Annals of Mathematics, **151** (2000), 1–33

Some spherical uniqueness theorems for multiple trigonometric series

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Abstract

We prove that if a multiple trigonometric series is spherically Abel summable everywhere to an everywhere finite function $f(x)$ which is bounded below by an integrable function, then the series is the Fourier series of $f(x)$ if the coefficients of the multiple trigonometric series satisfy a mild growth condition. As a consequence, we show that if a multiple trigonometric series is spherically convergent everywhere to an everywhere finite integrable function $f(x)$, then the series is the Fourier series of $f(x)$. We also show that a singleton is a set of uniqueness. These results are generalizations of a recent theorem of J. Bourgain and some results of V. Shapiro.

1. Introduction and summary of results

We start with the question of spherical uniqueness of multiple trigonometric series for integrable functions under Abel summability. Greek letters $\xi, \eta, \cdots$ will denote points of the d-dimensional lattice $\mathbb{Z}^d$, Roman letters $x, y, \cdots$ points of the d-dimensional torus $\mathbb{T}^d = [-\pi, \pi)^d$, $\langle \cdot, \cdot \rangle$ inner product, and $|\cdot|$ d-dimensional Euclidean norm. For a multiple trigonometric series $\sum_{\xi \in Z^d} a_\xi e^{i\langle x, \xi \rangle}$ where the coefficients a_ξ are arbitrary complex numbers, the Abel sum is defined to be the limit of the function

$$f(x,t) = \sum\nolimits_{\xi \in Z^d} a_\xi e^{i\langle x, \xi\rangle - |\xi| t}$$

*The research of both authors was partially supported by NSF grant DMS 9707011 and a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, De Paul University.

1991 *Mathematics Subject Classification.* Primary 42B05, 42B99; Secondary 42B08, 42B15, 42A63.

as $t \to 0^+$ if such limit exists. In general, denote

$$\begin{aligned} f^*(x) &= \limsup_{t\to 0^+} f(x,t) \\ &= \Re f^*(x) + i\Im f^*(x) \end{aligned}$$

$f_*(x) = \Re f_*(x) + i\Im f_*(x)$ is similarly defined with lim sup being replaced by lim inf.

It is well-known, when $d = 1$, that if $\sum a_\xi e^{i\xi x}$ is Abel summable to 0 everywhere and if $a_\xi = o(|\xi|)$, then all $a_\xi = 0$. See, for example, [Ve1] and [Ve2]. To see that this theorem is sharp, look at the one dimensional series $\delta'(x) = -\sum \xi \sin \xi x$, which may be thought of as the derivative of the Dirac delta function. It is easy to check that this series is Abel summable to 0, although the growth condition is just barely violated. Thinking of δ' as a degenerate d dimensional function, it is immediately clear that the hypothesis of a d dimensional uniqueness theorem concerning Abel summability will necessarily have to carry some growth condition. One generalization of this fact is due to Victor Shapiro, who extended one dimensional work of Verblunsky and of Rajchman and Zygmund ([Sh]).

THEOREM 1.1 (Shapiro). *Let $\sum a_\xi e^{i\langle \xi,x\rangle}$ be a multiple trigonometric series. Suppose that*

1. *the coefficients a_ξ satisfy the following growth rate condition:*

$$\sum_{R-1<|\xi|\le R} |a_\xi| = o(R) \text{ as } R\to\infty, \tag{1.1}$$

2. *$f^*(x)$ and $f_*(x)$ are finite for all x,*
3. *$a_\xi = \overline{a_{-\xi}}$ for all ξ, and*
4. *$\min\{\Re f_*(x), \Im f_*(x)\} \ge A(x)$ where $A(x)$ is in $L^1\left(\mathbb{T}^d\right)$.*

Then $f_*(x) \in L^1\left(\mathbb{T}^d\right)$ and $\sum a_\xi e^{i\langle\xi,x\rangle}$ is the Fourier series of f_*.

This theorem is sharp because the example δ' mentioned above just barely fails to meet condition (1.1). Nevertheless condition (1.1) is disappointingly strong in the sense that when Abel summability is replaced by regular convergence, condition (1.1) is not a direct consequence of convergence. However, there is a known theorem concerning the coefficients' growth rate for spherically convergent multiple trigonometric series. In fact, it is implied by the following Cantor-Lebesgue type theorem.

THEOREM 1.2 (Connes). *Let $\mathcal{O} \subset \mathbb{T}^d$ be a ball or a subset which has full measure and is of Baire second category relative to $\mathbb{T}^d$. If $\sum_{|\xi|=R} a_\xi e^{i\langle x,\xi\rangle}$ tends to 0 as $R \to \infty$ at every point of $\mathcal{O}$, then*

$$\varepsilon_R^2 = \sum_{|\xi|=R} |a_\xi|^2 = o(1) \qquad \text{as } R \to \infty. \tag{1.2}$$

Connes proved this theorem for dimension d in 1976, twenty years after Shapiro's Theorem 1.1. Cooke [C] and shortly thereafter Zygmund [Z] had completed the $d = 2$ case five years before Connes' work.

An easy corollary of Theorem 1.2 gives the coefficients' growth rate condition for spherically convergent multiple trigonometric series.

COROLLARY 1.3 (Connes). *Let $\mathcal{O} \subset \mathbb{T}^d$ be a ball or a subset which has full measure and is of Baire second category relative to $\mathbb{T}$. If $\lim_{R\to\infty} \sum_{|\xi|\leq R} a_\xi e^{i\langle x,\xi\rangle}$ exists (as a finite number) at each point of $\mathcal{O}$, then*

$$\varepsilon_R^2 = \sum_{|\xi|=R} |a_\xi|^2 = o(1) \qquad \text{as } R \to \infty. \tag{1.3}$$

The coefficients' growth rate condition given by (1.3) does not imply condition (1.1) when $d \geq 3$. To remedy this problem, we first prove the following analogue of Theorem 1.1 under the condition (1.3). We use notation $A \sim B$ to denote $B/2 \leq A < B$.

THEOREM 1.4. *Consider the multiple trigonometric series $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ where the coefficients a_ξ are arbitrary complex numbers. Suppose that*

1. *the coefficients of the series a_ξ satisfy*

$$\sum_{|\xi|\sim R} |a_\xi|^2 = \sum\nolimits_{R/2\leq|\xi|<R} |a_\xi|^2 = o(R^2) \text{ as } R \to \infty, \tag{1.4}$$

2. *$f^*(x)$ and $f_*(x)$ are finite for all x, and*
3. *$\min\{\Re f_*(x), \Im f_*(x)\}$ is bounded below by a function $A(x)$ in $L^1(\mathbb{T}^d)$.*

Then $f_*(x)$ is in $L^1(\mathbb{T}^d)$ and $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ is its Fourier series.

Note that condition (1.3) implies condition (1.4) since

$$\sum_{|\xi|\sim R} |a_\xi|^2 = \sum_{k=R^2/4}^{R^2-1} \sum_{|\xi|^2=k} |a_\xi|^2 = o\left(\sum_{k=R^2/4}^{R^2-1} 1\right) = o(R^2).$$

Since (1.1) implies

$$\sum_{|\xi|\sim R} |a_\xi| = o(R^2),$$

(1.1) implies (1.4) if a_ξ is bounded. But in general, and when $d \geq 3$, (1.1) and (1.4) do not relate to each other. Notice that (1.4) implies that

$$\sum_{|\xi|^2 \sim R} |a_\xi|^2 = \sum_{R/2 \leq |\xi|^2 < R} |a_\xi|^2 \leq \sum_{|\xi| \sim \sqrt{R}} |a_\xi|^2 = o(R).$$

As a consequence of Theorem 1.2 and Theorem 1.4, we obtain the following two spherical uniqueness theorems for multiple trigonometric series which are convergent to a function. These theorems make no assumption whatsoever about coefficient size.

THEOREM 1.5. *Let* $\sum_{\xi \in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ *be a trigonometric series which converges spherically everywhere to an everywhere finite function* $f(x)$; *i.e.*,

$$\lim_{R\to\infty} \sum_{|\xi|\leq R} a_\xi e^{i\langle x,\xi\rangle} = f(x) \text{ for all } x \in \mathbb{T}^d. \tag{1.5}$$

If $\min\{\Re f(x), \Im f(x)\} \geq g(x)$ *for all* x *and* $g(x) \in L^1(\mathbb{T}^d)$, *then* $f(x)$ *is in* $L^1(\mathbb{T}^d)$ *and* a_ξ *is the* ξ^{th} *Fourier coefficient of* $f(x)$ *for all* $\xi \in \mathbb{Z}^d$.

In particular,

THEOREM 1.6. *Let* $f(x) \in L^1(\mathbb{T}^d)$ *be finite at every* x. *If* $\sum_{\xi \in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ *is a trigonometric series which converges spherically to* $f(x)$ *at every point* x, *i.e.*

$$\lim_{R\to\infty} \sum_{|\xi|\leq R} a_\xi e^{i\langle x,\xi\rangle} = f(x) \text{ for all } x \in \mathbb{T}^d, \tag{1.6}$$

then a_ξ *is the* ξ^{th} *Fourier coefficient of* $f(x)$ *for all* $\xi \in \mathbb{Z}^d$.

Special cases of Theorem 1.6 have been proved by various people. When $d = 1$ and $f(x) \equiv 0$, this is the original uniqueness theorem of Cantor. For general $f(x) \in L^1(\mathbb{T}^1)$, it was first proved by de la Vallée-Poussin. When $d = 2$, Theorem 1.1 combined with the work of Cooke [C] implies Theorem 1.4 and thus, Theorem 1.5 and Theorem 1.6. The major breakthrough came when Bourgain [B] proved Theorem 1.6 for the special case of $f(x) \equiv 0$. For a survey on the uniqueness of multiple trigonometric series under various summation modes, as well as many open problems in this area, please refer to Ash and Wang [AW].

The proof of Theorem 1.4 is mainly based on Shapiro's framework [Sh]. To avoid assuming condition (1.1), we exploit an idea that Bourgain [B] used when he proved Theorem 1.6 for the special case $f(x) \equiv 0$. We refer to (1.4) hereafter as Bourgain's condition, in his honor. This condition simply asserts that Connes' condition holds "on the average."

