manifold, then $e(TM) \in H^n(M, \mathbb{Z})$ is even. Hence in this case (also in the case n is odd) $\Upsilon \in H^n(\Sigma M, \mathbb{Z})$.

To finish, let us now prove Theorem 0.2 from the introduction. Here again we use the integer coefficients.

First, it follows immediately, from the Gysin sequence of $\nu \oplus \xi$, that $\sigma^* : H^{n-1}(M) \to H^{n-1}(\hat{\xi})$ is an isomorphism. Hence so is $\alpha_0^* : H^{n-1}(\hat{\xi}) \to H^{n-1}(M)$.

Then from the cohomology exact sequence of the pair $(\hat{\xi}, \xi_\infty)$,

$$\cdots \longrightarrow H^{n-1}(\hat{\xi}) \xrightarrow{\alpha_0^*} H^{n-1}(M) \longrightarrow H^n(\hat{\xi}, \xi_\infty) \xrightarrow{\jmath^*} H^n(\hat{\xi}) \xrightarrow{\alpha_0^*} H^n(M) \longrightarrow \cdots$$

where we have replaced $H^j(\xi_\infty), j = n-1, n$ by $H^j(M)$, we see that $\jmath^* : H^n(\hat{\xi}, \xi_\infty) \to H^n(\hat{\xi})$ is injective, and its image is $\alpha_0^{*-1}(0)$.

By the definition of $\Upsilon(\xi)$, to prove Theorem 0.2, it is now sufficient to verify $\imath^*(\jmath^*(\gamma(\xi)))$ as the canonical generator of $H^n(S^n)$. But this easily follows from the characterization of the Thom class $\gamma(\xi)$.

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