Annals of Mathematics, **151** (2000), 125–150

Linking numbers and boundaries of varieties

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Introduction

The intersection index at a common point of two analytic varieties of complementary dimensions in $\mathbb{C}^n$ is positive. This observation, which has been called a "cornerstone" of algebraic geometry ([GH, p. 62]), is a simple consequence of the fact that analytic varieties carry a natural orientation. Recast in terms of linking numbers, it is our principal motivation. It implies the following: Let M be a smooth oriented compact 3-manifold in $\mathbb{C}^3$. Suppose that M bounds a bounded complex 2-variety V. Here "bounds" means, in the sense of Stokes' theorem, i.e., that $b[V] = [M]$ as currents. Let A be an algebraic curve in $\mathbb{C}^3$ which is disjoint from M. Consider the linking number $\mathrm{link}(M, A)$ of M and A. Since this linking number is equal to the intersection number (i.e. the sum of the intersection indices) of V and A, by the positivity of these intersection indices, we have $\mathrm{link}(M, A) \geq 0$. The linking number will of course be 0 if V and A are disjoint. (As A is not compact, this usage of "linking number" will be clarified later.) This reasoning shows more generally that $\mathrm{link}(M, A) \geq 0$ if M bounds a positive holomorphic 2-chain. Recall that a *holomorphic k-chain* in $\Omega \subseteq \mathbb{C}^n$ is a sum $\sum n_j[V_j]$ where $\{V_j\}$ is a locally finite family of irreducible k-dimensional subvarieties of Ω and $n_j \in \mathbb{Z}$ and that the holomorphic 2-chain is *positive* if $n_j > 0$ for all j. Our first result is that, conversely, the nonnegativity of the linking number characterizes boundaries of positive holomorphic 2-chains.

Theorem 1. *Let M be a smooth, oriented, compact, 3-manifold (not necessarily connected) in $\mathbb{C}^3$. Suppose that $\mathrm{link}(M, A) \geq 0$ whenever A is an algebraic curve in $\mathbb{C}^3$ disjoint from M. Then there exists a (unique) positive holomorphic 2-chain T in $\mathbb{C}^3 \setminus M$ of finite mass and with bounded support such that $[M] = b[T]$.*

We shall refer to the linking hypothesis in Theorem 1 as the linking condition. More generally, a smooth oriented compact manifold M in $\mathbb{C}^n$ of (odd) real dimension k satisfies the *linking condition* if

$$\mathrm{link}(M, A) \geq 0$$

for all algebraic subvarieties A of $\mathbb{C}^n$ disjoint from M of pure (complex) dimension $n-(k+1)/2$. Of course, the conclusion of Theorem 1 is closely related to the fundamental result of Harvey and Lawson [HL] that M bounds a bounded holomorphic 2-chain T if and only if M is maximally complex. In Theorem 1—unlike the Harvey-Lawson theorem—the holomorphic 2-chain is positive. This reflects the fact that "maximal complexity of M" is unaffected by a change of orientation of M, while our hypothesis on linking numbers is tied to a specific orientation. One of the main steps in our proof of Theorem 1 is indeed to verify that M is maximally complex.

If M bounds a holomorphic 2-chain $T = \sum n_j[V_j]$ as in the last theorem, then by the maximum principle $\operatorname{supp} T \setminus M = \bigcup V_j \subseteq \hat{M}$. Here $\hat{K}$, the polynomially convex hull of a compact set $K \subseteq \mathbb{C}^3$, is defined as the set $\{z \in \mathbb{C}^3 : |P(z)| \leq \sup_K |P| \text{ for all polynomials } P \text{ in } \mathbb{C}^3\}$. In general, $\hat{M}$ will be larger that $\bigcup V_j \cup M$. While the points in the polynomial hull are given explicitly by the definition just stated, the "individual" points of $\operatorname{supp} T$, with T the (unique) Harvey-Lawson solution to the equation $b[T] = [M]$ for a given maximally complex M, on the other hand, are not explicitly given. The next result determines these points in terms of linking numbers.

Theorem 2. *Let M be as given in Theorem* 1 *and let T be the unique bounded holomorphic* 2*-chain in $\mathbb{C}^3$ such that $b[T] = [M]$. Then for $x \in \mathbb{C}^3 \setminus M$, $x \in \operatorname{supp} T$ if and only if*

$$\operatorname{link}(M, A) > 0$$

for every algebraic curve A in $\mathbb{C}^3$ such that $x \in A$ and $A \cap M = \emptyset$.

Of course, half of this equivalence is trivial: it is merely the above-mentioned positivity of the intersection numbers. For the opposite implication, we shall show that if $x \notin \operatorname{supp} T$, then there exists A such that $x \in A$, $A \cap M = \emptyset$, and $\operatorname{link}(M, A) = 0$.

In order to prove these two theorems about 3-manifolds in $\mathbb{C}^3$, we need to establish the corresponding theorems for smooth oriented 1-manifolds γ, that are compact, but not necessarily connected. Thus γ is a finite disjoint union of oriented simple closed curves in $\mathbb{C}^n$. Recall that γ satisfies the *moment condition* if

$$\int_\gamma \phi = 0$$

for all holomorphic $(1,0)$-forms ϕ in $\mathbb{C}^n$. By Harvey and Lawson [HL], if γ satisfies the moment condition, then γ bounds (in the sense of Stokes' theorem) a (unique) bounded holomorphic 1-chain in $\mathbb{C}^n \setminus \gamma$.

THEOREM 3. *Let γ be a smooth compact oriented* 1*-manifold in* $\mathbb{C}^n$. *Suppose that* $\mathrm{link}(\gamma, A) \geq 0$ *for every algebraic hypersurface A in $\mathbb{C}^n$ such that $A \cap \gamma = \emptyset$. Then γ satisfies the moment condition and there exists a* (*unique*) *positive holomorphic* 1*-chain T in $\mathbb{C}^n \setminus \gamma$ of bounded support and finite mass such that* $b[T] = [\gamma]$.

Theorem 3 is the direct analogue for curves of Theorem 1. The analogue for Theorem 2 is the following:

THEOREM 4. *Let γ be given as in Theorem* 3 *and let T be the unique bounded positive homomorphic* 1*-chain in $\mathbb{C}^n \setminus \gamma$ such that* $b[T] = [\gamma]$. *Then for $x \in \mathbb{C}^n \setminus \gamma$, $x \in \operatorname{supp} T$ if and only if*

$$\mathrm{link}(\gamma, A) > 0$$

for every algebraic hypersurface A in $\mathbb{C}^n$ such that $x \in A$ and $A \cap \gamma = \emptyset$.

Theorems 1 and 2 of course suggest that corresponding results might hold for manifolds of odd dimension in all $\mathbb{C}^n$. This turns out to be true. However, the main steps in proving the general result are to establish the preliminary cases stated so far. For this reason, we have stated them separately, even though they are special cases of the general result, which we now state.

THEOREM 5. *Let M be a smooth oriented compact manifold in $\mathbb{C}^n$ of* (*odd*) *real dimension k with $3 \leq k \leq 2n-3$. Then M satisfies the linking condition*:

$$\mathrm{link}(M, A) \geq 0$$

for all algebraic subvarieties A of $\mathbb{C}^n$ disjoint from M of pure (*complex*) *dimension $n - (k+1)/2$ if and only if M is maximally complex and there exists a* (*unique*) *positive holomorphic k-chain T of dimension $(k+1)/2$ in $\mathbb{C}^n \setminus M$ of finite mass and bounded support such that $[M] = b[T]$. Moreover, for all $x \in \mathbb{C}^n \setminus M$, $x \in \operatorname{supp} T$ if and only if*

$$\mathrm{link}(M, A) > 0$$

for all algebraic subvarieties A of $\mathbb{C}^n$ disjoint from M of pure (*complex*) *dimension $n - (k+1)/2$ such that $x \in A$.*

It may be of interest to reformulate two of our results.

I) (Theorem 3 +Lemma 1.2). Let γ be a smooth oriented compact curve in $\mathbb{C}^n$. Then there exists a positive holomorphic 1-chain V in $\mathbb{C}^n \setminus \gamma$ of finite mass such that $[\gamma] = b[V]$ if and only if $\frac{1}{2\pi i}\int_\gamma \frac{dp}{p} \geq 0$ for any polynomial p in $\mathbb{C}^n$ such that $p|_\gamma \neq 0$.

The second much simpler part consists of the following statement, which is proved, but not explicitly formulated, below. It was conjectured by Dolbault and Henkin ([DH, p. 388]) and was recently also proved by Dinh [D]. (We thank the referee for these references.)

II) Let M be a smooth compact manifold in $\mathbb{C}^n$ of real dimension $2p-1 \geq 3$. Then there exists a holomorphic p-chain V in $\mathbb{C}^n \setminus M$ of finite mass such that $[M] = b[V]$ if and only if for almost any complex $(n-p+1)$-plane H in $\mathbb{C}^n$ the curve $\gamma = H \cap M$ bounds a holomorphic 1-chain in $H \setminus \gamma$.

We shall begin with some preliminary remarks; these include material on linking numbers, polynomial hulls of curves and the Arens-Royden theorem. We then establish first the theorems for curves. This relies on the theory of polynomial hulls of curves due to Wermer [W], Bishop and Stolzenberg [St]as well as on the Harvey-Lawson [HL] theorem for curves that involves the moment condition. This is the most difficult case, at least in the smooth case, in part because we do not know that $\hat{\gamma}$ is a 'nice' topological space for the most general smooth γ. When γ is real analytic, then $\hat{\gamma}$, as a topological space, is a finite simplicial complex, and the proof is much shorter than for the smooth case. From the curve result we deduce the theorem for 3-manifolds in $\mathbb{C}^3$. The remaining cases are then obtained by using projections for $k = 3$ and, for higher dimensions, by slicing and an inductive procedure. We shall use some standard facts and the notation for currents; for this we refer to Federer [F], Harvey[H] and Harvey-Shiffman[HS]. To avoid confusion with other uses of ∂, we denote the boundary of a current T by bT. Hausdorff k-dimensional measure will be denoted by $\mathcal{H}^k$. We want to thank Bruno Harris for some helpful conversations on algebraic topology.

1. Preliminaries

A. *Linking.* We shall briefly recall the definition of linking number and then derive a few of its properties. For more details we refer to Bott and Tu [BT] who give enlightening discussions of the linking number ([BT, pp. 231–235]) and the Poincaré dual. A very general definition of linking number for singular homology classes is given by Spanier ([Sp, p. 361]). Let M and Y be disjoint compact smooth oriented submanifolds of $\mathbb{R}^N$ of respective dimensions s and t. Suppose that $s + t = N - 1$. Then the linking number $\text{link}(M, Y)$ can be defined as follows: Let Σ be a compact oriented $(s+1)$-chain in $\mathbb{R}^N$ such that $M = b\Sigma$ and such that Σ and Y meet transversally. Then $\text{link}(M, Y)$ is the intersection number $\#(\Sigma, Y)$.