The detailed proof of Theorem 1.4 is given in Sections 2 through 5.

At the end of the paper, we begin the study of sets of uniqueness for spherical convergence. As a first step toward establishing this theory, we show that any singleton is a set of uniqueness.

THEOREM 1.7. *Let q be a point on $\mathbb{T}^d$. Suppose that a multiple trigonometric series $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ spherically converges everywhere except at q to a function $f(x)\in L^1(\mathbb{T}^d)$. Furthermore, suppose $f(x)$ is finite for all x except q. Then $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ is the Fourier series of $f(x)$.*

It is easily deduced from Theorem 1.2 and the following fact about Abel summability, which is an analogue of a theorem of Shapiro [Sh, §6]:

THEOREM 1.8. *Consider the multiple ($d\geq 2$) trigonometric series $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ where the coefficients a_ξ are arbitrary complex numbers. Let q be a point on $\mathbb{T}^d$. Suppose that*

1. $\sum_{|\xi|\sim R}|a_\xi|^2=o(R^2)$ *as* $R\to\infty$,
2. $f^*(x)$ *and* $f_*(x)$ *are finite for all* x *except* q, *and*
3. $f^*(x)$ *and* $f_*(x)$ *are functions in* $L^1(\mathbb{T}^d)$.

Then $\sum_{\xi\in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ is the Fourier series of $f_*(x)$.

Note that the theorem is false when $d=1$ since the trigonometric series $\sum e^{i\xi x}$ is Abel convergent to 0 everywhere in $\mathbb{T}\setminus\{0\}$.

2. Proof of Theorem 1.4

We may assume $d\geq 3$ since the cases $d=1$ and $d=2$ are known.

We need some preliminary results and some notation before we start the proof. Without loss of generality, by considering the real and imaginary parts separately, we may assume that $a_\xi=\overline{a}_{-\xi}$, where $\overline{a}$ is the conjugate of the complex number a. Thus $f(x,t), f^*(x)$ and $f_*(x)$ are all real functions. In addition, we may assume that $a_0=0$.

Define

$$f_1(x,t) = -\sum_{\xi\neq 0}\frac{a_\xi}{|\xi|}e^{i\langle x,\xi\rangle-|\xi|t} \tag{2.1}$$

$$f_2(x,t) = -\sum_{\xi\neq 0}\frac{a_\xi}{|\xi|^2}e^{i\langle x,\xi\rangle-|\xi|t}.$$

Under the condition (1.4), it is easy to see that for each $x\in\mathbb{T}^d$ and $t>0$, $f(x,t), f_1(x,t)$ and $f_2(x,t)$ converge absolutely and hence are infinitely differentiable as functions of $t>0$. Thus by the mean value theorem, for $t_1>t_2>0$, there exist $t_3,t_4\in(t_2,t_1)$ such that $f_1(x,t_1)-f_1(x,t_2)=f(x,t_3)(t_1-t_2)$, and

$f_2(x,t_1)-f_2(x,t_2)=-f_1(x,t_4)(t_1-t_2)$. Since for each x, $f^*(x)$ and $f_*(x)$ are finite, $f(x,t)$ is bounded for all $t>0$. The bound depends on x in general. Thus, for each x, there exist finite-valued functions $f_1(x)$ and $f_2(x)$ such that

$$f_1(x,t)\to f_1(x) \text{ and } f_2(x,t)\to f_2(x) \text{ as } t\to 0^+. \tag{2.2}$$

On the other hand, if we define the Riemann function $F(x)$ by

$$F(x)=-\sum_{\xi\neq 0}\frac{a_\xi}{|\xi|^2}e^{i\langle x,\xi\rangle}, \tag{2.3}$$

then, because of the Bourgain condition (1.4), $F(x)\in L^2(\mathbb{T}^d)$ and $f_2(x,t)\overset{L^2}{\to}F(x)$ as $t\to 0^+$. In fact, observe that there is an absolute constant C such that

$$\begin{aligned} ||f_2(x,t)-F(x)||_2^2 &= \sum_{\xi\neq 0}\frac{|a_\xi|^2}{|\xi|^4}\left(1-e^{-2|\xi|t}\right) \\ &\le C\sum_{k=1}^{\infty}2^{-2k}\left(1-e^{-2^{k/2+1}t}\right)\sum_{|\xi|^2\sim 2^k}|a_\xi|^2 \\ &\le C\sum_{k=1}^{\infty}2^{-k}\left(1-e^{-2^{k/2+1}t}\right) \\ &\to 0 \text{ as } t\to 0^+. \end{aligned} \tag{2.4}$$

Thus,

$$f_2(x)=F(x) \quad \text{a.e.} \tag{2.5}$$

The key to the proof is to show that $\Delta f_2(x)=f_*(x)$ almost everywhere. To this end, we need to use a generalized Laplacian.

Let $B(x,\rho)$ be an open ball in $\mathbb{T}^d$ centered at $x\in\mathbb{T}^d$ with radius $\rho>0$ and $m(B(x,\rho))$ the volume of $B(x,\rho)$. Then $m(B(x,\rho))=v_d\rho^d$, where v_d is the volume of the unit ball in $\mathbb{R}^d$. For any locally integrable function $g(x)$, the average of g over $B(x,\rho)$ is

$$\begin{aligned} A_\rho g(x) &= \frac{1}{m(B(x,\rho))}\int_{B(x,\rho)}g(y)\,dy \\ &= \frac{1}{v_d\rho^d}\int_{B(x,\rho)}g(y)\,dy. \end{aligned}$$

Let

$$I(x)=\frac{I_{B(0,1)}(x)}{m(B(0,1))},$$

where $I_{B(0,1)}(x)$ is the characteristic function of the unit ball. Denote $\hat{I}(\xi)$ to be the Fourier transform of $I(x)$. Then $\hat{I}(\rho\xi)$ satisfies the following properties:

$$\lim_{\rho\to 0}\frac{\hat{I}(\rho\xi)-1}{\rho^2|\xi|^2}=-\frac{1}{2}\int_{B(0,1)}x_1^2I(x)\,dx=c_d<0, \tag{2.6}$$

and for $|\xi| = 1$,

$$\int_0^\infty \left| \partial_r \left[\frac{\hat{I}(\rho r\xi) - 1}{\rho^2 r^2} \right] \right| dr = \int_0^\infty \left| \partial_r \left[\frac{\hat{I}(r\xi) - 1}{r^2} \right] \right| dr < c. \tag{2.7}$$

Note that the constant c in (2.7) is independent of ρ.

The above two equalities are standard. In fact, to see (2.6), rotate (choose the first coordinate axis to be in the direction of ξ) and use polar coordinates to get

$$\begin{aligned} \hat{I}(\rho\xi) - 1 &= \frac{1}{v_d} \int_{|x|\le 1} (e^{i\langle x, \rho\xi\rangle} - 1)\, dx \\ &= \frac{1}{v_d} \int_{|x|\le 1} (e^{ix_1\rho|\xi|} - 1)\, dx \\ &= \frac{v_{d-1}}{v_d} \int_{-1}^{1} (\cos(\rho|\xi|x_1) - 1)(1 - x_1^2)^{\frac{d-1}{2}}\, dx_1. \end{aligned} \tag{2.8}$$

Since for any $x \in \mathbb{T}$, $|\cos x - 1| \le \frac{x^2}{2}$, and $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$, by the bounded convergence theorem and (2.8), we have

$$\begin{aligned} \lim_{\rho\to 0} \frac{\hat{I}(\rho\xi) - 1}{\rho^2|\xi|^2} &= -\frac{v_{d-1}}{2v_d} \int_{-1}^{1} x_1^2 (1 - x_1^2)^{\frac{d-1}{2}}\, dx_1 \\ &= -\frac{1}{2} \int_{B(0,1)} x_1^2 I(x)\, dx = c_d < 0. \end{aligned}$$

Observe that the above argument shows $\hat{I}(\xi_1) = \hat{I}(\xi_2)$ if $|\xi_1| = |\xi_2|$. Thus, we may abuse our notation and write $\hat{I}(\xi) = \hat{I}(|\xi|)$.

Inequality (2.7) also follows similarly. If $|\xi| = 1$, then

$$\partial_r \left[\frac{\hat{I}(\rho r\xi) - 1}{\rho^2 r^2} \right] = -2\frac{v_{d-1}}{v_d} \int_{-1}^{1} \frac{(1 - \frac{1}{2} ir\rho x_1)e^{ir\rho x_1} - 1}{r^3\rho^2} (1 - x_1^2)^{\frac{d-1}{2}}\, dx_1.$$

Thus, for $c = -2v_{d-1}/v_d$,

$$\begin{aligned} &\int_0^\infty \left| \partial_r \left[\frac{\hat{I}(\rho r\xi) - 1}{\rho^2 r^2} \right] \right| dr \\ &\quad = c \int_0^\infty \left| \int_{-1}^{1} \frac{(1 - \frac{1}{2} ir\rho x_1)e^{ir\rho x_1} - 1}{r^3\rho^2} (1 - x_1^2)^{\frac{d-1}{2}}\, dx_1 \right| dr \\ &\quad = c \int_0^\infty \left| \int_{-1}^{1} \frac{(1 - \frac{1}{2} irx_1)e^{irx_1} - 1}{r^3} (1 - x_1^2)^{\frac{d-1}{2}}\, dx_1 \right| dr \\ &\quad = \int_0^\infty \left| \partial_r \left[\frac{\hat{I}(r\xi) - 1}{r^2} \right] \right| dr \end{aligned}$$

by the simple change of variable argument: $\rho r \to r$. The above integral is finite since

$$\left|\int_{-1}^{1} \frac{(1-\frac{1}{2}irx_1)e^{irx_1}-1}{r^3}(1-x_1^2)^{\frac{d-1}{2}}\,dx_1\right| \le \frac{c}{r^2} \text{ as } r\to\infty,$$

and

$$(1-\frac{1}{2}irx_1)e^{irx_1}-1 = \frac{1}{2}irx_1 + O(r^3) \text{ as } r\to 0.$$