There is a useful alternate equivalent definition of the linking number which uses the Poincaré dual: Let η_M and η_Y be compact Poincaré duals of

M and Y supported on disjoint neighborhoods of M and Y respectively. Thus η_M is a closed $(N-s)$-form with compact support in $\mathbb{R}^N$ and so there exists a compactly supported $(N-s-1)$-form ω_M in $\mathbb{R}^N$ such that $d\omega_M = \eta_M$. Then

$$\text{link}(M,Y) = \int \omega_M \wedge \eta_Y. \tag{1.1}$$

The integration on the right-hand side of (1.1) is over all of $\mathbb{R}^N$, but of course $\omega_M \wedge \eta_Y$ has compact support. The Poincaré dual η_Y can be "localized" to have support in an arbitrary neighborhood W of Y. Its fundamental property is that $\int_W \phi \wedge \eta_Y = \int_Y \phi$ for all *closed* t-forms ϕ on W. Choosing ϕ to be the restriction of ω_M to W (ω_M is closed as a form on W) we get

$$\text{link}(M,Y) = \int \omega_M \wedge \eta_Y = \int_W \omega_M \wedge \eta_Y = \int_Y \omega_M. \tag{1.2}$$

We shall use the linking number in a somewhat more general setting. Namely, we need $\text{link}(M,A)$ when M is a compact oriented k-manifold (k odd) in $\mathbb{C}^n$ and A is an algebraic subvariety of $\mathbb{C}^n$ with its natural orientation and of complex dimension s so that $k+2s = 2n-1$. Then A is not compact and the above definitions of linking number need to be extended. One approach is to modify A outside of a large ball $B(r)$, centered at 0 of radius r, and containing M, so that A becomes compact as follows: Let $R = A \cap bB(r)$, a compact oriented $(2s-1)$- chain (for almost all r) contained in the sphere $bB(r)$, and let A'' be a $2s$-chain in $bB(r)$ so that $bA'' = R$. Then $A' = A \cap B(r) - A''$ is a (compact) $2s$-cycle in $\mathbb{C}^n$ which agrees with A inside $B(r)$. We take $\text{link}(M,A)$ to be $\text{link}(M,A')$; it is independent of the choices of r and R. Alternatively, we can apply the first definition above, taking $\text{link}(M,A)$ as $\#(\Sigma,A)$ where $M = b\Sigma$ and $\Sigma \subseteq B(r)$ is such that Σ and A meet transversally. This yields the same linking number. The definition in terms of differential forms can also be adapted to this setting as follows. Let $[A]$ be the current of integration over A, a positive (s,s)-current. We can extend the definition of (1.2) to the following:

$$\text{link}(M,A) = \int_A \omega_M = [A](\omega_M). \tag{1.3}$$

Lemma 1.1. *Let M be a smooth real k dimensional compact oriented manifold in $\mathbb{C}^n$ and let H be a complex hyperplane in $\mathbb{C}^n$ given as $\{F = \lambda\}$, where F is a complex linear function on $\mathbb{C}^n$; we view H as a copy of $\mathbb{C}^{n-1}$. Suppose that $Q = M \cap H$ is a smooth $k-2$ manifold, oriented as the slice of M by the map F. Let A be an algebraic variety of pure complex dimension $n-(k+1)/2$ contained in H and disjoint from Q. Then* $\text{link}(M,A)$, *the "link" taken in $\mathbb{C}^n$, agrees with* $\text{link}(Q,A)$, *the "link" taken in H and well-defined in H since $2n-k-1 = 2(n-1)-(k-2)-1$.*

Proofs. (1) Let $j : H \to \mathbb{C}^n$ be the inclusion map. Let η_M be a Poincaré dual of M in $\mathbb{C}^n$. We can choose η_M to have compact support disjoint from A. By a basic functorial property of Poincaré duals ([BT, p. 69]) we have $j^*(\eta_M) = \eta_{j^{-1}(M)} = \eta_{M\cap H} = \eta_Q$, a Poincaré dual of Q in H. Let ω_M be a compactly supported $(2n-k-1)$-form in $\mathbb{C}^n$ such that $d\omega_M = \eta_M$. Set $\omega_Q = j^*(\omega_M)$. Now, $d\omega_Q = d(j^*(\omega_M)) = j^*(d\omega_M) = j^*(\eta_M) = \eta_Q$. Hence, by two applications of (1.3),

$$\operatorname{link}(Q, A) = \int_{j^{-1}(A)} \omega_Q = \int_{j^{-1}(A)} j^*(\omega_M) = \int_A \omega_M = \operatorname{link}(M, A). \qquad \square$$

(2) Let G be a $(k+1)$-chain in $\mathbb{C}^n$ such that $bG = M$ and such that G and $G \cap H$ intersect A transversally. One checks that $\#(G, A)$ in $\mathbb{C}^n$ equals $\#(G \cap H, A)$ in H. This implies that $\operatorname{link}(Q, A) = \operatorname{link}(M, A)$. $\square$

Let γ be a smooth 1-cycle in $\mathbb{C}^n$ and let A be an algebraic hypersurface in $\mathbb{C}^n$ that is disjoint from γ.

Lemma 1.2. *If $A = Z(P)$, where P is a polynomial in $\mathbb{C}^n$, then*

$$\operatorname{link}(\gamma, A) = \frac{1}{2\pi i}\int_\gamma dP/P.$$

Proofs. (1) We give first a proof based on the Poincaré-Lelong formula

$$[A] = -i/2\pi d\partial \log |P|^2,$$

where $[A]$ is the $(n-1, n-1)$-current of integration over A. Then $\psi = -i/2\pi\partial \log |P|^2$ is a current such that $d\psi = [A]$. Off of the zero set $A = Z(P)$, we have $\psi = \frac{1}{2\pi i} dP/P$. We can obtain a smooth form cohomologous to $[A]$ ([GH, p. 393]), by taking the convolution of $[A]$ with a smooth function and this smooth form then is a Poincaré dual η_A to A. Corresponding to η_A is a smoothing ω_A of ψ such that $d\omega_A = \eta_A$ and such that ω_A is cohomologous to ψ. Let η_γ be a compact Poincaré dual of γ, a $(2n-2)$-form, supported on a small neighborhood of γ that is disjoint from the support of η_A. We have, since ω_A is cohomologous to $\frac{1}{2\pi i}\frac{dP}{P}$ off of A,

$$\operatorname{link}(\gamma, A) = \operatorname{link}(A, \gamma) = \int \omega_A \wedge \eta_\gamma = \int_\gamma \omega_A = \frac{1}{2\pi i}\int_\gamma \frac{dP}{P}. \qquad \square$$

(2) Consider the map $\psi : \mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $\psi(z) = (z, P(z))$. Set $\gamma' = \psi(\gamma)$ and $A' = \{w \in \mathbb{C}^{n+1} : w_{n+1} = 0\}$. Then $\operatorname{link}(\gamma, A)$ in $\mathbb{C}^n$ equals $\operatorname{link}(\gamma', A')$ in $\mathbb{C}^{n+1}$. One can continuously deform γ' in $\mathbb{C}^{n+1}$ to the curve $\gamma'' = (0 \in \mathbb{C}^n) \times P(\gamma)$ in $\mathbb{C}^{n+1}$ by curves $\gamma_t = \{(tz, P(z)) : z \in \gamma\}$, $0 \le t \le 1$, that are disjoint from A' in $\mathbb{C}^{n+1}$. Hence $\operatorname{link}(\gamma', A') = \operatorname{link}(\gamma'', A')$ in $\mathbb{C}^{n+1}$. Finally $\operatorname{link}(\gamma'', A')$ in $\mathbb{C}^{n+1}$ equals $\operatorname{link}(P(\gamma), \{0\})$ in $\mathbb{C}$ and this last linking number in $\mathbb{C}$ is just the winding number of $P(\gamma)$ about 0 which is $\frac{1}{2\pi i}\int_\gamma \frac{dP}{P}$. $\square$

Remark. It may be of interest to observe, although we shall not need it, that Lemma 1.2 extends to the higher dimensional setting of Theorem 5. Namely, suppose that M has dimension k and that A, of complex dimension s, where $2s+k = 2n-1$, is a complete intersection in $\mathbb{C}^n$ given as the common zero set of polynomials $P_1, P_2, \cdots, P_{n-s}$. Let $P = (P_1, P_2, \cdots, P_{n-s}) : \mathbb{C}^n \to \mathbb{C}^{n-s}$ and let β_{n-s} be the Bochner-Martinelli [GH] $(2(n-s)-1)$-form in $\mathbb{C}^{n-s}$ with singularity at 0. Then $k = 2(n-s)-1$ and

$$\operatorname{link}(M, A) = \int_M P^*(\beta_{n-s}).$$

This can be easily verified by adapting the second proof of Lemma 1.2.

B. *Polynomial hulls of curves.* For K a compact subset of $\mathbb{C}^n$, $\mathbf{P}(K)$ will denote the uniform closure on K of the polynomials in $\mathbb{C}^n$ and $\hat{K}$ will denote the polynomially convex hull of K, defined as the set $\{z \in \mathbb{C}^n : |f(z)| \leq \sup_K |f| \text{ for all polynomials } f \text{ in } \mathbb{C}^n\}$. The maximal ideal space of $\mathbf{P}(K)$ can be identified with $\hat{K}$. Then the Shilov boundary of $\mathbf{P}(K)$ is identified with a subset of K.

LEMMA 1.3. *Let Γ_1 be a finite union of smooth curves in $\mathbb{C}^n$ and let β be a smooth arc in $\mathbb{C}^n$ which is disjoint from Γ_1. Then*

$$(\widehat{\Gamma_1 \cup \beta}) = \widehat{\Gamma_1} \cup \beta.$$

Proof. We need only to show that $(\widehat{\Gamma_1 \cup \beta}) \subseteq \widehat{\Gamma_1} \cup \beta$, the opposite inclusion being trivial. We know by Stolzenberg [St] that $\widehat{\Gamma_1 \cup \beta} \setminus \Gamma_1 \cup \beta$ is a possibly empty, one-dimensional subvariety V of $\mathbb{C}^n \setminus \Gamma_1 \cup \beta$. We claim that $V \subseteq \widehat{\Gamma_1}$. Suppose not. Then there exists a polynomial f in $\mathbb{C}^n$ such that $|f| < 1/2$ on $\widehat{\Gamma_1}$ and $f(p) = 1$ for some $p \in V$. We can adjust f so that $f \neq 1$ on β. Set $g = 1 - f$. Then $\operatorname{Re}(g) > 0$ on Γ_1 and so g has a continuous logarithm on Γ_1. As $g \neq 0$ on β, g also has a continuous logarithm on the arc β. Hence, Γ_1 and β being disjoint, g has a logarithm on $\Gamma_1 \cup \beta \supseteq bV = \bar{V} \setminus V$. Thus, by the argument principle [St], g has no zeros on V. But $g(p) = 0$, a contradiction. Hence the claim $V \subseteq \widehat{\Gamma_1}$. Therefore $\widehat{\Gamma_1 \cup \beta} \subseteq \widehat{\Gamma_1} \cup \beta$ and the lemma follows. $\square$

LEMMA 1.4. *Let Γ be a finite union of smooth disjoint simple closed curves in $\mathbb{C}^n$. Suppose that*

(a) Γ *is contained in the closure of* $\hat{\Gamma} \setminus \Gamma$, *and*

(b) Γ *is the Shilov boundary of* $\mathbf{P}(\hat{\Gamma})$.

Let E be the complement in Γ of the set of points $p \in \Gamma$ such that the pair $(\hat{\Gamma}, \Gamma)$ is locally a smooth 2-manifold with boundary (contained in Γ) in a neighborhood of p. Then $E \subseteq \Gamma$ is compact with $\mathcal{H}^1(E) = 0$.