Define the generalized Laplacian operator on $g(x)\in L^1$ to be

$$\tilde{\Delta}g(x) = \lim_{\rho\to 0} -\frac{1}{c_d}\frac{A_\rho g(x)-g(x)}{\rho^2}$$

if such a limit exists (not necessarily finite), where $c_d < 0$ is the constant given in (2.6). We can also define the upper and lower generalized Laplacians $\tilde{\Delta}^* g(x)$ and $\tilde{\Delta}_* g(x)$ by replacing lim by lim sup and lim inf respectively when the function $g(x)$ is real-valued. It is clear that all three of these generalized Laplacians agree with the usual Laplacian when applied to a C^2 function. Recall that $a_0 = 0$. For $f_2(x,t)$ given by (2.1), we have for $x \in \mathbb{T}^d$,

$$\begin{aligned}
\frac{A_\rho f_2(x,t)-f_2(x,t)}{\rho^2} &= -\sum_{\xi\neq 0}\frac{a_\xi}{|\xi|^2}\frac{\hat{I}(\rho\xi)-1}{\rho^2}e^{i\langle x,\xi\rangle-|\xi|t} \\
&= -\sum_{k\ge 1}\frac{\hat{I}(\rho\sqrt{k})-1}{\rho^2 k}\sum_{|\xi|^2=k}a_\xi e^{i\langle x,\xi\rangle-|\xi|t} \\
&= -\sum_{k\ge 1}\left(\sum_{|\xi|^2\le k}a_\xi e^{i\langle x,\xi\rangle-|\xi|t}\right) \\
&\qquad \times\left(\frac{\hat{I}(\rho\sqrt{k})-1}{\rho^2 k}-\frac{\hat{I}(\rho\sqrt{k+1})-1}{\rho^2(k+1)}\right) \\
&\to -f(x,t)c_d \text{ as } \rho\to 0
\end{aligned} \tag{2.9}$$

since by the fundamental theorem of calculus and (2.7),

$$\sum_{k\ge 1}\left|\frac{\hat{I}(\rho\sqrt{k})-1}{\rho^2 k}-\frac{\hat{I}(\rho\sqrt{k+1})-1}{\rho^2(k+1)}\right| \le \int_0^\infty \left|\partial_r\left[\frac{\hat{I}(\rho r\xi)-1}{\rho^2 r^2}\right]\right| dr < b < \infty$$

for a constant b independent of ρ. Thus, the above argument shows that

$$\tilde{\Delta}f_2(x,t) = \lim_{\rho\to 0} -\frac{1}{c_d}\frac{A_\rho f_2(x,t)-f_2(x,t)}{\rho^2} = f(x,t) \tag{2.10}$$

for $x\in\mathbb{T}^d$ and $t>0$.

To pass to the limit as $t\to 0^+$, we need the following lemma of Shapiro (Lemma 7 of [Sh2]). To see that Shapiro's lemma applies, note that $F\in L^2(\mathbb{T}^d)$ implies $F\in L^1(\mathbb{T}^d)$.

LEMMA 2.1. *If* (1.4) *holds, then at every point x where $f_*(x)$ and $f^*(x)$ are finite,*

$$\tilde{\Delta}_* f_2(x) \le f^*(x) \quad \text{and} \quad f_*(x) \le \tilde{\Delta}^* f_2(x). \tag{2.11}$$

The following classical results on the Green's function $G(x)$ appear with proof as Lemma 8 of Shapiro [Sh2]. (Also see Theorem 6 of Bochner [Bo].)

LEMMA 2.2. *There is a function $G(x)$ in $L^1(\mathbb{T}^d)$ whose Fourier series is given by $\sum_{|\xi|\neq 0} |\xi|^{-2} e^{i\langle x,\xi\rangle}$. Further, $G(x)$ has the following properties:*

1. *$G(x)$ is in class $C^\infty(\mathbb{T}^d)$ away from 0 and $\Delta G(x) = 1$ for $x \neq 0$.*
2. *$G(x) = \Phi(x) + H^*(x)$ where H^* is continuous on $\mathbb{T}^d$ and $\Delta H^*(x) = 1$ for $x \in \mathbb{T}^d \setminus \{0\}$ and where $\Phi(x) = C_d |x|^{-(d-2)}$ for $d \ge 3$ with $C_d = 2^{d-1}\pi^{d/2}\Gamma(d/2)/(d-2)$ and $\Phi(x) = -2\pi \log |x|$ when $d = 2$.*
3. *Let $u(x)$ be an upper semi-continuous function on $\mathbb{T}^d$ which is also in $L^1(\mathbb{T}^d)$. Define $U(x) = (2\pi)^{-d} \int_{T^d} G(x-y)u(y)\,dy$ and $u_0 = (2\pi)^{-d} \int_{T^d} u(y)\,dy$. Then $U(x)$ is upper semi-continuous on $\mathbb{T}^d$, $U(x) \in L^1(\mathbb{T}^d)$, and $\tilde{\Delta}_* U(x) \ge -u(x) + u_0$ for $x \in \mathbb{T}^d$. Moreover, $\tilde{\Delta}^* U(x) = \tilde{\Delta}_* U(x) = -u(x) + u_0$ almost everywhere in $\mathbb{T}^d$.*

A consequence of Lemma 2.2 is that for any integrable function u, the Fourier series of $U(x) = (2\pi)^{-d} \int_{T^d} G(x-y)u(y)\,dy$ is $\sum_{|\xi|\neq 0} u_\xi |\xi|^{-2} e^{i\langle x,\xi\rangle}$, where $u_0 + \sum_{|\xi|\neq 0} u_\xi e^{i\langle x,\xi\rangle}$ is the Fourier series of u.

We now state the following key lemma which will be proved in Section 3. The function $\overline{U}$ will not in general be periodic, so we have to work in $\mathbb{R}^d$, rather than in $\mathbb{T}^d$.

LEMMA 2.3. *Let $f_2(x)$ be as given in* (2.2) *where $f(x,t)$ satisfies the conditions in Theorem* 1.4. *Suppose that $\overline{U}(x)$ is an upper semi-continuous function and that it is in $L^1_{\text{loc}}(\mathbb{R}^d)$. Let $S(x) = f_2(x) + \overline{U}(x)$. If $\tilde{\Delta}^* S(x) \ge 0$, then $S(x)$ is subharmonic in $\mathbb{R}^d$.*

Remark 2.1. By modifying the proof of Lemma 2.3 in Section 3, Lemma 2.3 can be shown to hold locally. Explicitly, we can replace $\mathbb{R}^d$ everywhere in Lemma 2.3 by any open ball $B \subset \mathbb{R}^d$ and $L^1_{\text{loc}}(\mathbb{R}^d)$ by $L^1(B)$.

We now are ready to prove Theorem 1.4.

Since $A(x) \in L^1(\mathbb{T}^d)$, there exists an upper semi-continuous function $u(x)$ (see p.75 of [S], for example) such that $u(x) \le A(x)$. As in Lemma 2.2, define $U(x) = (2\pi)^{-d} \int_{T^d} G(x-y)u(y)\,dy$, $u_0 = (2\pi)^{-d} \int_{T^d} u(y)\,dy$ and $S(x) = f_2(x) + U(x) - u_0|x|^2/(2d)$. Then by Lemma 2.1, $\tilde{\Delta}^* f_2(x) \ge f_*(x) \ge A(x) \ge u(x)$. Consequently, by periodicity, Lemmas 2.2 and 2.3, $S(x)$ is subharmonic in $\mathbb{R}^d$. Therefore, by Riesz's representation for subharmonic functions and a theorem of Saks [S], $\tilde{\Delta}^* S(x) = \tilde{\Delta}_* S(x)$ almost everywhere and is in L^1 locally. Since $\tilde{\Delta}^* U(x) = \tilde{\Delta}_* U(x)$ almost everywhere and is in L^1 locally, this shows that $\tilde{\Delta}^* f_2(x) = \tilde{\Delta}_* f_2(x)$ almost everywhere and is in L^1 locally. Thus by assumption and Lemma 2.1, $f_*(x)$ is in L^1 locally.

Let $B(x) = \min\{f^*(x), \tilde{\Delta}^* f_2(x)\}$. Then by Lemma 2.1, $\tilde{\Delta}_* f_2(x) \leq B(x) \leq \tilde{\Delta}^* f_2(x)$. Consequently, $B(x) = \tilde{\Delta}^* f_2(x)$ almost everywhere, and is in $L^1_{\text{loc}}(\mathbb{R}^d)$. By a theorem of Vitali-Carathéodory (p. 75 of [S]), there exists a nondecreasing sequence of upper semi-continuous functions $\{u^k(x)\}$ on $\mathbb{R}^d$, which are also in $L^1_{\text{loc}}(\mathbb{R}^d)$, such that each $u^k(x)$ is bounded above and $u^k(x) \leq B(x)$ for all $x \in \mathbb{R}^d$,

$$\lim_{k\to\infty} u^k(x) = B(x) \text{ for almost all } x \in \mathbb{R}^d \tag{2.12}$$

and

$$\lim_{k\to\infty} \int_E u^k(y)\, dy = \int_E B(y)\, dy \tag{2.13}$$

for any bounded set $E \subset \mathbb{R}^d$. Set $U^k(x) = (2\pi)^{-d} \int_{T^d} G(x-y) u^k(y)\, dy$ and $u_0^k = (2\pi)^{-d} \int_{T^d} u^k(y)\, dy$. Then (2.13) implies that u_0^k is convergent to $b_0 = (2\pi)^{-d} \int_{T^d} B(y)\, dy$ as $k \to \infty$. By Lemma 2.1–Lemma 2.3, we have $S^k(x) = f_2(x) + U^k(x) - u_0^k |x|^2/(2d)$ is subharmonic in $\mathbb{R}^d$.

Note that $0 \leq B(x) - u^k(x) \leq B(x) - u^1(x)$. Since $B(x)$ and $u^1(x)$ are locally integrable on $\mathbb{R}^d$, by Lemma 2.2, (2.12), and the dominated convergence theorem,

$$\lim_{k\to\infty} U^k(x) = U(x) = (2\pi)^{-d} \int_{T^d} G(x-y) B(y)\, dy \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d)$$

and hence there exists a subsequence, still called U^k for notational simplicity, such that

$$\lim_{k\to\infty} U^k(x) = U(x) \quad \text{a.e.}$$

Since for any sequence of subharmonic functions convergent in L^1, there is a subharmonic function which is almost everywhere the L^1 limit of that sequence (see p. 20 of [R]); $S(x) = f_2(x) + U(x) - b_0 |x|^2/(2d)$ is almost everywhere equal to a subharmonic function $S_*(x)$ in $\mathbb{R}^d$.

Similarly, there exists a sequence of nonincreasing lower semi-continuous functions $v^k(x)$ on $\mathbb{R}^d$, which are also in $L^1_{\text{loc}}(\mathbb{R}^d)$, such that each $v^k(x)$ is bounded below and $v^k(x) \geq B(x)$,

$$\lim_{k\to\infty} v^k(x) = B(x) \text{ for almost all } x \in \mathbb{R}^d$$

and

$$\lim_{k\to\infty} \int_E v^k(y)\, dy = \int_E B(y)\, dy$$

for any bounded set $E \subset \mathbb{R}^d$. Since $-v^k(x)$ is nondecreasing the above arguments show that there exists a superharmonic function $S^*(x)$, which is almost everywhere equal to $S(x)$.

Therefore $S_*(x) = S^*(x)$ almost everywhere. The subharmonicity of S_* and superharmonicity of S^* show that at every x

$$S_*(x) \le A_1 S_*(x) = A_1 S(x) = A_1 S^*(x) \le S^*(x). \tag{2.14}$$

In addition, if both $S_*(x)$ and $S^*(x)$ are finite, for any $\epsilon > 0$, there exists $\delta > 0$, such that

$$S_*(y) \le S_*(x) + \epsilon \text{ and } S^*(y) \ge S^*(x) - \epsilon \tag{2.15}$$

for all $y \in B(x, \delta)$. Thus the fact that $S_* = S^*$ almost everywhere and (2.15) imply that $S^*(x) \le S_*(x) + 2\epsilon$. So $S^*(x) \le S_*(x)$. In fact, a similar argument shows that for all x, $S_*(x) > -\infty$ and $S^*(x) < \infty$ since $S_*(x) < \infty$ and $S^*(x) > -\infty$ by sub- or superharmonicity. Thus $S_*(x)$ and $S^*(x)$ are finite for all x and $S^*(x) \le S_*(x)$. Consequently, by (2.14) $S^*(x) = S_*(x)$ everywhere and hence it is harmonic in $\mathbb{R}^d$.

But then,

$$\begin{aligned} S^*(x) &= \frac{1}{v_d} \int_{|y-x|\le 1} S^*(y)\, dy \\ &= \frac{1}{v_d} \int_{|y-x|\le 1} S(y)\, dy \\ &= \frac{1}{v_d} \int_{|y-x|\le 1} f_2(y)\, dy + \frac{1}{v_d} \int_{|y-x|\le 1} U(y)\, dy \\ &\qquad - \frac{b_0}{2d} \frac{1}{v_d} \int_{|y-x|\le 1} |y|^2\, dy \\ &= I + II + III. \end{aligned}$$

Since f_2 and U are periodic, I and II are bounded. Thus, $S^*(x) = O(|x|^2)$. By the penultimate inequality of Section 2.13 of [PW] it follows that every second order partial derivative of S^* is a bounded harmonic function and hence constant, so that S^* itself is a quadratic polynomial. (An alternative argument can be based on expanding $S^*(x)$ into spherical harmonics.) Thus, the periodic function $f_2 + U$ is almost everywhere equal to a quadratic polynomial $Q(x) = c_{1,0,\cdots,0} x_1^2 + \cdots$. A simple countability argument shows that for almost every $x \in \mathbb{R}^d$ we have $(f_2 + U)(x + 2\pi n e_1) = Q(x + 2\pi n e_1)$ for $n = 1, 2, 3, \cdots$, where $e_1 = (1, 0, \cdots, 0)$. Let $n \to \infty$ to see that $c_{1,0,\cdots,0} = 0$. Similar reasoning shows that $Q(x)$ reduces to a constant K. Consequently, we have $f_2(x) = -U(x) + K$ almost everywhere. However, both U and F are integrable over $\mathbb{T}^d$. The integrals of U and f_2 over $\mathbb{T}^d$ are both 0 by (2.5) and Lemma 2.2. So $K = 0$.

Hence,

$$\begin{aligned} f_2(x) &= -(2\pi)^{-d} \int_{T^d} G(x-y)B(y)\,dy \quad \text{a.e.} \\ &= -(2\pi)^{-d} \int_{T^d} G(x-y)\tilde{\Delta}^* f_2(y)\,dy \quad \text{a.e.} \end{aligned}$$

Finally, by (2.5), we have

$$F(x) = -(2\pi)^{-d} \int_{T^d} G(x-y)\tilde{\Delta}^* f_2(y)\,dy \quad \text{a.e.}$$

Comparing the Fourier series of both sides, we see that the a_ξ are the Fourier coefficients of $\tilde{\Delta}^* f_2(x) - K_1$ for some constant K_1. The Fourier series of the integrable function $\tilde{\Delta}^* f_2(x) - K_1$, $\sum a_\xi e^{i\langle \xi, x\rangle}$ is Abel summable to $\tilde{\Delta}^* f_2(x) - K_1$ almost everywhere (Theorem 2 of [Sh2]). Thus, from the definition of $f_*(x)$, $f_*(x) = \tilde{\Delta}^* f_2(x) - K_1$ almost everywhere. Therefore $f_*(x) \in L^1(\mathbb{T}^d)$ and a_ξ is the ξ^{th} Fourier coefficient of $f_*(x)$. In fact, $K_1 = 0$ by Lemma 2.1. This completes the proof that Lemma 2.3 will imply Theorem 1.4.

We end this section with the following observation. It is well known that if $u(x)$ is an upper semi-continuous function in $B = B(x_0, h_0) \subset \mathbb{T}^d$ and G_B denotes the Green function of B, then when $d \geq 3$, the function

$$U'(x) = \frac{1}{\sigma_d(d-2)} \int_B G_B(x,y)u(y)\,dy,$$

where σ_d is the surface area of the unit ball in $\mathbb{R}^d$, satisfies

$$\tilde{\Delta}_* U'(x) \geq -u(x), \text{ for all } x \in B(x_0, h_0).$$

Replacing U everywhere by U' in the proof of Theorem 1.4, we have the following lemma. Notice that we include the case where a_0 may not be zero.

Lemma 2.4. *Let $\sum_{\xi \in Z^d} a_\xi e^{i\langle x, \xi\rangle}$ be a multiple ($d \geq 3$) trigonometric series with $\overline{a}_\xi = a_{-\xi}$. Suppose that the coefficients a_ξ satisfy condition* (1.4);

1. *$f^*(x)$ and $f_*(x)$ are finite for all $x \in B$ where $B \subset \mathbb{T}^d$ is a ball*; *and*
2. *$f_*(x) \geq A(x)$ for almost all $x \in B$, where $A(x)$ is in $L^1(B)$.*

Then for any ball $B_1 \subset \overline{B_1} \subsetneq B$, f_* is in $L^1(B_1)$. Moreover,

$$\overline{f}_2(x) = f_2(x) + \frac{1}{\sigma_d(d-2)} \int_{B_1} G_{B_1}(x,y) f_*(y)\,dy + a_0|x|^2/(2d)$$

is finite everywhere and is almost everywhere equal to a function $h(x)$ harmonic in B_1. In addition, if $f^*(x) = f_*(x)$ everywhere in B, is in $L^1(B)$, and is continuous in B, then

$$f_2(x) + \frac{1}{\sigma_d(d-2)} \int_B G_B(x,y) f_*(y)\,dy + a_0|x|^2/(2d)$$

is harmonic on B.

Note that under condition (1.4),

$$F(x) = -\sum_{\xi \neq 0} \frac{a_\xi}{|\xi|^2} e^{i\langle \xi, x\rangle}$$

is in $L^2(\mathbb{T}^d)$ and $F(x) = f_2(x)$ almost everywhere in $\mathbb{T}^d$. Combining the above lemma with Lemma 5 of Shapiro [Sh1], we have the following analogue of Lemma 3 of Shapiro [Sh4].

Let B^o denote the interior of B.

LEMMA 2.5. *Let $\sum_{\xi \in Z^d} a_\xi e^{i\langle x,\xi\rangle}$ be a multiple trigonometric series with $\overline{a}_\xi = a_{-\xi}$. Suppose that the coefficients a_ξ satisfy condition* (1.4),

1. *$f^*(x)$ and $f_*(x)$ are finite for all $x \in B$ where $B \subset \mathbb{T}^d$ is a ball* (*open or closed*), *and*
2. *$f_*(x) = 0$ for almost all $x \in B$.*

Then for any ball $B_1 \subset \overline{B_1} \subset B^o$, $f(x,t)$ converges to 0 as $t \to 0^+$ uniformly in B_1. In particular, $f^*(x) = f_*(x) = 0$ in B.

3. Proof of Lemma 2.3

The proof of Lemma 2.3 is so difficult that this section will be given the following preface.

The proof of Lemma 2.3 is extremely delicate, incorporating all the subtle ideas from Bourgain's landmark work [B] as well as an additional Baire category argument that overcomes the unpleasant fact that an upper semicontinuous function on a compact set need not be uniformly upper semicontinuous. Some of the difficulty is pushed into Lemmas 3.2 and 3.3. The proof of Lemma 3.2 contains a great deal of hard analysis. Even after so much of the work in Lemmas 3.2 and 3.3 has been hidden, the reasoning involved in the proof of Lemma 2.3 is still tortuous, and so we will provide an overview here.