Proof. The set of points of Γ where $(\hat{\Gamma}, \Gamma)$ is locally a smooth 2-manifold is open in Γ and so E is compact. By [St], $\hat{\Gamma} \setminus \Gamma$ is a nonempty 1-dimensional subvariety of $\mathbb{C}^n \setminus \Gamma$. We argue by contradiction and suppose that $\mathcal{H}^1(E) > 0$. Let $p \in E$ be a point of density in Γ of E. Choose a polynomial f such that p is a regular point of $f|\Gamma$. Hence there is a subarc τ of Γ such that $\mathcal{H}^1(\tau \cap E) > 0$ and such that f maps τ diffeomorphically to an arc $\tau' \subseteq \mathbb{C}$. Since the set of singular values of $f|\Gamma$ has $\mathcal{H}^1$-measure zero, by shrinking τ and τ' we can further assume that τ' contains no singular values of $f|\Gamma$ and that $f^{-1}(\tau') \cap \Gamma$ is the union of s arcs $\tau_1, \tau_2, \cdots \tau_s$ such that $\tau_1 = \tau$ and each τ_j is mapped by f diffeomorphically to τ'. Choose a small neighborhood ω of τ' in $\mathbb{C}$ such that $f(\Gamma) \cap \omega = \tau'$ and $\omega \setminus \tau'$ is the union of two components Ω_1 and Ω_2. Then $f|(f^{-1}(\Omega_j) \cap \hat{\Gamma})$ is a branched analytic cover of Ω_j of some finite order, $j = 1, 2$. Therefore, after possibly shrinking τ', we can choose a neighborhood $\mathcal{U}$ of τ in $\mathbb{C}^n$ such that $f|(f^{-1}(\Omega_j) \cap (\hat{\Gamma} \cap \mathcal{U}))$ is a branched analytic cover of Ω_j of order $m_j \geq 0$ with m_j *at most* equal to 1. Hypothesis (a) implies that not both m_j can be equal to 0. If $m_1 = 1$ and $m_2 = 1$ then $f^{-1}(\Omega_j) \cap (\hat{\Gamma} \cap \mathcal{U})$ is a graph of an analytic map F_j on Ω_j for $j = 1$ and $j = 2$. The graphs F_1 and F_2 have identical boundary values on τ' equal to $(f|\tau)^{-1}$ and therefore continue analytically across τ' to give a single analytic map F on ω. This implies that $\hat{\Gamma} \cap \mathcal{U}$ is an analytic variety and this means that τ is disjoint from the Shilov boundary of $\mathbf{P}(\hat{\Gamma})$. This contradicts the hypothesis (b). Thus we are left only with the case that exactly one of the $m_j = 1$ and the other multiplicity is 0. Then the map F_j extends smoothly to τ' and parametrizes $(\hat{\Gamma}, \Gamma)$ near points of τ as a 2-manifold with boundary. Therefore τ is disjoint from E. This is a contradiction and the lemma follows. $\square$

LEMMA 1.5. *Let γ be a finite union of smooth curves in $\mathbb{C}^n$ and let $x \in \hat{\gamma} \setminus \gamma$. There exists a polynomial P in $\mathbb{C}^n$ such that $P(x) = 0$ and $P \neq 0$ on $\hat{\gamma} \setminus \{x\}$.*

Proof. We claim that there exists a complex linear map $\phi = (\phi_1, \phi_2) : \mathbb{C}^n \to \mathbb{C}^2$ such that $(\phi|\hat{\gamma})^{-1}(\phi(x)) = \{x\}$. Set $V = \hat{\gamma} \setminus \gamma$; by [St], $\hat{\gamma} \setminus \gamma$ is a 1-dimensional subvariety of $\mathbb{C}^n \setminus \gamma$. First choose a linear function ϕ_1 so that $\phi_1(x) \notin \phi_1(\gamma)$. Set $q_1 = \phi_1(x)$. Then $\phi_1^{-1}(q_1) \cap V$, being a 0-variety bounded away from γ, is a finite set $\{y_1 = x, y_2, \cdots, y_m\} \subseteq \mathbb{C}^n$. Choose a linear function ϕ_2 such that ϕ_2 separates the m points $\{y_1, y_2, \cdots, y_m\} \subseteq \mathbb{C}^n$. Then $\phi = (\phi_1, \phi_2) : \mathbb{C}^n \to \mathbb{C}^2$ satisfies $(\phi|\hat{\gamma})^{-1}(\phi(x)) = \{x\}$.

Set $\gamma_0 = \phi(\gamma) \subseteq \mathbb{C}^2$, $V_0 = \phi(V) \subseteq \mathbb{C}^2$ and $q = \phi(x)$. By the maximum principle, $V_0 \subseteq \widehat{\gamma_0}$; also $q \in \widehat{\gamma_0} \setminus \gamma_0$ and $(\phi|\hat{\gamma})^{-1}(q) = \{x\}$.

Let ℓ be an affine complex line in $\mathbb{C}^2$ such that q is an isolated point in $\ell \cap \widehat{\gamma_0}$. (Recall that $\widehat{\gamma_0} \setminus \gamma_0$ is a 1-dimensional subvariety of $\mathbb{C}^2 \setminus \widehat{\gamma_0}$.) Since $\widehat{\gamma_0}$ is polynomially convex, we can find Runge domains in $\mathbb{C}^2$ that decrease down to $\widehat{\gamma_0}$. In particular, there exists a Runge domain Ω containing $\widehat{\gamma_0}$ such that, L, the *connected component* of $\Omega \cap \ell$ that contains q, satisfies $L \cap \widehat{\gamma_0} = \{q\}$ and L is a hypersurface in Ω. By Serre [S] and Andreotti-Narasimhan[AN], since Ω is Runge in $\mathbb{C}^2$, $\check{H}^2(\Omega, \mathbb{Z}) = 0$. Hence by the Cousin II problem, there exists a function F_0 holomorphic on Ω such that $L = \{z \in \Omega : F_0(z) = 0\}$. In particular, $F_0 \neq 0$ on $\widehat{\gamma_0} \setminus \{q\}$ and $dF_0(q) \neq 0$. Set $F = F_0 \circ \phi$. Then, since $V_0 \subseteq \widehat{\gamma_0}$, F is a holomorphic function on a neighborhood of $\hat{\gamma}$ and the only zero of F on $\hat{\gamma}$ occurs at x. Approximating F uniformly on a neighborhood of $\hat{\gamma}$ by polynomials then gives the desired P. □

C. *The Arens-Royden theorem.* Let K be a compact space. We denote the algebra of continuous complex-valued functions on K by $\mathbf{C}(K)$ and denote the invertible elements (i.e. nonvanishing functions) in $\mathbf{C}(K)$ by $\mathbf{C}^{-1}(K)$. Then $\mathbf{C}^{-1}(K)$ is an abelian group under multiplication and contains the subgroup $\exp(\mathbf{C}(K)) = \{e^f : f \in \mathbf{C}(K)\}$. By a theorem of Bruschlinsky the quotient group $\mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$ is naturally isomorphic to $\check{H}^1(K, \mathbb{Z})$, the first Čech cohomology group with integer coefficients.

Let A be a uniform algebra on K. Denote the invertible elements in A by A^{-1}. (If K is the maximal ideal space of A then A^{-1} is just the set of $f \in A$ such that $f \neq 0$ on K.) The multiplicative group A^{-1} contains the subgroup $\exp(A) = \{e^f : f \in A\}$. The Arens-Royden theorem states that if K is the maximal ideal space of A, then the quotient group $A^{-1}/\exp(A)$ is naturally isomorphic to $\check{H}^1(K, \mathbb{Z})$. Moreover, this isomorphism factors through the previous one in the sense that the natural map $A^{-1}/\exp(A) \to \mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$ induced by the inclusion $A \to \mathbf{C}(K)$ is an isomorphism.

For K a compact subset of $\mathbb{C}^n$, if K is polynomially convex, then K is the maximal ideal space of $\mathbf{P}(K)$ and we have the natural isomorphism $j : \mathbf{P}^{-1}(K)/\exp(\mathbf{P}(K)) \to \mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$ provided by the Arens-Royden theorem. (In this setting, an easy proof of the Arens-Royden theorem can be obtained by approximating K by Runge domains Ω, applying the fact ([GR, Th. 7, p. 250]) that $\check{H}^1(\Omega, \mathbb{Z}) \simeq \check{H}^0(\Omega, \mathcal{O}^*)/\exp(\check{H}^0(\Omega, \mathcal{O}))$, and taking the inductive limit over Ω.) The isomorphism j reduces the problem of finding a polynomial on K with certain periods to producing a nonvanishing continuous function with those periods.

Lemma 1.6. *Let K be a polynomially convex compact subset of $\mathbb{C}^n$ and let σ be a* 1*-cycle contained in K. Let $f \in \mathbf{C}^{-1}(K)$. Then there exists a polynomial P in $\mathbb{C}^n$ such that*

$$\Delta_\sigma(\arg P) = \Delta_\sigma(\arg f).$$

Remark. The notation $\Delta_\sigma(\arg f)$ denotes the variation of the argument of f along the oriented 1-cycle σ. If f and σ are smooth, $i\Delta_\sigma(\arg f) = \int_\sigma df/f$. Alternatively, $\Delta_\sigma(\arg f)/2\pi$ is the degree of $f/|f|$ as a map $\sigma \to S^1$.

Proof. Let $[f]$ be the class of f in $\mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$. Since the natural map $\mathbf{P}^{-1}(K)/\exp(\mathbf{P}(K)) \to \mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$ is surjective, there exists $F \in \mathbf{P}^{-1}(K)$ such that $[F] = [f]$, where $[F] \in \mathbf{C}^{-1}(K)/\exp(\mathbf{C}(K))$. Hence there exists $u \in \mathbf{C}(K)$ such that $F = fe^u$. Since $\Delta_\sigma(\arg e^u) = 0$, we get $\Delta_\sigma(\arg F) = \Delta_\sigma(\arg f) + \Delta_\sigma(\arg e^u) = \Delta_\sigma(\arg f)$. Finally we can approximate F uniformly on K by a polynomial P so that $|\Delta_\sigma(\arg P) - \Delta_\sigma(\arg F)| < 2\pi$. Therefore $\Delta_\sigma(\arg P) = \Delta_\sigma(\arg F)$, since σ, being a cycle ("closed"), each Δ_σ term is an integral multiple of 2π. Hence $\Delta_\sigma(\arg P) = \Delta_\sigma(\arg F) = \Delta_\sigma(\arg f)$. □

To apply Lemma 1.6 we shall need the following explicit version of the Bruschlinsky theorem.