We assume that the set W where S fails to be upper semi-continuous is nonempty and then reason down a path which eventually divides into two paths each ending in a contradiction. First, a Baire category argument produces a nonempty portion Z of W ($Z = W \cap B$ for some ball B) such that S restricted to Z is "very good," f_2 restricted to Z is "very good," et cetera.

Next, for each $\varepsilon > 0$, let W_ε be the points of W where S has a jump of at least ε:

$$\limsup_{y \to x} S(y) \geq S(x) + \varepsilon, \text{ for every } x \in W_\varepsilon.$$

For each $\varepsilon > 0$ and each $x \in B \setminus \overline{W}_\varepsilon$ consider the harmonic measure ω of ∂W_ε with respect to $B \setminus \overline{W}_\varepsilon$ at x. Our path splits depending on whether the harmonic measure is "thin:"

$$\omega(B \setminus \overline{W}_\varepsilon, \partial W_\varepsilon, x) = 0 \text{ for all pairs } (\varepsilon, x) \text{ with } \varepsilon > 0 \text{ and } x \in B \setminus \overline{W}_\varepsilon, \tag{3.1}$$

or whether it is "thick:"

$$\omega(B \setminus \overline{W}_\varepsilon, \partial W_\varepsilon, x) > 0 \text{ for some } \varepsilon > 0 \text{ and some } x \in B \setminus \overline{W}_\varepsilon. \tag{3.2}$$

If (3.1) holds, from Lemma 3.2 it follows that S is bounded above and Lemma 3.3 then applies and asserts that $W \cap B = \emptyset$, a contradiction.

On the other hand, if (3.2) is the case, we apply a second Baire category argument to strengthen assumption (3.2) by producing an $\varepsilon > 0$ and a subset Z_ε of W_ε so that U is "uniformly $\varepsilon/40$ subharmonic" when restricted to Z_ε. Furthermore, the set Z_ε is still "thick:" $\omega(B \setminus \overline{Z}_\varepsilon, \partial Z_\varepsilon, x) > 0$.

Finally a very careful procedure involving picking balls within balls within balls is used to find a point p_1 of W_ε and a very nearby point p_2 of B so that $S(p_2) - S(p_1)$ is small relative to ε because of Lemma 3.2, but large relative to ε because of S having large (relative to ε) jumps at each point of W_ε. This contradiction will complete the proof of Lemma 2.3 which we begin here.

Since $\tilde{\Delta}^* S(x) \geq 0$ and $S(x)$ is in L^1 locally, we have $S(x) < \infty$ for all $x \in \mathbb{R}^d$. $S(x) \not\equiv -\infty$ since $S(x)$ is in L^1 locally. We first show that $S(x) = f_2(x) + \overline{U}(x)$ is upper semi-continuous in $\mathbb{R}^d$.

Let

$$W_\varepsilon = \left\{ x \in \mathbb{R}^d : \sup_{|x-y|<\delta} S(y) - S(x) > \varepsilon \text{ for all } \delta > 0 \right\}. \tag{3.3}$$

Then the set where $S(x)$ in $\mathbb{R}^d$ is not upper semi-continuous is given by

$$W = \bigcup_{\varepsilon > 0} W_\varepsilon.$$

If $W = \emptyset$, then $S(x)$ is upper-semicontinuous. Now we assume $W \neq \emptyset$ and construct the set Z. Bourgain's condition (1.4) implies that $f(x,t) = \sum a_\xi e^{i\langle \xi, x\rangle - |\xi| t}$ is a uniform limit of its partial sums and hence is continuous on $\mathbb{T}^d \times [\frac{1}{j}, \infty)$ for every positive integer j. Taking periodicity into account, we see that for each k, $f(x,t)$ is uniformly continuous on $\mathbb{R}^d \times [\frac{1}{k+1}, \frac{1}{k}]$. So we may partition $[\frac{1}{2}, 1]$ into $1 = t_1 > t_2 > \cdots > t_r = \frac{1}{2}$ so that for $i = 1, 2, \cdots, r-1$,

$$\sup_{x \in R^d} \sup_{t_i \geq t \geq t_{i+1}} |f(x,t_i) - f(x,t)| \leq 1\ . \tag{3.4}$$

Then partition $[\frac{1}{3}, \frac{1}{2}]$ into $\frac{1}{2} = t_r > t_{r+1} > \cdots > t_s = \frac{1}{3}$ so that inequality (3.4) holds for $i = r, r+1, \cdots, s-1$ and so on, thereby producing a sequence $\mathcal{T} = \{t_n\}$ satisfying $1 = t_1 > t_2 > \cdots$, $\lim_{k\to\infty} t_k = 0$, and

$$\sup_{x \in R^d} \sup_{t_i \geq t \geq t_{i+1}} |f(x,t_i) - f(x,t)| \leq 1 \tag{3.5}$$

holds for all k. Since $f(x,t)$ is bounded as $t \to 0^+$ for each $x \in \mathbb{R}^d$,

$$\bigcup_{n\geq 1}\bigcap_{t\in\mathcal{T}}\left\{x \in \mathbb{R}^d : |f(x,t)| \leq n\right\} = \mathbb{R}^d.$$

Therefore,

$$\bigcup_{n\geq 1}\bigcap_{t\in\mathcal{T}}\left\{x \in \overline{W} : |f(x,t)| \leq n\right\} = \overline{W}.$$

Since for each positive integer n and each $t \in \mathcal{T}$, the set

$$\left\{x \in \overline{W} : |f(x,t)| \leq n\right\}$$

is relatively closed with respect to $\overline{W}$, by Baire's category theorem applied to the space $\overline{W}$ (the intersection of countably many relatively open dense sets is not empty), for some $N_0 \geq 1$,

$$\bigcap_{t\in\mathcal{T}}\left\{x \in \overline{W} : |f(x,t)| \leq N_0\right\}$$

has a nonempty interior relative to $\overline{W}$. This means that there exist an open ball $B(p,\rho_0), p \in W$, and a constant N_0 such that

$$\sup_{t\in\mathcal{T}}\ \sup_{x\in B(p,\rho_0)\cap\overline{W}} |f(x,t)| \leq N_0 < \infty. \tag{3.6}$$

Bourgain's condition implies that $\sup_{x\in\mathbb{R}^d}|f(x,t)| \leq C$, $\sup_{x\in\mathbb{R}^d}|f_1(x,t)| \leq C$, and $\sup_{x\in\mathbb{R}^d}|f_2(x,t)| \leq C$ whenever $t \geq 1$. Use

$$f_1(x,t) = \int_1^t f(x,s)\,ds + f_1(x,1),$$

(3.5) and (3.6) to see that there is a constant $N > 0$ such that

$$\sup_{\substack{x\in Z\\ t>0}} |f_1(x,t)| \leq N, \tag{3.7}$$

where $Z = B(p,\rho_0) \cap \overline{W}$. Similarly, since

$$f_2(x,t) = -\int_0^t f_1(x,s)\,ds + f_2(x)$$

by (2.2),

$$\sup_{\substack{x\in Z\\ t>0}} |f_2(x) - f_2(x,t)| \leq Nt. \tag{3.8}$$

Therefore, $f_2(x)$ is continuous when restricted to Z. It follows that $S(x)$ is upper semi-continuous restricted to Z.

We will show a contradiction if $W \neq \emptyset$. Once S is everywhere upper semi-continuous, it is subharmonic since $\tilde{\Delta}_* S \geq 0$. For this, see p.14 of [R]. This will complete the proof of Lemma 2.3.

The following lemmas are needed in proving $W = \emptyset$.

For a bounded open set G and Borel measurable set F, we denote $\omega(G,F,x)$ to be the harmonic measure of a Borel set F relative to G at $x \in G$. Harmonic measure is closely related to Brownian motion. Let $(\{X_t\}_t, \mathcal{F}_t, P)$ be the standard Brownian motion in $\mathbb{R}^d$. For $x \in G$, let T be the exiting time of X_t from G:

$$T = \inf\{t \geq 0 : X_t \notin G\}.$$

Then $X_T \in \partial G$ since X_t is continuous in t. Let P^x denote the probability measure such that $X_0 = x$ almost everywhere. Then the harmonic measure $\omega(G,F,x) = P^x(X_T \in F)$.

The following properties of harmonic measure are well-known. We summarize them as a preliminary lemma.

Lemma 3.1. *Let $F_0 \subset F_1 \subset F_2$ be closed subsets of a bounded open set G. Then for $x \in G \setminus F_2$,*

$$\begin{aligned} \omega(G \setminus F_2, \partial F_2, x) &\geq \omega(G \setminus F_1, \partial F_1, x) \\ &\geq \omega(G \setminus F_1, \partial F_0, x) \geq \omega(G \setminus F_2, \partial F_0, x). \end{aligned} \tag{3.9}$$

To see the last inequality, let $T_i = \inf\{t \geq 0 : X_t \notin G \setminus F_i\}$, $i = 1, 2$. Then $T_2 \leq T_1$. Note that on $\{T_2 < T_1\}$, we must have $X_{T_2} \in \overline{G} \setminus F_1$. Otherwise, $X_{T_2} \in \overline{G} \setminus (\overline{G} \setminus F_1) = F_1$. Thus by definition $T_2 \geq T_1$, a contradiction. But $\{T_2 < T_1\} \subset \{X_{T_2} \in \overline{G} \setminus F_1\}$ implies that $\{X_{T_2} \in F_1\} \subset \{T_1 = T_2\}$. Consequently, $\{X_{T_2} \in \partial F_0\} \subset \{X_{T_2} \in F_0\} \subset \{X_{T_2} \in F_1\} \subset \{T_1 = T_2\}$. This proves the inequality since

$$P^x(X_{T_2} \in \partial F_0) = P^x(X_{T_2} \in \partial F_0, T_1 = T_2) \leq P^x(X_{T_1} \in \partial F_0).$$

The middle inequality is simply the monotonicity of the harmonic measure. To see the left inequality, observe that on $\{X_{T_1} \in \partial F_1\}$, $X_{T_2} \in \partial F_2$. Otherwise, $X_{T_2} \in \partial G$ and $T_2 \leq T_1$ imply that $X_{T_1} \in \partial G$, a contradiction. Consequently, $\{X_{T_1} \in \partial F_1\} \subset \{X_{T_2} \in \partial F_2\}$. This completes the proof.