Lemma 1.7. *Let S_0 be a compact bordered Riemann surface, not necessarily connected, and let F be a finite subset of S_0. Let S be obtained from S_0 by identifying points in the classes of some partition of F. Let γ be a disjoint union of Jordan curves in S such that $[\gamma] \neq 0$ in $H_1(S, \mathbb{Z})$. Then there exists $f \in \mathbf{C}^{-1}(S)$ such that $\Delta_\gamma(\arg f) < 0$.*

Proof. We claim that $H_1(S, \mathbb{Z})$ is torsion-free. Let $p : S_0 \to S$ be the identification map. We verify the claim in three steps. (a) $H_1(S_0, F; \mathbb{Z})$ is torsion-free. This follows from the exact sequence

$$0 \to H_1(S_0, \mathbb{Z}) \to H_1(S_0, F, \mathbb{Z}) \to H_0(F, \mathbb{Z})$$

and the fact that $H_1(S_0, \mathbb{Z})$ is torsion-free, as is $H_0(F, \mathbb{Z})$. (b) The induced map $p_* : H_1(S_0, F; \mathbb{Z}) \to H_1(S, p(F); \mathbb{Z})$ is an isomorphism, as is easily checked using Mayer-Vietoris sequences to localize at $p(F)$ and to separate the branches of S. (c) From (a) and (b) we conclude that $H_1(S, p(F); \mathbb{Z})$ has no torsion and hence our claim follows from the exact sequence

$$0 \to H_1(S, \mathbb{Z}) \to H_1(S, p(F); \mathbb{Z}).$$

We write $[S, S^1]$ for the set of homotopy equivalence classes of continuous functions $f : S \to S^1 \subseteq \mathbb{C}$. In this case homotopy equivalence is the same as equivalence mod $e^{\mathbf{C}(S)}$ in $\mathbf{C}^{-1}(S)$. As S is a CW complex (even a finite simplicial complex) we can apply a classification theorem (see Spanier, [Sp, Th. 8.1.8, p. 427]) to conclude that there is a natural isomorphism $\psi : [S, S^1] \to H^1(S, \mathbb{Z})$ given by $\psi([f])([\beta]) = \Delta_\beta(\arg f)$, for all continuous functions $f : S \to S^1 \subseteq \mathbb{C}$ and all 1-cycles β in S. Hence, for all $T \in H^1(S, \mathbb{Z}) = \mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{Z})$, there exists $f \in \mathbf{C}^{-1}(S)$ such that for all 1-cycles β in S, $T([\beta]) = \Delta_\beta(\arg f)$.

Finally since $[\gamma] \neq 0$ in $H_1(S, \mathbb{Z})$ and since $H_1(S, \mathbb{Z})$ is torsion-free, there exists $T \in H^1(S, \mathbb{Z}) = \mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{Z})$ such that $T([\gamma]) \neq 0$. By the previous paragraph, there exists $f \in \mathbf{C}^{-1}(S)$ such that $\Delta_\gamma(\arg f) = T([\gamma]) \neq 0$. If $\Delta_\gamma(\arg f) < 0$ we are done; otherwise we replace f by $1/f$. □

2. Proof of Theorem 3

LEMMA 2.1. *Let γ be a smooth compact oriented* 1*-chain in* $\mathbb{C}^n$ *satisfying the moment condition. Let*

$$T = \sum n_j [V_j]$$

be the unique holomorphic 1*-chain in* $\mathbb{C}^n \setminus \gamma$ *such that* $bT = [\gamma]$ *whose existence is given by the Harvey-Lawson theorem. If* γ *satisfies the linking condition, then* T *is positive*; *i.e.*, $n_j > 0$ *for all* j.

Proof. Fix an index k and a point $x \in V_k$ such that $x \notin \overline{V_j}$ for $j \neq k$. By the maximum principle, supp $T \subseteq \hat{\gamma}$. By Lemma 1.5 there exists a polynomial P in $\mathbb{C}^n$ such that $P(x) = 0$ and $P \neq 0$ on $\hat{\gamma} \setminus \{x\}$. Hence for $A = \mathbf{Z}(P)$,

$$0 \leq \operatorname{link}(\gamma, A) = \sum n_j \cdot \#(V_j, A).$$

For $j \neq k$, $A \cap V_j = \emptyset$ and so $\#(V_j, A) = 0$. We have therefore $0 \leq n_k \cdot \#(V_k, A)$. As $P(x) = 0$, $\#(V_k, A) > 0$ and we get that $0 \leq n_k$. We conclude that $0 < n_k$. ◻

We first prove Theorem 3 in two special cases.

Case (i). *γ is a simple closed oriented smooth curve.*

Proof. If γ is polynomially convex, then $\mathbf{P}(\gamma) = \mathbf{C}(\gamma)$. Hence, first choosing an $f \in \mathbf{C}(\gamma)$ such that f maps γ to the unit circle with $\frac{1}{2\pi}\Delta_\gamma(\arg f) = -1$, we get a polynomial P such that $\frac{1}{2\pi}\Delta_\gamma(\arg P) = -1$. Therefore, by Lemma 1.2, $\operatorname{link}(\gamma, A) = -1$ where $A = \mathbf{Z}(P)$, and this contradicts the linking condition. We conclude that γ is not polynomially convex. It follows that $V = \hat{\gamma} \setminus \gamma$ is a 1-variety of finite area in $\mathbb{C}^n \setminus \gamma$ and $b[V] = [\gamma]$, as currents; cf. Lemma 2.4 below. Let ψ be a holomorphic $(1,0)$-form in $\mathbb{C}^n$. Then, since $[V]$ is a $(1,1)$-current and $d\psi$ is a $(2,0)$-form, we get

$$\int_\gamma \psi = [V](d\psi) = 0.$$

This says that γ satisfies the moment condition, proving the theorem in case (i).

Case (ii). *γ is a real analytic* 1*-cycle.*

Proof. We claim that $[\gamma] = 0$ in $H_1(\hat{\gamma}, \mathbb{Z})$. Suppose, by way of contradiction, that $[\gamma] \neq 0$ in $H_1(\hat{\gamma}, \mathbb{Z})$. Since $\hat{\gamma}$ is a compact bordered Riemann surface with a finite number of points identified, we can apply Lemma 1.7 to obtain an $f \in \mathbf{C}^{-1}(\hat{\gamma})$ such that $\frac{1}{2\pi}\Delta_\gamma \arg f < 0$. By the Arens-Royden theorem in the form of Lemma 1.6, there is a polynomial P such that $\frac{1}{2\pi}\Delta_\gamma \arg P = \frac{1}{2\pi}\Delta_\gamma \arg f < 0$. This contradicts the linking condition. We conclude that $\gamma \sim 0$ in $\hat{\gamma}$.

We claim that γ satisfies the moment condition. Then, by Lemma 2.1, Theorem 3 follows in this case. Let ψ be a holomorphic 1-form in $\mathbb{C}^n$. Then $d\psi$ is a $(2,0)$-form and so $d\psi = 0$ on the one-dimensional analytic set $\hat{\gamma}$. But, since $\gamma \sim 0$ in $\hat{\gamma}$, $\gamma = b\Sigma$ where Σ is a 2-chain in $\hat{\gamma}$. Hence by Stokes' theorem, $\int_\gamma \psi = \int_\Sigma d\psi = 0$. This is the moment condition.

Case (iii). *The general case.*

Proof. Arguing as in case (i) we see that γ cannot be polynomially convex. Thus we can suppose that γ is not polynomially convex in the general case. Choose a minimal subfamily $\mathcal{F} \subseteq \{\gamma_j\}$ such that the polynomial hull of the sum $\Gamma = \sum\{\gamma_j : \gamma_j \in \mathcal{F}\}$ satisfies $\hat{\Gamma} \setminus \gamma = \hat{\gamma} \setminus \gamma$; then $\mathcal{F} \neq \emptyset$ because γ is not polynomially convex. Let $\sigma = \sum\{\gamma_j : \gamma_j \notin \mathcal{F}\}$. We get a partition of γ as $\gamma = \Gamma + \sigma$. We are abusing language somewhat, since we write γ, Γ and σ as oriented 1-cycles and also, when we take the polynomially convex hull, as the corresponding underlying sets in $\mathbb{C}^n$. Let $V = \hat{\Gamma} \setminus \Gamma$; V is a one dimensional subvariety of $\mathbb{C}^n \setminus \Gamma$. Let $\mathfrak{S}$ denote the Shilov boundary of the uniform algebra $\mathbf{P}(\hat{\Gamma})$.

Lemma 2.2. (a) $\Gamma = \mathfrak{S}$ *and* (b) $\Gamma \subseteq \overline{\hat{\Gamma} \setminus \Gamma}$.

Proof. (a) Clearly $\mathfrak{S} \subseteq \Gamma$. We need only show that $\Gamma \subseteq \mathfrak{S}$. Arguing by contradiction, we suppose otherwise. Then there exists an open subarc τ of some $\gamma_k \in \mathcal{F}$ such that $\tau \subset \widehat{\Gamma \setminus \tau}$. Put $\mathcal{F}_1 = \mathcal{F} \setminus \gamma_k$, $\Gamma_1 = \sum\{\gamma_j : \gamma_j \in \mathcal{F}_1\}$ and $\beta = \gamma_k \setminus \tau$. Then $\Gamma \setminus \tau = \Gamma_1 \cup \beta$. By Lemma 1.3, $(\widehat{\Gamma_1 \cup \beta}) = \widehat{\Gamma_1} \cup \beta$ and so $\widehat{\Gamma_1} \cup \beta$ is polynomially convex. Hence $\Gamma \subseteq \widehat{\Gamma_1} \cup \beta$. Therefore $\hat{\Gamma} \setminus \gamma = \widehat{\Gamma_1} \setminus \gamma$. Thus, as $\mathcal{F}_1$ is a proper subset of $\mathcal{F}$, this contradicts the minimality of $\mathcal{F}$. Part (a) follows.

(b) We argue by contradiction and suppose that there exists an open subarc τ of some $\gamma_k \in \mathcal{F}$ such that τ is disjoint from $\overline{\hat{\Gamma} \setminus \Gamma}$. Then, by the local maximum modulus principle, $\hat{\Gamma} \setminus \Gamma \subseteq \widehat{\Gamma \setminus \tau}$. As in part (a), $\Gamma \setminus \tau = \Gamma_1 \cup \beta$ and $\hat{\Gamma} \setminus \gamma = \widehat{\Gamma_1} \setminus \gamma$. Again this contradicts the minimality of $\mathcal{F}$ and part (b) follows. □

The next lemma is due essentially to Lawrence [L], who treats the case of a simple closed rectifiable curve Γ. We shall briefly indicate how his proof adapts to our setting, in which Γ is smooth, but not connected.

Lemma 2.3. *The* 1*-variety* $V = \hat{\Gamma} \setminus \Gamma$ *has finite area* ($\mathcal{H}^2$ *measure*) *and the corresponding positive* $(1,1)$*-current* $[V]$ (*oriented by the natural orientation of* V) *satisfies*

$$b[V] = \sum\{\varepsilon_j[\gamma_j] : \gamma_j \in \mathcal{F}\}, \tag{2.1}$$

where each $\varepsilon_j = \pm 1$.

Remark. We do not use the linking hypothesis in Lemma 2.3. Without that hypothesis, it is, in general, not true that $b[V] = [\Gamma]$ for $V = \hat{\Gamma} \setminus \Gamma$ with Γ a 1-cycle in $\mathbb{C}^n$. For example, take Γ to be the unit circle in $\mathbb{C}$ with the clock-wise orientation; then V is the open unit disk and $b[V] = -[\Gamma]$. It is the addition of the linking hypothesis for γ that will yield the correct orientation in Lemma 2.4.