The next three lemmas are essential to the proof of Lemma 2.3.

Lemma 3.2. *Let S, f_2, and $\overline{U}$ be as given in Lemma 2.3. Let W be the set where S is not upper semi-continuous. Assume that there is a open ball $B(p, \rho_0)$, $p \in W$, such that when restricted to $Z = B(p, \rho_0) \cap \overline{W}$, $f_2(x)$ is continuous and (3.8) holds. Then, for $p_1 \in W, B(p_1, \rho_1) \subset B(p, \frac{1}{2}\rho_0)$ and $p_2 \in B(p_1, \frac{1}{2}\rho_1)$, there exists a constant $c > 0$ such that for almost all such ρ_1,*

$$\begin{aligned} S(p_2) - S(p_1) \leq\ & c\Bigg(\Big[|f_2(p_1)| + \rho_1^{-\frac{3}{4}(d-1)}\Big] \\ & \times\ [1 - \omega(B(p_1,\rho_1) \setminus \overline{W}, \partial(W \cap B(p_1,\rho_1)), p_2)]^{\frac{1}{4}} \end{aligned} \tag{3.10}$$

$$+ \sup_{q \in B(p_1, 2\rho_1) \cap \overline{W}} |f_2(q) - f_2(p_1)|$$

$$+ 2 \sup_{q \in B(p_1, 2\rho_1)} \Big(\overline{U}(q) - \overline{U}(p_1) \Big) \Big).$$

Lemma 3.2 is a one-sided version of Bourgain's key lemma in [B]. The proof is also similar and is given in Section 4. It follows from Lemma 3.2 that $S(x)$ is bounded from above in $B(p, \frac{\rho_0}{4})$ when $p_1 = p$.

Lemma 3.3. *Assume $\overline{U}$ is defined on $\overline{B}(p,r)$ and is upper semi-continuous on $B(p,r)$. Let f_2 be a function in $\overline{B}(p,r)$ such that $S(x) = f_2(x) + \overline{U}(x)$ is bounded from above in $\overline{B}(p,r)$, in $L^1(\overline{B}(p,r))$, and satisfies*

$$\tilde{\Delta}^* S(x) \geq 0, \text{ and } \tilde{\Delta}_* f_2(x) < \infty \tag{3.11}$$

for each $x \in B(p,r)$. If S is upper semi-continuous when restricted to $\overline{W} = \{x \in B(p,r) : S(x) \text{ is not upper semi-continuous}\}$, and for all $x \in B(p,r) \setminus \overline{W}_\varepsilon$ the harmonic measure

$$\omega(B(p,r) \setminus \overline{W}_\varepsilon, B(p,r) \cap W_\varepsilon, x) = 0 \tag{3.12}$$

for all $\varepsilon > 0$ where W_ε is given by (3.3), then W must be empty and $S(x)$ is subharmonic on $B(p,r)$.

The proof of Lemma 3.3 is given in Section 5. The special case when $\overline{U} \equiv 0$ was proved by Bourgain.

The next lemma provides a harmonic measure version of a point density.

Lemma 3.4. *Let $B(p_0, r)$ be a ball in $\mathbb{R}^d$ and F a closed set such that $B(p_0, r) \cap F \neq \emptyset$. Suppose for some $x \in B(p_0, r) \setminus F$,*

$$\omega(B(p_0, r) \setminus F, \partial(B(p_0, r) \cap F), x) > 0.$$

Then there exists $p_1 \in B(p_0, r) \cap F$, such that

$$\inf_{\delta_1 > 0} \liminf_{\delta_2 \to 0} \inf_{x \in B(p_1, \delta_2)} \omega(B(p_1, \delta_1) \setminus F, \partial(B(p_1, \delta_1) \cap F), x) = 1. \tag{3.13}$$

The proof of Lemma 3.4 is outlined in [B]. For a detailed proof, see the proof of Theorem 3.14 in [AW].

We now return to the proof of Lemma 2.3. By (2.5) and the fact that $F(x) \in L^2(\mathbb{T}^d)$, we have S is in $L^1_{\text{loc}}(\mathbb{R}^d)$. For the duration of this proof, we abbreviate $B(p, \rho_0/8)$ to B. There are two cases.

Case one: for all $\epsilon > 0$,

$$\omega(B \setminus \overline{W_\epsilon}, \partial(B \cap W_\epsilon), x) = 0$$

for all $x \in B \setminus \overline{W_\epsilon}$. Then by Lemma 3.2, S is bounded from above and by Lemma 2.1, $\tilde{\Delta}_* f_2 < \infty$ everywhere. Also $B(p, p_0)$ was chosen so that S is upper semi-continuous when restricted to $B(p, p_0) \cap \overline{W}$. Thus all the hypotheses of Lemma 3.3 are satisfied and $W \cap B = \emptyset$, which is a contradiction.

Case two: for some $\epsilon > 0$ and for some $x_0 \in B \setminus \overline{W_\epsilon}$, we have

$$\omega(B \setminus \overline{W_\epsilon}, \partial(B \cap W_\epsilon), x_0) > 0. \tag{3.14}$$

Even though $\overline{U}$ is upper semi-continuous everywhere, it may not be uniformly upper semi-continuous on W_ϵ. This presents a problem which did not arise at the corresponding point in Bourgain's proof. To deal with this, we now introduce a subset of W_ϵ called Z_ϵ, on a portion of which there holds a kind of uniform upper semi-continuity.

Let

$$Z_\epsilon = \{y \in B \cap W_\epsilon : \omega(B \setminus \overline{W_\epsilon}, \overline{B(y, \delta) \cap W_\epsilon}, x_0) > 0, \text{ for all } \delta > 0\}.$$

Then,

$$\omega(B \setminus \overline{W_\epsilon}, \partial Z_\epsilon, x_0) > 0. \tag{3.15}$$

In fact, by definition, for each $z \in B \cap W_\epsilon \setminus \overline{Z_\epsilon}$, there exists a ball $B(z, \delta_z)$, such that

$$\omega(B \setminus \overline{W_\epsilon}, \overline{B(z, \delta_z) \cap W_\epsilon}, x_0) = 0. \tag{3.16}$$

The open cover $\{B(z, \delta_z)\}$ of $B \cap W_\epsilon \setminus \overline{Z_\epsilon}$ has a countable subcover $\{B(z_i, \delta_{z_i})\}$. Thus, (3.16) implies

$$\omega(B \setminus \overline{W_\epsilon}, B \cap W_\epsilon \setminus \overline{Z_\epsilon}, x_0) = 0.$$

So (3.15) follows from (3.14) as $\omega(B \setminus \overline{W_\epsilon}, \overline{Z_\epsilon}, x_0) = \omega(B \setminus \overline{W_\epsilon}, \partial Z_\epsilon, x_0)$.

Since $\overline{U}$ is upper semi-continuous,

$$\bigcup_{m \geq 1} \left\{ y \in \overline{Z_\epsilon} : \sup_{|z-y| \leq 2/m} \overline{U}(z) - \overline{U}(y) \leq \frac{\epsilon}{40} \right\} = \overline{Z_\epsilon}.$$

Apply Baire's category theorem to the space $\overline{Z_\epsilon}$ to see that there exists $m \geq 1$ and an open ball $B(q, \rho) \subset B, q \in Z_\epsilon$, such that

$$B(q, \rho) \cap \overline{Z_\epsilon} \subset \overline{\left\{ y \in \overline{Z_\epsilon} : \sup_{|z-y| \leq 2/m} \overline{U}(z) - \overline{U}(y) \leq \frac{\epsilon}{40} \right\}}.$$

Equivalently, for any fixed $y \in B(q, \rho) \cap \overline{Z_\epsilon}$, there exists a sequence $y_n \in B(q, \rho) \cap Z_\epsilon$ convergent to y such that

$$\sup_{|z-y_n| \leq 2/m} \overline{U}(z) - \overline{U}(y_n) \leq \frac{\epsilon}{40}. \tag{3.17}$$

However, $\overline{U}$ is upper semi-continuous. So there exists $0 < \delta < \frac{1}{m}$ such that for $|y_n - y| < \delta$,

$$\overline{U}(y_n) - \overline{U}(y) < \frac{\epsilon}{40}.$$

Thus, for $|z-y| < \frac{1}{m}$, since $|z - y_n| < \frac{2}{m}$ if $|y_n - y| < \delta$,

$$\sup_{|z-y|<\frac{1}{m}} \overline{U}(z) < \overline{U}(y_n) + \frac{\epsilon}{40} < \overline{U}(y) + \frac{\epsilon}{20}. \tag{3.18}$$

Without loss of generality, we assume that $\frac{1}{m} \le \frac{\rho}{2}$.

Because $q \in Z_\epsilon$, we also have

$$\omega(B \setminus \overline{W_\epsilon}, \overline{B(q, \frac{\rho}{2}) \cap W_\epsilon}, x_0) = \omega(B \setminus \overline{W_\epsilon}, \partial(B(q, \frac{\rho}{2}) \cap W_\epsilon), x_0) > 0. \tag{3.19}$$

Set $F_\epsilon = B(q, \frac{\rho}{2}) \cap W_\epsilon$. Then the rightmost inequality of Lemma 3.1 and (3.19) imply that

$$\omega(B \setminus \overline{F}_\epsilon, \partial F_\epsilon, x_0) \ge \omega(B \setminus \overline{W}_\epsilon, \partial F_\epsilon, x_0) > 0.$$

From Lemma 3.4, there exists $p' \in \overline{F}_\epsilon$ such that

$$\inf_{\delta_1 > 0} \liminf_{\delta_2 \to 0} \inf_{x \in B(p', \delta_2)} \omega(B(p', \delta_1) \setminus \overline{F}_\epsilon, \partial(B(p', \delta_1) \cap F_\epsilon), x) = 1. \tag{3.20}$$

Notice that Lemma 3.4 requires the set F to be closed, so we cannot be sure that $p' \in F_\varepsilon$. Although F_ε may not be closed, the uniformity implied by (3.20) allows us to continue.

Since f_2 restricted to $\overline{W} \cap B$ is continuous, we may select $1/(8m) > \delta_1 > 0$ such that

$$|f_2(z) - f_2(y)| \le \frac{\varepsilon}{10} \tag{3.21}$$

for all $y, z \in B(p', 8\delta_1) \cap \overline{W}$.

Let $\eta > 0$ be any positive number. From (3.20), it follows that there exists $0 < \delta_2 = \delta_2(\eta, \delta_1) < \delta_1$ such that

$$\omega(B(p', \delta_1) \setminus \overline{F}_\epsilon, \partial(B(p', \delta_1) \cap F_\epsilon), y) > 1 - \eta \text{ for all } y \in B(p', \delta_2).$$

We may also assume that $\delta_1 + \delta_2 = \delta_3'$ satisfies $B(p', \delta_3') \subset B(p, \frac{\rho_0}{2})$. Pick any δ_3 bigger than δ_3' but small enough to force $B(p', \delta_3) \subset B(p, \frac{\rho_0}{2})$. Note that $p' \in \overline{F}_\varepsilon$ implies that there exists $p_1 \in B(p', \frac{\delta_2}{2}) \cap F_\epsilon$. Since $B(p', \delta_1) \subset B(p_1, \delta_3)$,

$$B(p_1, \delta_3) \setminus \overline{[F_\varepsilon \cup \{B(p_1, \delta_3) \setminus B(p', \delta_1)\}]} = B(p', \delta_1) \setminus \overline{F}_\varepsilon.$$

So by the rightmost inequality of Lemma 3.1,

$$\begin{aligned}
&\omega(B(p_1, \delta_3) \setminus \overline{F}_\epsilon, \partial(B(p', \delta_1) \cap F_\epsilon), y) \\
&\quad \ge \omega(B(p_1, \delta_3) \setminus \overline{[F_\varepsilon \cup \{B(p_1, \delta_3) \setminus B(p', \delta_1)\}]}, \partial(B(p', \delta_1) \cap F_\epsilon), y) \\
&\quad = \omega(B(p', \delta_1) \setminus \overline{F}_\epsilon, \partial(B(p', \delta_1) \cap F_\epsilon), y) \\
&\quad \ge 1 - \eta \text{ for all } y \in B(p_1, \frac{\delta_2}{2}).
\end{aligned}$$

Consequently,

$$\omega(B(p_1,\delta_3)\setminus\overline{F}_\epsilon,\partial(B(p_1,\delta_3)\cap F_\epsilon),y)\geq 1-\eta \text{ for all } y\in B(p_1,\frac{\delta_2}{2}),$$

since $B(p',\delta_1)\cap F_\epsilon\subset B(p_1,\delta_3)\cap F_\epsilon$. Finally, by the left inequality of Lemma 3.1

$$\omega(B(p_1,\delta_3)\setminus\overline{W},\partial(B(p_1,\delta_3)\cap W),y)\geq\omega(B(p_1,\delta_3)\setminus\overline{F}_\epsilon,\partial(B(p_1,\delta_3)\cap F_\epsilon),y).$$

We therefore have

$$\omega(B(p_1,\delta_3)\setminus\overline{W},\partial(B(p_1,\delta_3)\cap W),y)>1-\eta \text{ for all } y\in B(p_1,\frac{\delta_2}{2}). \tag{3.22}$$

By definition, $p_1\in W_\epsilon$ implies that there exists $p_2\in B(p_1,\frac{\delta_2}{2})$ such that

$$S(p_2)-S(p_1)\geq\frac{\varepsilon}{2}.$$

Apply Lemma 3.2 at p_1, p_2, and $\rho_1=\delta_3$ where the inequality (3.10) holds for δ_3. Then by (3.18), (3.21), (3.22), and the above inequality, we have

$$\frac{\varepsilon}{2}\leq S(p_2)-S(p_1)\leq c[|f_2(p_1)|+\delta_3^{-\frac{3}{4}(d-1)}]\eta^{1/4}+\frac{\varepsilon}{5}. \tag{3.23}$$

Note here that p_1, p_2, and δ_3 depend on η. However, since f_2 is continuous and hence bounded on $B(p,\rho_0)\cap\overline{W}$ and δ_3 is bounded below by δ_1 as $\eta\to 0$, so (3.23) becomes a contradiction upon choosing η sufficiently small.

4. Proof of Lemma 3.2

For any bounded measurable function $f(x)$ defined on ∂G,

$$H_f(x)=\int_{\partial G} f(z)\,\omega(G,dz,x) \tag{4.1}$$

is harmonic in G. If every point on ∂G satisfies the exterior cone condition and f is continuous at $x\in\partial G$, then

$$\lim_{\substack{y\to x\\ y\in G}} H_f(y)=f(x).$$

Since any upper semi-continuous function is the limit of a decreasing sequence of continuous functions, so the maximum principle for subharmonic functions and (4.1) imply that

$$f(x)\leq\int_{\partial G} f(z)\,\omega(G,dz,x) \tag{4.2}$$

for any function f subharmonic on an open set $\tilde{G}\supset\overline{G}\supset G$.

We need only to consider $p_2\notin\overline{W}$. Let $\tau=\mathrm{dist}(p_2,\overline{W})\leq\frac{1}{2}\rho_1$. For $\kappa\ll\tau$, define

$$G_\kappa=\{x\in B(p_1,\rho_1):\mathrm{dist}(x,\overline{W})<\kappa\}.$$

Clearly $\overline{W} \cap B(p_1, \rho_1) \subset G_\kappa$. We know that S is upper semi-continuous and $\tilde{\Delta}^* S(x) \geq 0$ on $B(p, \rho_0) \setminus \overline{W}$. This is the hypothesis of a classical theorem (see for example [R, p. 14]) which concludes that S is subharmonic on $B(p, \rho_0) \setminus \overline{W}$. Thus, $S(x) - S(p_1)$ is subharmonic on $B(p, \rho_0) \setminus \overline{W}$. In particular, $S(x) - S(p_1)$ is subharmonic on an open set containing $\overline{B(p_1, \rho_1) \setminus \overline{G}_\kappa}$. Note that $B(p_1, \rho_1) \setminus \overline{G}_\kappa$ satisfies the exterior cone condition everywhere on the boundary. So by (4.2), we have

$$\begin{aligned} S(p_2) - S(p_1) \leq & \int_{\partial(B(p_1,\rho_1)\setminus \overline{G}_\kappa)} [S(x) - S(p_1)]\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ = & \int_{\partial B(p_1,\rho_1)\setminus(B(p_1,\rho_1)\cap \partial G_\kappa)} [S(x) - S(p_1)]\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ & + \int_{B(p_1,\rho_1)\cap \partial G_\kappa} [S(x) - S(p_1)]\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ = & \; I_1 + I_2. \end{aligned}$$

We first estimate I_1. When $p_2 \in B(p_1, \rho_1/2)$, a classical result on harmonic measure shows that $\omega = \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2)$ is absolutely continuous with respect to the surface Lebesgue measure σ when restricted to the sphere $B(p_1, \rho_1)$. (See [D] or (4.39) of [AW].) By (2.5), $f_2(x) = F(x)$ almost everywhere with respect to Lebesgue measure; thus, for almost every $\rho_1 > 0$, $f_2(x) = F(x)$ almost everywhere with respect to the surface Lebesgue measure on $B(p_1, \rho_1)$ and hence with respect to the harmonic measure ω for all $p_2 \in B(p_1, \rho_1/2)$. Consequently,

$$\begin{aligned} I_1 \leq & \int_{\partial B(p_1,\rho_1)} |F(p_2) - f_2(p_1)|\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ & + \sup\nolimits_{q \in B(p_1,\rho_1)} \overline{U}(q) - \overline{U}(p_1) \\ = & \; I_3 + I_4. \end{aligned}$$

A result of Bourgain [B] (see also Lemma 4.5 of [AW]) shows that

$$I_3 \leq c \left(\left[|f_2(p_1)| + \rho_1^{-\frac{3}{4}(d-1)} \right] [1 - \omega(B(p_1, \rho_1) \setminus \overline{W}, \partial(W \cap B(p_1, \rho_1)), p_2)]^{\frac{1}{4}} \right).$$

Remark 4.1. In fact, Lemma 4.5 of [AW] was based on Connes' condition (1.3). But a careful reading of the proof shows the conclusion of Lemma 4.5 holds true under Bourgain's condition (1.4) since inequalities (4.18) and (4.19) in Lemma 4.2 and Corollary 4.3 respectively can be replaced by

$$\sup_k \frac{1}{2^k} \sum_{|\xi|^2 \sim 2^k} |c_\xi|^2 \leq M$$

as they are used only in (4.21).

This gives the first half of (3.10). Now we estimate I_2.