Proof. Lawrence's argument that $\hat{\Gamma} \setminus \Gamma$ has finite area is valid when Γ is a finite union of simple closed smooth curves. Hence the $(1,1)$-current $[V]$ exists with supp$([V] \subseteq \Gamma$. Lawrence's arguments, together with Lemma 2.2, imply that $b[V] = \mathcal{H}^1 \llcorner \Gamma \wedge \eta$ where η is a Borel measurable *unit* tangent vectorfield to Γ; in particular $b[V]$ has multiplicity 1 at almost every point of Γ. Finally Lawrence's argument shows that $(b[V]) \llcorner \gamma_j = \pm[\gamma_j]$ for each $\gamma_j \in \mathcal{F}$, since $[\gamma_j]$ is an indecomposable integral current. This gives (2.1). □

LEMMA 2.4. *With V as in Lemma* 2.3,

$$b[V] = [\Gamma].$$

Proof. We need to show that $\varepsilon_j = 1$ for all j. Fix an index k with $\gamma_k \in \mathcal{F}$. Since $\gamma_k \subseteq \mathfrak{S}$ by Lemma 2.2 (a), we can choose a polynomial F so that $F(x) = 1$ for some $x \in \gamma_k$ and $|F| < 1/2$ on the set $\gamma \setminus \gamma_k$. Choose, by Lemma 2.2 (b), a point $\lambda \in F(\hat{\gamma}) \setminus F(\gamma)$ with $|\lambda| > 1/2$ and set $A = \mathbf{Z}(F - \lambda)$, a complex hypersurface in $\mathbb{C}^n$. Then A is disjoint from γ and, by the linking hypothesis on γ, $\operatorname{link}(\gamma, A) \geq 0$.

On all γ_j, $j \neq k$, $|F| < 1/2 < |\lambda|$; hence $F - \lambda$ has a logarithm on γ_j and so

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{d(F-\lambda)}{F-\lambda} = 0. \tag{2.2}$$

Hence

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d(F-\lambda)}{F-\lambda} = \sum_j \frac{1}{2\pi i} \int_{\gamma_j} \frac{d(F-\lambda)}{F-\lambda} = \frac{1}{2\pi i} \int_{\gamma_k} \frac{d(F-\lambda)}{F-\lambda}.$$

Therefore we have

$$0 \leq \operatorname{link}(\gamma, A) = \frac{1}{2\pi i} \int_{\gamma} \frac{d(F-\lambda)}{F-\lambda} = \frac{1}{2\pi i} \int_{\gamma_k} \frac{d(F-\lambda)}{F-\lambda}. \tag{2.3}$$

From this we will deduce that $\varepsilon_k > 0$. We can assume that λ is a regular value of $F|V$ and that V is s-sheeted over the component Ω of $\mathbb{C} \setminus F(\gamma)$ containing λ, $s \geq 1$. Hence there exists a small closed disk $\Delta \subseteq \Omega$ centered at λ such that $F^{-1}(\Delta) \cap V$ is the disjoint union of s components $\Delta_1, \Delta_2, \cdots, \Delta_s$ each of which is mapped biholomorphically to Δ. By (2.1),

$$b[V \setminus \cup_{i=1}^s \Delta_i] = b[V] - \sum_{i=1}^s b[\Delta_i] = \sum \{\varepsilon_j[\gamma_j] : \gamma_j \in \mathcal{F}\} - \sum_{i=1}^s b[\Delta_i].$$

Hence we get, as $\omega = \frac{1}{2\pi i}\frac{d(F-\lambda)}{F-\lambda}$ is a closed 1-form on $V \setminus \cup_{i=1}^s \Delta_i$,

$$\begin{aligned} 0 = [V \setminus \cup_{i=1}^s \Delta_i](d\omega) &= b[V \setminus \cup_{i=1}^s \Delta_i](\omega) \\ &= \sum \{\varepsilon_j[\gamma_j](\omega) : \gamma_j \in \mathcal{F}\} - \sum_{i=1}^s b[\Delta_i](\omega). \end{aligned}$$

By the Cauchy integral formula,

$$b[\Delta_i](\omega) = \frac{1}{2\pi i}\int_{b\Delta} \frac{dz}{z-\lambda} = 1$$

for each index i. Thus, applying (2.2), we get

$$0 = \varepsilon_k \frac{1}{2\pi i}\int_{\gamma_k} \frac{d(F-\lambda)}{F-\lambda} - s. \tag{2.4}$$

Since $s \neq 0$, (2.4) implies that

$$\frac{1}{2\pi i}\int_{\gamma_k} \frac{d(F-\lambda)}{F-\lambda} \neq 0.$$

Hence, by (2.3),

$$\frac{1}{2\pi i}\int_{\gamma_k} \frac{d(F-\lambda)}{F-\lambda} > 0.$$

Now (2.4) implies that $\varepsilon_k > 0$. Therefore $\varepsilon_k = 1$ and this gives the lemma. □

Now consider the above partition γ as $\gamma = \Gamma + \sigma$, where $\hat{\Gamma} \setminus \gamma = \hat{\gamma} \setminus \gamma$. We set $V = \hat{\Gamma} \setminus \Gamma$. Consider the two cases:

1. $\sigma \not\subseteq V$, or 2. $\sigma \subseteq V$.

Case 1. Fix $x \in \sigma$ with $x \notin \hat{\Gamma}$. Then $x \in \gamma_k$ for some γ_k which is not one of the curves which comprise Γ. We construct a smooth complex-valued function f on $\hat{\gamma}$ as follows: first take $f \equiv 1$ on all of $\hat{\gamma}$ except for a small subarc v of γ_k such that $x \in v$ and $\bar{v} \cap \hat{\Gamma} = \emptyset$. We can then extend f so that the image of f on v winds once negatively about the unit circle. Then f is nonvanishing on $\hat{\gamma}$. As $f \equiv 1$ on $\hat{\Gamma}$, we have $f \in \mathbf{P}(\hat{\Gamma})$. By the hypothesis for case 1, $\hat{\gamma}$ is the union of $\hat{\Gamma}$ and some of the σ curves which are not contained in $\hat{\Gamma}$. Hence (see [St]) $f \in \mathbf{P}(\hat{\gamma})$. By our construction

$$\frac{1}{2\pi i}\int_{\gamma_k} \frac{df}{f} = \frac{1}{2\pi i}\int_{v} \frac{df}{f} = -1$$

since $f \equiv 1$ on $\gamma_k \setminus v$, and

$$\frac{1}{2\pi i}\int_{\gamma_j} \frac{df}{f} = 0$$

for $j \neq k$, since $f \equiv 1$ on these γ_j. Hence

$$\frac{1}{2\pi i}\int_{\gamma} \frac{df}{f} = -1.$$

Approximating $f \in \mathbf{P}(\hat{\gamma})$ we get a polynomial P such that $P \neq 0$ on $\hat{\gamma}$ and

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dP}{P} = -1.$$

By Lemma 1.6, this gives $\operatorname{link}(\gamma, A) = -1$ where $A = \{z \in \mathbb{C}^3 : P(z) = 0\}$. This contradicts the linking condition and we conclude that Case 1 cannot arise.

Case 2. Let E be the set of "bad" points of Γ given in Lemma 1.4. Locally at each point of $\Gamma \setminus E$, $\hat{\Gamma} \setminus \Gamma$ is a 2-manifold with boundary. Choose ψ a real-valued $\mathcal{C}^\infty$ function on $\mathbb{C}^n$ such that $\psi \geq 0$ on $\mathbb{C}^n$ and $\psi = 0$ on E. Choose, by Sard's theorem, $\varepsilon > 0$ ("admissible") so that

(a) ε is a regular value of $\psi|\Gamma$,

(b) ε is a regular value of $\psi|V_{\text{reg}}$,

(c) $\psi \neq \varepsilon$ on V_{sing}.

Set $D_\varepsilon = \{z \in \hat{\Gamma} : \psi \geq \varepsilon\}$, $Q_\varepsilon = \{z \in \hat{\Gamma} : \psi \leq \varepsilon\}$ and $\alpha_\varepsilon = \{z \in \hat{\Gamma} : \psi = \varepsilon\}$. Then $\alpha_\varepsilon = \tau_\varepsilon + \rho_\varepsilon$ where τ_ε is a finite set of closed curves in V_{reg} and ρ_ε is a finite union of arcs joining two points of Γ and, except for its endpoints, lying in V_{reg}. Except for the finite set $V_{\text{sing}} \cap D_\varepsilon$, D_ε is a topological manifold with boundary bD_ε, where bD_ε is piecewise smooth consisting of the oriented curves τ_ε and other oriented curves, whose sum we denote by κ_ε; thus κ_ε is a sum of some subarcs of Γ-curves and the arcs of ρ_ε. Thus $bD_\varepsilon = \tau_\varepsilon + \kappa_\varepsilon$.

We consider two subcases:

Case 2a: There exists an (admissible) $\varepsilon > 0$, such that $[\sigma] \neq 0$ in $H_1(D_\varepsilon, \tau_\varepsilon; \mathbb{Z})$ or

Case 2b: For all (admissible) $\varepsilon > 0$, $[\sigma] = 0$ in $H_1(D_\varepsilon, \tau_\varepsilon; \mathbb{Z})$.

Before considering case 2a we need two lemmas. For a nonvanishing continuous complex-valued function h defined on an oriented 1-cycle C, we denote the index of h on C by $\operatorname{Ind}(h, C)$. This equals both $\frac{1}{2\pi}\Delta_C(\arg h)$ and the winding number of the curve $h(C)$ about the origin.

LEMMA 2.5. *Let A be the planar annulus $\{z \in \mathbb{C} : a \leq |z| \leq b\}$, $a < b$, and let $\Gamma_a = \{z \in \mathbb{C} : |z| = a\}$ and $\Gamma_b = \{z \in \mathbb{C} : |z| = b\}$, both positively oriented.*

(a) *Let h be a nonvanishing continuous complex-valued function defined on $bA = \Gamma_b - \Gamma_a$ such that* $\operatorname{Ind}(h, \Gamma_a) = \operatorname{Ind}(h, \Gamma_b)$. *Then h extends to be a nonvanishing continuous complex-valued function H defined on A.*

(b) *Let g be a nonvanishing continuous complex-valued function defined on $\Gamma_b \cup S$ where S is a proper closed subset of Γ_a. Then g extends to be a nonvanishing continuous complex-valued function defined on A.*

Proof. (a) This is a special case of a more more general extension theorem of Hopf. We give a short proof for our special case. Let $k = \mathrm{Ind}(h, \Gamma_a) \in \mathbb{Z}$. Set $q = hz^{-k}$ on bA. Then q satisfies $\mathrm{Ind}(q, \Gamma_a) = \mathrm{Ind}(q, \Gamma_b) = 0$. Hence q has a complex-valued logarithm u on bA, i.e. $q = e^u$ on bA. By Tietze's extension theorem, we can extend u to be a continuous complex-valued function Φ on A. Now take $H = z^k e^\Phi$ on A.

(b) By part (a) it suffices to extend g to a nonvanishing continuous complex-valued function on Γ_a such that $\mathrm{Ind}(g, \Gamma_a) = \mathrm{Ind}(g, \Gamma_b)$. Let τ be an open subarc of Γ_a whose closure is disjoint from S. Then $\Gamma_a \setminus \tau$ is a closed interval containing S. Hence we can extend g from S to be a continuous complex-valued nonvanishing function on $\Gamma_a \setminus \tau$ with $g = 1$ on the two endpoints of $\Gamma_a \setminus \tau$. Since $g = 1$ on the two endpoints of $\Gamma_a \setminus \tau$, $\frac{1}{2\pi}\Delta_{\Gamma_a\setminus\tau}(\arg g) \in \mathbb{Z}$. Hence $j = \mathrm{Ind}(g, \Gamma_b) - \frac{1}{2\pi}\Delta_{\Gamma_a\setminus\tau}(\arg g) \in \mathbb{Z}$. Now we can extend g over τ such that, on τ, g is a map covering the unit circle j times; that is, g on τ has complex values of modulus one and satisfies $\frac{1}{2\pi}\Delta_\tau(\arg g) = j$. Thus $\mathrm{Ind}(g, \Gamma_a) = \frac{1}{2\pi}\Delta_\tau(\arg g) + \frac{1}{2\pi}\Delta_{\Gamma_a\setminus\tau}(\arg g) = \mathrm{Ind}(g, \Gamma_b)$, as desired. □

LEMMA 2.6. *Let A be the planar annulus $\{z \in \mathbb{C} : a \le |z| \le b\}$, $a < b$, and let $\Gamma_a = \{z \in \mathbb{C} : |z| = a\}$ and $\Gamma_b = \{z \in \mathbb{C} : |z| = b\}$, both positively oriented.*

(a) *Let h be a nonvanishing continuous complex-valued function defined on $|z| \le b$. Then there exists a continuous complex-valued function f defined on $|z| \le b$ such that*

i) $f \neq 0$ *on* $|z| \le b$,

ii) $f = 1$ *on* $|z| \le a$, *and*

iii) $f = h$ *on* Γ_b.

(b) *Let h be a nonvanishing continuous complex-valued function defined on A. Let S be a proper subset of Γ_a. Then there exists a continuous complex-valued function f defined on A such that*

i) $f \neq 0$ *on* A,

ii) $f = 1$ *on* S, *and*

iii) $f = h$ *on* Γ_b.

Proof. (a) Set $f = 1$ on $|z| \le a$. Since $\mathrm{ind}(h, \Gamma_b) = 0$, we can use (a) of Lemma 2.5 to extend f to A so that $f = h$ on Γ_b.

(b) Define g on $\Gamma_b \cup S$ by $g \equiv 1$ on S and $g = h$ on Γ_b. Then apply (b) of Lemma 2.5 to extend g as the desired function on A. $\square$

We first consider Case 2a: We assume that $[\sigma] \neq 0$ in $H_1(D_\varepsilon, \tau_\varepsilon; \mathbb{Z})$ the first relative singular homology group. Let $\tilde{D}_\varepsilon$ be obtained from D_ε by attaching a disk at each of the components of τ_ε. Then $\tilde{D}_\varepsilon$ is a finite simplicial complex. Let $\tilde{\tau}_\varepsilon$ be the union of these closed disks in $\tilde{D}_\varepsilon$. There is a natural map

$$H_1(D_\varepsilon, \tau_\varepsilon; \mathbb{Z}) \to H_1(\tilde{D}_\varepsilon, \tilde{\tau}_\varepsilon; \mathbb{Z})$$

induced by the inclusion $(D_\varepsilon, \tau_\varepsilon) \subseteq (\tilde{D}_\varepsilon, \tilde{\tau}_\varepsilon)$ and, by excision, this map is an isomorphism. Hence $[\sigma] \neq 0$ in $H_1(\tilde{D}_\varepsilon, \tilde{\tau}_\varepsilon; \mathbb{Z})$. From the exact homology sequence

$$H_1(\tilde{D}_\varepsilon; \mathbb{Z}) \to H_1(\tilde{D}_\varepsilon, \tilde{\tau}_\varepsilon; \mathbb{Z})$$

we conclude that $[\sigma] \neq 0$ in $H_1(\tilde{D}_\varepsilon; \mathbb{Z})$. Since $\tilde{D}_\varepsilon$ is a compact bordered Riemann surface with a finite number of points identified, we can apply Lemma 1.7 to obtain an $f \in \mathbf{C}^{-1}(\tilde{D}_\varepsilon)$ such that $\frac{1}{2\pi}\Delta_\sigma \arg f < 0$. By Lemma 2.6, since $\tilde{\tau}_\varepsilon$ is a disjoint union of closed disks disjoint from σ, we can arrange that $f \equiv 1$ on $\tilde{\tau}_\varepsilon$. Since $\sigma \subseteq D_\varepsilon \subseteq \tilde{D}_\varepsilon$, by restricting f to D_ε we get $f \in \mathbf{C}^{-1}(D_\varepsilon)$ such that $\Delta_\sigma(\arg f) < 0$ and $f \equiv 1$ on $\tau_\varepsilon = D_\varepsilon \cap \tilde{\tau}_\varepsilon$.

Since $\kappa_\varepsilon \cap \rho_\varepsilon$ is a *proper* subset of each closed curve in κ_ε, we can apply Lemma 2.6 to arrange that $f \equiv 1$ on ρ_ε. Thus $f \equiv 1$ on $\tau_\varepsilon + \rho_\varepsilon = \alpha_\varepsilon = bQ_\varepsilon$ and thus we can extend f to be a continuous function on $\hat{\Gamma}$ by setting $f \equiv 1$ on Q_ε. To summarize: we have $f \in \mathbf{C}^{-1}(\hat{\Gamma})$ such that $\Delta_\sigma(\arg f) < 0$. Applying the Arens-Royden theorem we get a polynomial P such that $P \neq 0$ on $\hat{\Gamma}$ and $\Delta_\sigma(\arg P) < 0$. This contradicts the linking condition.

We next consider Case 2b. We assume that for all (admissible) $\varepsilon > 0$, $[\sigma] = 0$ in $H_1(D_\varepsilon, \tau_\varepsilon; \mathbb{Z})$. We construct a compact subset K of $\hat{\Gamma} \setminus E$ by adjoining to the set σ, for each component of σ, a path in $\hat{\Gamma} \setminus E$ joining that component to some point of $\Gamma \setminus E$. If necessary we can further enlarge K so that each of the finitely many components of $\hat{\Gamma} \setminus \Gamma$ contains an arc joining one endpoint to a point of $\Gamma \setminus E$. We choose a sequence of admissible ε_k decreasing to 0 and we write D_k for D_{ε_k}, Q_k for Q_{ε_k}, etc. We choose ε_1 small enough so that $K \subseteq D_k$ for all k.

Then, since $[\sigma] = 0$ in $H_1(D_k, \tau_k; \mathbb{Z})$ for all k, there exists a 2-chain Σ_k in D_k and a 1-cycle μ_k in τ_k such that $b\Sigma_k = \sigma + \mu_k$. For $x \in D_k \setminus (\tau_k \cup \sigma)$, the multiplicity of Σ_k at x, defined as the intersection number of homology classes $[\Sigma_k]$ and $[x]$, is constant on each component of $D_k \setminus (\tau_k \cup \sigma)$. We write $\mathrm{mult}(\Sigma_k, x)$ for the multiplicity of Σ_k at $x \in D_k \setminus (\tau_k \cup \sigma)$. Let T_k be the (1,1)-current associated to Σ_k. Then T_k is just integration over the finite number of components of $D_k \setminus (\tau_k \cup \sigma)$ with integral weights given by the multiplicity of Σ_k. We have $bT_k = [\sigma] + [\mu_k]$.

Let N_k be the maximum of $|\mathrm{mult}(\Sigma_k, x)|$ (as x varies over the finite number of connected components of $D_k \setminus (\tau_k \cup \sigma)$).

Lemma 2.7. *$N_k \leq N_1$ for all k.*

Proof. We can assume that the 2-cycle Σ_k is contained in the union of those components of D_k which meet σ, since homology is the sum of the homology of the (path) components. Fix $x \in D_k \setminus (\tau_1 \cup \sigma)$ such that $\mathrm{mult}(\Sigma_k, x) \neq 0$. We shall show that $|\mathrm{mult}(\Sigma_k, x)| \leq N_1$.

Let Ω be the component of D_k which contains x. Then Ω meets σ and so contains components of σ. Let δ be an arc in Ω joining x to a component of σ and otherwise disjoint from σ. Then that component of σ and a path in $\hat{\Gamma} \setminus E$ joining that component to some point of $p \in \Gamma \setminus E$ are contained in a component Ω_1 of $D_1 \setminus \tau_1$ with $\Omega_1 \subseteq \Omega$.

Since $b(\Sigma_k - \Sigma_1) = \mu_k - \mu_1$ is disjoint from Ω_1, the multiplicity of $\Sigma_k - \Sigma_1$ in Ω_1 is constant. We claim that this multiplicity is 0. In fact, near $p \in \Gamma \setminus E$, there is a deformation retraction of Ω_1 to a smaller set which is disjoint from a small neighborhood ω of p. This retraction moves Σ_k and Σ_1 off of ω but does not change $b\Sigma_k$ or $b\Sigma_1$ and so does not change the multiplicity of $\Sigma_k - \Sigma_1$ on Ω_1. As the modified 2-chains each have multiplicity 0 on ω we conclude that $\Sigma_k - \Sigma_1$ has multiplicity 0 on Ω_1. This gives the claim. We have therefore

$$\mathrm{mult}(\Sigma_k, y) = \mathrm{mult}(\Sigma_1, y)$$

for each $y \in \Omega_1$.

Since the multiplicity of Σ_k is constant on components of $\Omega \setminus \sigma$ we have

$$\mathrm{mult}(\Sigma_k, x) = \mathrm{mult}(\Sigma_k, y_0),$$

for some $y_0 \in \Omega_1$— we can just choose a point y_0 on $\delta \cap \Omega_1$. We conclude that

$$\mathrm{mult}(\Sigma_k, x) = \mathrm{mult}(\Sigma_1, y_0).$$

Thus $|\mathrm{mult}(\Sigma_k, x)| \leq |\mathrm{mult}(\Sigma_1, y_0)| \leq N_1$. □

We denote the mass norm of a current T by $\mathbf{M}(T)$. In particular, $\mathbf{M}([Q_\varepsilon])$ is then the "area" ($\mathcal{H}^2$ measure) of Q_ε.

Lemma 2.8. *$\mathbf{M}([Q_\varepsilon]) \to 0$ as $\varepsilon \to 0$.*

Proof. $Q_\varepsilon \subseteq \hat{\Gamma}$ and $\mathcal{H}^2(\hat{\Gamma}) < \infty$. Since $Q_\varepsilon \to E$ as $\varepsilon \to 0$ and $\mathcal{H}^2(E) = 0$, we conclude that $\mathcal{H}^2(Q_\varepsilon) \to 0$ as $\varepsilon \to 0$. □

Lemma 2.9. *$\{T_k\}$ is Cauchy in mass norm.*

Proof. Let $\varepsilon > 0$. Choose a compact subset L contained in $\hat{\Gamma} \setminus E$ such that $\mathcal{H}^2(\hat{\Gamma} \setminus L) < \varepsilon/(2N_1)$ and such that each component of L meets K. Since $\hat{\Gamma} \setminus \Gamma$ consists of a finite number of components each of which meets K, we only need to choose L, using Lemma 2.8, to meet each component C of $\hat{\Gamma} \setminus \Gamma$ in a sufficiently large connected set which meets K at some point of C.

We claim that there exists s_0 such that $s > s_0$ implies that L is contained in the union of the components of D_s which meet K. Indeed, set $\eta = \inf_L \psi$. Choose s_0 so that $\varepsilon_{s_0} < \eta$ and $s > s_0$. Let $x \in L$ and let Ω be the component of D_s which contains x. By the choice of η, $D_s \supseteq L$ and so Ω contains the component of L through x. Hence Ω meets K, by the construction of L. This gives the claim.

Now suppose that $s_0 < j < k$ and consider $T_k - T_j$. We claim that $\mathrm{mult}(\Sigma_k - \Sigma_j, x) = 0$ if $x \in L$. Assume this for the moment. Hence $\mathrm{supp}(T_k - T_j) \subseteq \hat{\Gamma} \setminus L$. Therefore, since $|\mathrm{mult}(\Sigma_k - \Sigma_j, x)| \leq N_k + N_j \leq 2N_1$ by Lemma 2.7, we get

$$\mathbf{M}(T_k - T_j) \leq 2N_1 \mathcal{H}^2(\hat{\Gamma} \setminus L) < \varepsilon.$$

This gives the lemma.

It remains to verify the last claim. Let $x \in L$. Let Ω be the component of D_k containing x. Let Ω_1 be the component of $D_j \setminus \tau_j$ containing x. Then $\Omega_1 \subseteq \Omega$. Since $b(\Sigma_k - \Sigma_j) = \mu_k - \mu_j$ is disjoint from Ω_1, $\Sigma_k - \Sigma_j$ has constant multiplicity in Ω_1. By the construction of s_0, Ω_1 meets K and so there is a path in Ω_1 to the point in $\Gamma \setminus E$. Now we can argue just as in the proof of Lemma 2.7 to conclude that the multiplicity of $\Sigma_k - \Sigma_j$ is 0 in Ω_1. This yields the claim. $\square$

Let $T = \lim T_k$. By Lemma 2.9, T is the limit in mass norm of the normal currents T_k and therefore is a flat current in $\mathbb{C}^n$ (see [F]). From $bT_k = [\sigma] + [\mu_k]$ we conclude that $bT = [\sigma] + S$, where S is a 1-current supported on E, since $\mu_k \subseteq Q_k$ and $\bigcap Q_k = E$. Now, since T is flat, bT is flat. Hence S is flat. Since E is contained in the 1-manifold Γ with $\mathcal{H}^1(E) = 0$ and since $dS = 0$, we conclude, by Federer's support theorem, that $S = 0$. Hence $bT = [\sigma]$.

Now let ϕ be a holomorphic (1,0)-form in $\mathbb{C}^n$. We have

$$\int_\sigma \phi = bT(\phi) = T(d\phi) = \lim T_k(d\phi) = \lim \int_{\Sigma_k} d\phi.$$

But $d\phi = 0$ on the 1-variety $\hat{\Gamma} \setminus \Gamma \supseteq \Sigma_k$. Therefore $\int_\sigma \phi = 0$. Since $b[V] = [\Gamma]$ as currents, we have $\int_\Gamma \phi = \int_V d\phi = 0$, as $d\phi = 0$ on V_{reg}. Finally since $\gamma = \Gamma + \sigma$, we have $\int_\gamma \phi = 0$. This is the moment condition. This completes the proof of Theorem 3. $\square$

Remark. The proof shows that Cases 1 and 2a are not possible. Hence only Case 2b can occur and then $[\gamma] = [\Gamma] + [\sigma] = b[V] + bT = b([V] + T)$. That is, $[\gamma]$ is the boundary of the (1,1) current $[V] + T$ supported in $\hat{\gamma}$. This gives a replacement to the simpler condition that $\gamma \sim 0$ in $\hat{\gamma}$, when γ is real analytic.

3. Proof of Theorem 1

LEMMA 3.1. *Let M be a* 3*-manifold in $\mathbb{C}^3$ satisfying the hypotheses of Theorem* 1*. Let ϕ be a complex linear function on $\mathbb{C}^3$ and let H_λ be the affine complex hyperplane $\{z \in \mathbb{C}^3 : \phi(z) = \lambda\}$ for $\lambda \in \mathbb{C}$. Then, for almost all λ, $H_\lambda \cap M$ is* (*empty or*) *a smooth oriented* 1*-cycle γ_λ* (*i.e. a finite set of disjoint closed curves*) *that bounds a positive holomorphic* 1*-chain in $H_\lambda \setminus \gamma_\lambda$.*

Proof. For almost all λ, by Sard's theorem, $M \cap H_\lambda$ is a smooth 1-cycle, call it γ_λ, and γ_λ carries an orientation as a slice of M by the map $z \mapsto \phi(z)$. We view H_λ as a copy of $\mathbb{C}^2$. We claim that γ_λ satisfies the linking condition hypothesis for Theorem 3. Let A be an algebraic curve in $\mathbb{C}^2 = H_\lambda$ that is disjoint from γ_λ. We can also view A as an algebraic curve in $\mathbb{C}^3 \supseteq H_\lambda$ that is disjoint from M. By Lemma 1.1, $\operatorname{link}(A, M)$, with A a curve in $\mathbb{C}^3$, agrees with $\operatorname{link}(A, \gamma_\lambda)$, with A a curve in $\mathbb{C}^2$. As $\operatorname{link}(A, M) \geq 0$ by the hypothesis of Theorem 1, $\operatorname{link}(A, \gamma_\lambda) \geq 0$. Now Theorem 3 implies that γ_λ bounds a positive holomorphic 1-chain in $H_\lambda \setminus \gamma_\lambda$. □

Proof of Theorem 1. We want to show that M is maximally complex. This means we need to show [HL] that $\int_M \psi = 0$ for every global (p, q)-form ψ with $p + q = 3$ and $|p - q| > 1$. This yields either $(3, 0)$ or $(0, 3)$. By complex conjugation, we can thus assume that ψ is a $(3, 0)$-form; i.e., it suffices to show that

$$\int_M \alpha \, dz_1 \wedge dz_2 \wedge dz_3 = 0 \tag{3.1}$$

for all smooth functions α on M. Without loss of generality we can assume that $\alpha \in \mathcal{C}_0^\infty(\mathbb{C}^3)$. Using the inverse Fourier transform, we can write

$$\alpha(z) = \int_{\mathbb{C}^3} e^{i(z\cdot\zeta + \overline{z\cdot\zeta})} \beta(\zeta) d\mathcal{L}^6(\zeta),$$

where $z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 + z_3\zeta_3$, $\mathcal{L}^6$ is Lebesgue measure on $\mathbb{C}^3$, and $\beta \in \mathcal{S}$ ($\mathcal{S}$ is the class of "rapidly decreasing" functions). Putting this integral expression for α into (3.1) and interchanging the order of integration, we see that it suffices to show that

$$\int_M e^{i(z\cdot\zeta + \overline{z\cdot\zeta})} dz_1 \wedge dz_2 \wedge dz_3 = 0 \tag{3.2}$$

for $\zeta \neq 0$, $\zeta \in \mathbb{C}^3$.

Now fix $\zeta \neq 0$. We introduce a complex linear change of variable in $\mathbb{C}^3$ so that $w_1 = z \cdot (i\zeta), w_2, w_3$ are the new variables. Viewing w_1, w_2, w_3 as functions on $\mathbb{C}^3$, we get $dw_1 \wedge dw_2 \wedge dw_3 = c dz_1 \wedge dz_2 \wedge dz_3$ for some complex constant $c \neq 0$. Thus (3.2) is equivalent to the equation $I = 0$ where

$$I = \int_M e^{w_1 - \bar{w_1}} dw_1 \wedge dw_2 \wedge dw_3.$$

Let Σ be a rectifiable 4-chain in $\mathbb{C}^3$ such that $b[\Sigma] = [M]$—for example, a cone in $\mathbb{C}^3$ with base M. Applying Stokes' theorem to I we get

$$I = \int_{\Sigma} d(e^{w_1 - \bar{w_1}} dw_1 \wedge dw_2 \wedge dw_3) = -\int_{\Sigma} e^{w_1 - \bar{w_1}} d\overline{w_1} \wedge dw_1 \wedge dw_2 \wedge dw_3.$$

Now slice Σ by the map $w = (w_1, w_2, w_3) \mapsto w_1$ and get

$$I = -\int_{\lambda \in \mathbb{C}} e^{\lambda - \bar{\lambda}} \left(\int_{\Sigma_\lambda} dw_2 \wedge dw_3 \right) d\lambda \wedge d\bar{\lambda},$$

where Σ_λ is the slice of Σ. Set

$$J_\lambda = \int_{\Sigma_\lambda} dw_2 \wedge dw_3.$$

It suffices to show that $J_\lambda = 0$ for almost all λ.

Let H_λ be the affine hyperplane $\{w_1 = \lambda\}$. Then by Lemma 3.1, for almost all λ, $H_\lambda \cap M = \gamma_\lambda$ is a 1-cycle for which there exists a holomorphic 1-chain $[V_\lambda]$ with $b[V_\lambda] = [\gamma_\lambda]$. Also $b[\Sigma_\lambda] = [\gamma_\lambda]$ a.e.; this is because (a) by a remark of Harvey and Shiffman ([HS, 1.3.9, p. 567]), slicing commutes a.e. with the d operator, (b) $bT = -dT$ for p currents with p even, and (c) Σ and Σ_λ are a 4-current and a 2-current, respectively. Therefore, by two applications of Stokes' theorem,

$$J_\lambda = \int_{\Sigma_\lambda} dw_2 \wedge dw_3 = \int_{\Sigma_\lambda} d(w_2 \wedge dw_3) = \int_{\gamma_\lambda} w_2 \wedge dw_3 = \int_{V_\lambda} dw_2 \wedge dw_3.$$

Since $[V_\lambda]$ is a $(1,1)$-current, the last integral equals 0. This completes the proof that M is maximally complex. The following lemma then yields Theorem 1. $\square$

Remark. The idea of applying the Fourier transform in connection with slicing is due to Globevnik and Stout [GS]. They used it to show that if a function satisfies the Morera property on the boundary of a domain in $\mathbb{C}^n$, then it satisfies the weak tangential Cauchy-Riemann equations.

LEMMA 3.2. *Let M be a (not necessarily connected) smooth compact oriented k-dimensional manifold in $\mathbb{C}^n$ and suppose that M is maximally complex. Let*

$$T = \sum n_j [V_j]$$

be the unique holomorphic s-chain (with $2s - 1 = k$) in $\mathbb{C}^n \setminus M$ such that $bT = [M]$ whose existence is given by the Harvey-Lawson theorem. If M satisfies the linking condition, then T is positive; i.e., $n_j > 0$ for all j.

Proof. We proceed by induction on odd k. The case $k = 1$ is Lemma 2.1. Assume that $k \geq 3$. Let F be a complex linear function that is not locally constant on any of the varieties V_j. We consider slices of rectifiable currents by the map $F : \mathbb{C}^n \to \mathbb{C}$. Put $H_\lambda = \{z \in \mathbb{C}^n : F(z) = \lambda\}$, a complex hyperplane.

For almost all λ we have that $\langle [V_j], F, \lambda\rangle = [V_j \cap H_\lambda]$, $\langle M, F, \lambda\rangle = [M \cap H_\lambda]$ and $M \cap H_\lambda$ is a smooth oriented $(k-2)$-manifold satisfying the linking condition in $H_\lambda = \mathbb{C}^{n-1}$.

Fix an index m. We can choose λ such that $V_m \cap H_\lambda$ contains an analytic branch W (of complex dimension $s-1$) such that W is not contained in any of the V_j for $j \neq m$. We can choose λ such that $\langle [V_j], F, \lambda\rangle = [V_j \cap H_\lambda]$ for all j, $\langle M, F, \lambda\rangle = [M \cap H_\lambda]$, and $M \cap H_\lambda$ is a smooth oriented $(k-2)$-manifold satisfying the linking condition in $H_\lambda = \mathbb{C}^{n-1}$. Also $M \cap H_\lambda$ is maximally complex, since $[M \cap H_\lambda]$ is the boundary of the holomorphic chain $S = \langle T, F, \lambda\rangle$ (see the next paragraph). Therefore, by induction, the holomorphic $(s-1)$-chain S has *positive* multiplicities.

We have $[M] = bT$ and since slicing commutes almost everywhere with the boundary operator, we can also choose λ so that $\langle bT, F, \lambda\rangle = b\langle T, F, \lambda\rangle$. Hence $[M \cap H_\lambda] = b(\sum n_j[V_j \cap H_\lambda])$. And so S is the chain $\sum n_j[V_j \cap H_\lambda]$, where some of the $V_j \cap H_\lambda$ may be empty or reducible. Thus $n_m[W]$ is one of the terms in the holomorphic $(s-1)$-chain S when S is written as a sum of irreducible varieties with integral multiplicities. Hence, S being positive, $n_m > 0$. □

Remark. In order to proceed by induction, we need to consider disconnected M. However when M is connected, one can argue more directly, since in that case , the Harvey-Lawson result for maximally complex M is that $[M] = \pm b[V]$, for V an irreducible s-variety in $\mathbb{C}^n \setminus M$. The linking condition then obviously implies that $[M] = b[V]$.

4. Proofs of Theorem 4 and 2

Proof of Theorem 4. It is clear that $x \in \operatorname{supp} T \setminus \gamma$ implies

$$\operatorname{link}(\gamma, A) > 0.$$

For the converse suppose that $x \notin \operatorname{supp} T \setminus \gamma$. We shall show that there exists an algebraic hypersurface A in $\mathbb{C}^n$ such that $x \in A$ and

$$\operatorname{link}(\gamma, A) = 0.$$

First suppose that $x \notin \hat{\gamma}$. Then there exists a polynomial P such that $P(x) = 1 > ||P||_{\hat{\gamma}}$. Then $A = \mathbf{Z}(P-1)$ gives the desired algebraic hypersurface.

Finally suppose that $x \in \hat{\gamma} \setminus \{\gamma \cup \operatorname{supp} T\}$. Then, by Lemma 1.5, there exists a polynomial P in $\mathbb{C}^n$ such that $P(x) = 0$ and $P \neq 0$ on $\operatorname{supp} T \cup \gamma$. Hence $A = \mathbf{Z}(P)$ gives an algebraic hypersurface such that $x \in A$ and $\operatorname{link}(\gamma, A) = 0$, since $P \neq 0$ on $\operatorname{supp} T \cup \gamma$. This gives Theorem 4. □

We next show that Theorem 4 implies Theorem 2: It is clear that $x \in \operatorname{supp} T$ implies

$$\operatorname{link}(M, A) > 0.$$

For the converse take $x \notin \operatorname{supp} T$ and choose an affine complex linear hyperplane $H_\lambda = \{F = \lambda\}$ through x such that $H_\lambda \cap M$ is an oriented 1-manifold γ so that $[\gamma] = T_\lambda$, the λ-slice of T by F. We view H_λ as a copy of $\mathbb{C}^2$ and apply Theorem 4 (taking T_λ as the T in Theorem 4). Since $x \notin \operatorname{supp} T$, we conclude that there exists an algebraic curve A in H_λ such that $x \in A$, $A \cap \gamma = \emptyset$ and

$$\operatorname{link}(\gamma, A) = 0.$$

Then A is an algebraic curve in $\mathbb{C}^3$ such that $A \cap M = \emptyset$ and, by Lemma 1.1,

$$\operatorname{link}(M, A) = \operatorname{link}(\gamma, A) = 0.$$

This yields Theorem 2. □

5. Proof of Theorem 5

LEMMA 5.1. *Let T be an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$ of odd real dimension $k > 3$ that is not maximally complex. Then there exists a complex linear hyperplane H (through 0) in $\mathbb{C}^n$ such that $T \cap H$ is not maximally complex.*

Proof. Let $E = T \cap iT$ be the maximal complex linear subspace of T. Let F be the Hermitian orthogonal complement of E. Then E and F are complex linear subspaces of $\mathbb{C}^n$ such that $\mathbb{C}^n = E \oplus F$. Hence

$$T = E \oplus S$$

where $S = T \cap F$ is totally real; i.e. $S \cap iS = \{0\}$. Since T is not maximally complex and of odd dimension, $\dim S \geq 3$.

We consider two cases:

(a) $E \neq \{0\}$, or

(b) $E = \{0\}$, i.e. $T = S$ is totally real.

Case (a) . Take $u \neq 0$ in E and set H equal to the Hermitian orthogonal complement of $\mathbb{C}[u]$, the $\mathbb{C}$-span of u. Then $T \cap H = E' \oplus S$, where $E' = E \ominus u$ and $i(T \cap H) = E' \oplus iS$. Hence $(T \cap H) \cap i(T \cap H) = E'$ has real codimension in $T \cap H$ equal to $\dim S$, which is greater than or equal to 3, and so $T \cap H$ is not maximally complex.

Case (b). For any complex linear hyperplane H, $T \cap H$ is totally real, since T is totally real. Since $\dim T \cap H \geq k - 2 \geq 3$ as $k \geq 5$, it follows that $T \cap H$ is not maximally complex.

Remark. The condition that a linear space be totally real is an "open" condition. This means that for all $\mathbb{R}$-linear subspaces T' of $\mathbb{C}^n$ sufficiently close to T and with the same dimension and for all complex hyperplanes H' sufficiently close to the hyperplane constructed in the proof of the lemma, $H' \cap T'$ is also not maximally complex.

Proof of Theorem 5. Assuming that M satisfies the linking condition, we prove the maximal complexity by induction on (odd) k, starting with $k = 3$. We have already done the case $k = 3$ and $n = 3$ as Theorem 1. Assume that $k = 3$ and $n > 3$ and fix $x \in M$. We claim that $T_x(M)$, the tangent space of M at x, is maximally complex. There exists a complex linear map $\phi : \mathbb{C}^n \to \mathbb{C}^3$ such that x is a regular point of $\phi|M$ and $\phi^{-1}(\phi(x)) \cap M = \{x\}$ (see Harvey [H, proof of Lemma 3.5, p. 349]). Let $\mathcal{M}$ denote the image $\phi(M)$; $\mathcal{M}$ is a an oriented immersed 3-manifold with singularities (the "scar set") in $\mathbb{C}^3$ such that integration over M gives a current $[\mathcal{M}]$ such that $\phi_*([M]) = [\mathcal{M}]$ (see Harvey ([H, p. 368]).

Let H be a complex hyperplane in $\mathbb{C}^3$ such that (a) $L = \phi^{-1}(H)$ is a complex hyperplane in $\mathbb{C}^n$ that intersects M in a smooth 1-cycle $\tilde{\gamma}$ and (b) $H \cap \mathcal{M}$ is a 1-cycle $\gamma = \phi(\tilde{\gamma})$. As M satisfies the linking condition, our previous arguments show that $\tilde{\gamma}$ satisfies the linking condition in L (with L viewed as a copy of $\mathbb{C}^{n-1}$). Hence $\tilde{\gamma}$ bounds a holomorphic 1-chain $\tilde{V}$. It follows that γ bounds the holomorphic 1-chain $V = \phi(\tilde{V})$. As (a) and (b) hold for almost all hyperplanes H, we conclude, as in the proof of Theorem 1, that

$$\int_{\mathcal{M}} \alpha \; dz_1 \wedge dz_2 \wedge dz_3 = 0$$

for all $\mathcal{C}^\infty$ functions α on $\mathbb{C}^3$. This implies that $T_y(\mathcal{M})$ is maximally complex at points $y \in \mathcal{M}$ where $\mathcal{M}$ is smooth, since α can be chosen to have support in arbitrarily small neighborhoods of y. It follows that $T_x(M)$ is maximally complex, because x is a regular point of $\phi|M$. As x is an arbitrary point of M, we conclude that M is maximally complex. By Lemma 3.2 this completes the case $k = 3$.

Now consider the case $k > 3$. We argue by contradiction and suppose that M is not maximally complex. It follows that there exists a point $p \in M$ such that $T_p(M)$, the tangent space of M viewed as a real linear subspace of $\mathbb{C}^n$, is not maximally complex. Then by Lemma 5.1, there is a complex hyperplane H in $\mathbb{C}^n$ such that $T_p(M) \cap H_p$ is not maximally complex. Suppose, for the moment, that the translate $H_p = H + p$ of H through p in $\mathbb{C}^n$, given say by $H_p = \mathbf{Z}(F)$ where F is an affine complex linear function, is such that $Q = M \cap H_p$ is a smooth $(k-2)$-manifold, oriented as the slice of the map $F : M \to \mathbb{C}$. Then $T_p(Q) = T_p(M) \cap H$ is not maximally complex. On the other hand, the linking condition for M implies that Q satisfies the linking condition in H_p (viewed as a copy of $\mathbb{C}^{n-1}$). Hence, by induction on k, since

$k-2 \geq 3$, we conclude that Q is a maximally complex manifold. Therefore $T_p(Q)$ is a maximally complex linear space. This is a contradiction.

From this we can conclude that M is maximally complex—except for the assumption above that $Q = M \cap H_p$ is a smooth $(k-2)$-manifold. If this is not the case, we can, by Sard's theorem, translate H_p a small amount so that the intersection with M becomes smooth. Our above argument, together with the remark after Lemma 5.1, then show that $T_q(M)$ is maximally complex for some $q \in M$ arbitrarily close to p. It follows that, in the limit, $T_p(M)$ is maximally complex. Again Lemma 3.2 yields the first part of Theorem 5 for general $k > 0$.

Finally, we prove by induction on k that if $x \in \mathbb{C}^n \setminus M$ and $x \notin \operatorname{supp} T$, then there exists an algebraic subvariety A of $\mathbb{C}^n$ with $x \in A$ and

$$\operatorname{link}(M, A) = 0.$$

The case $k = 3$ is just Theorem 2.

Suppose that $k > 3$. We argue as in the proof of Theorem 2. Choose an affine complex linear hyperplane $H_\lambda = \{F = \lambda\}$ through x such that $H_\lambda \cap M$ is an oriented $(k-2)$-manifold Q so that $[Q] = b(T_\lambda)$, where T_λ is the slice of T by F. We view H_λ as a copy of $\mathbb{C}^{n-1}$. As $x \notin Q$, we can apply the induction hypothesis to get an algebraic subvariety A of H_λ such that A is disjoint from Q, $x \in A$ and $\operatorname{link}(Q, A) = 0$, where the linking number is computed in $\mathbb{C}^{n-1}$. Then A is an algebraic subvariety in $\mathbb{C}^n$ such that A is disjoint from M and $\operatorname{link}(M, A) = 0$. This gives the theorem. □

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(Received March 16, 1998)