For any $x \in B(p_1, \rho_1) \cap \partial G_\kappa$, there exists $\tilde{x} \in \overline{W} \cap B(p_1, 2\rho_1)$, such that $|x - \tilde{x}| = \kappa$. Since S is subharmonic at x,

$$S(x) \leq A_\kappa f_2(x) + A_\kappa \overline{U}(x). \tag{4.3}$$

Since $\tilde{x} \in \overline{W} \cap B(p, \rho_0)$, by assumption,

$$|f_2(\tilde{x}) - f_2(\tilde{x}, \kappa)| \leq N\kappa. \tag{4.4}$$

Thus combining (4.3) and (4.4), we have

$$\begin{aligned} S(x) - S(\tilde{x}) \quad &\leq \quad A_\kappa f_2(x) - A_\kappa f_2(\tilde{x}, \kappa) + A_\kappa f_2(\tilde{x}, \kappa) - f_2(\tilde{x}, \kappa) \\ &\quad + A_\kappa \overline{U}(x) - \overline{U}(\tilde{x}) + N\kappa. \end{aligned}$$

Consequently,

$$\begin{aligned} S(x) - S(p_1) &= S(x) - S(\tilde{x}) + S(\tilde{x}) - S(p_1) \\ &\leq A_\kappa f_2(x) - A_\kappa f_2(\tilde{x}, \kappa) + A_\kappa f_2(\tilde{x}, \kappa) - f_2(\tilde{x}, \kappa) \\ &\quad + f_2(\tilde{x}) - f_2(p_1) + A_\kappa \overline{U}(x) - \overline{U}(p_1) + N\kappa \\ &\leq A_\kappa f_2(x) - A_\kappa f_2(\tilde{x}, \kappa) + A_\kappa f_2(\tilde{x}, \kappa) - f_2(\tilde{x}, \kappa) \\ &\quad + \sup\nolimits_{q \in B(p_1, 2\rho_1) \cap \overline{W}} |f_2(q) - f_2(p_1)| \\ &\quad + \sup\nolimits_{q \in B(p_1, 2\rho_1)} \overline{U}(q) - \overline{U}(p_1) + N\kappa. \end{aligned}$$

From (2.5) and the definition of $A_\rho f_2(x)$, we have $A_\rho f_2(x) = A_\rho F(x)$ for all x. Thus

$$\begin{aligned} I_2 \quad &\leq \quad \int\limits_{B(p_1, \rho_1) \cap \partial G_\kappa} |A_\kappa F(x) - A_\kappa f_2(\tilde{x}, \kappa)|\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ &\quad + \int\limits_{B(p_1, \rho_1) \cap \partial G_\kappa} |A_\kappa f_2(\tilde{x}, \kappa) - f_2(\tilde{x}, \kappa)|\, \omega(B(p_1, \rho_1) \setminus \overline{G}_\kappa, dx, p_2) \\ &\quad + \sup\nolimits_{q \in B(p_1, 2\rho_1) \cap \overline{W}} |f_2(q) - f_2(p_1)| \\ &\quad + \sup\nolimits_{q \in B(p_1, 2\rho_1)} \overline{U}(q) - \overline{U}(p_1) + N\kappa \\ &= \quad I_5 + I_6 + \sup\nolimits_{q \in B(p_1, 2\rho_1) \cap \overline{W}} |f_2(q) - f_2(p_1)| \\ &\quad + \sup\nolimits_{q \in B(p_1, 2\rho_1)} \overline{U}(q) - \overline{U}(p_1) + N\kappa. \end{aligned} \tag{4.5}$$

It is enough to show that $I_5 \to 0$ and $I_6 \to 0$ as $\kappa \to 0$. Observe that

$$\begin{aligned} I_5 &\leq \int_{B(p_1,\rho_1)\cap\partial G_\kappa} |A_\kappa F(x) - A_\kappa F(\tilde{x})|\, \omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, dx, p_2) \\ &\quad + \int_{B(p_1,\rho_1)\cap\partial G_\kappa} |A_\kappa F(\tilde{x}) - A_\kappa f_2(\tilde{x},\kappa)|\, \omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, dx, p_2) \\ &= I_7 + I_8. \end{aligned}$$

A result of Bourgain [B] (again, see also Lemma 4.4 of [AW]), shows that $I_7 \to 0$ as $\kappa \to 0$. However, the same proof of Lemma 4.4 in [AW] shows that if a_ξ satisfies Bourgain's condition (1.4), then

$$\lim_{\kappa\to 0} \kappa \int_{B(p_1,\rho_1)\cap\partial G_\kappa} \left| \sum_{|\xi|\neq 0} \frac{a_\xi}{|\xi|} \hat{I}(\kappa|\xi|) e^{i\langle \overline{x},\xi\rangle} \right| \omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, dx, p_2) = 0, \tag{4.6}$$

where $|\overline{x} - x| \leq \kappa$. Note that

$$A_\kappa F(\tilde{x}) - A_\kappa f_2(\tilde{x},\kappa) = -\sum_{|\xi|\neq 0} \frac{a_\xi}{|\xi|^2} \hat{I}(\kappa|\xi|) e^{i\langle \tilde{x},\xi\rangle}(1 - e^{-|\xi|\kappa}),$$

while by the mean value theorem, for each $\xi \neq 0$, there exists $t_\xi > 0$ such that

$$\sum_{|\xi|\neq 0} \frac{a_\xi}{|\xi|^2} \hat{I}(\kappa|\xi|) e^{i\langle \tilde{x},\xi\rangle}(1 - e^{-|\xi|\kappa}) = \kappa \sum_{|\xi|\neq 0} \frac{a_\xi e^{-|\xi| t_\xi}}{|\xi|} \hat{I}(\kappa|\xi|) e^{i\langle \tilde{x},\xi\rangle}.$$

Since $e^{-|\xi|t_\xi} < 1$, $\{a_\xi e^{-|\xi|t_\xi}\}$ satisfies Bourgain's condition (1.4) as $\{a_\xi\}$ does. Thus by (4.6), $I_8 \to 0$ as $\kappa \to 0$. This shows that $I_5 \to 0$ as $\kappa \to 0$.

The method that Bourgain used to prove that $I_7 \to 0$ as $\kappa \to 0$ can also be used to prove $I_6 \to 0$ as $\kappa \to 0$. To establish this, we will use the following lemma of Bourgain [B]. (See also the proof of Corollary 4.3 of [AW].)

LEMMA 4.1. *Let $k \geq 1, \gamma > 0, \eta \leq 2^{-k}$. Let $E_{k,\gamma,\eta}$ be a set of η-separated points $x \in B(p,q) \subset \mathbb{R}^d$ satisfying*

$$\left| \sum_{|\xi|\sim 2^k} \frac{b_\xi}{|\xi|^2} e^{i\langle x,\xi\rangle} \right| \geq \gamma.$$

Then, the cardinality of $E_{k,\gamma,\eta}$ satisfies

$$|E_{k,\gamma,\eta}| \leq c\gamma^{-2}\eta^{-d} 2^{-2k} \nu_k^2,$$

where c is an absolute constant and

$$\nu_k^2 = 2^{-2k} \sum_{|\xi|\sim 2^k} |b_\xi|^2.$$

Let

$$\alpha_k = \begin{cases} c\alpha[\log(1+2^k\kappa)]^{-2}, & \text{for } 2^k \geq \kappa^{-1} \\ c\alpha[\log(1+2^{-k}\kappa^{-1})]^{-2}, & \text{for } 2^k < \kappa^{-1}. \end{cases}$$

The positive constant c is chosen so that $\sum_{k\geq 1}\alpha_k \leq 2c\alpha\sum_{n\geq 0}(\log(1+2^n))^{-2} = \alpha$ for all $\alpha > 0$. Clearly, c is an absolute constant. For $\alpha > 0$, let

$$S_{\kappa,\alpha} = \{x \in B(p_1,\rho_1)\cap\partial G_\kappa : |A_\kappa f_2(\tilde{x},\kappa) - f_2(\tilde{x},\kappa)| > \alpha\}.$$

Then

$$I_6 = \int_0^\infty \omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, S_{\kappa,\alpha}, p_2)\,d\alpha. \tag{4.7}$$

Let

$$\begin{aligned} S_{\kappa,k,\alpha_k} &= \{x \in B(p_1,\rho_1)\cap\partial G_\kappa : |A_\kappa f_{2,k}(\tilde{x},\kappa) - f_{2,k}(\tilde{x},\kappa)| > \alpha_k\}, \\ S'_{\kappa,k,\alpha_k} &= \{x \in B(p_1,\rho_1) : |A_\kappa f_{2,k}(x,\kappa) - f_{2,k}(x,\kappa)| > \alpha_k\}, \end{aligned}$$

where

$$\begin{aligned} f_{2,k}(x,\kappa) &= \sum_{|\xi|\sim 2^k} \frac{a_\xi}{|\xi|^2} e^{i\langle x,\xi\rangle - \kappa|\xi|}, \text{ and} \\ A_\kappa f_{2,k}(x,\kappa) &= \sum_{|\xi|\sim 2^k} \frac{a_\xi}{|\xi|^2}\hat{I}(\kappa|\xi|)e^{i\langle x,\xi\rangle-\kappa|\xi|}. \end{aligned}$$

Then

$$S_{\kappa,\alpha} \subset \bigcup_{k\geq 1} S_{\kappa,k,\alpha_k}. \tag{4.8}$$

Since $|x-\tilde{x}| = \kappa$, observe that a collection of balls of radius $\eta \leq 2^{-k}$ centered at points in S'_{κ,k,α_k} covering S'_{κ,k,α_k} will cover S_{κ,k,α_k} if the radius of each ball is enlarged by κ.

Bourgain's condition (1.4) may be restated as $\delta_k \to 0$, where

$$\delta_k^2 = 2^{-2k}\sum_{|\xi|\sim 2^k}|a_\xi^2|.$$

In particular, δ_k^2 is bounded for all k. Now apply Lemma 4.1 and use the fact that $\hat{I}(|\xi|) = O(|\xi|^{-(d+1)/2})$ as $|\xi| \to \infty$ and also use (2.6). We find that the number of balls of radius $\eta \leq 2^{-k}$ centered at S'_{κ,k,α_k} covering S'_{κ,k,α_k} is at most

$$\begin{aligned} |E_{k,\alpha_k,\eta}| &\leq c\alpha_k^{-2}\eta^{-d}2^{-2k}e^{-\kappa 2^k}\delta_k^2 \sup_{2^{k-1}\leq j<2^k}|\hat{I}(\kappa j)-1|^2 \\ &\leq \begin{cases} c\alpha_k^{-2}\delta_k^2\eta^{-d}2^{-2k}e^{-\kappa 2^k}, & \kappa 2^k \geq 1 \\ c\alpha_k^{-2}\delta_k^2\eta^{-d}2^{-2k}(\kappa 2^k)^4, & \kappa 2^k < 1. \end{cases} \end{aligned} \tag{4.9}$$

We estimate $\omega(B(p_1,\rho_1)\setminus\overline{G}_\kappa, S_{\kappa,k,\alpha_k}, p_2)$ according to the size of k.

Case (i): $\kappa 2^k \geq 1$. By (4.9) with $\eta = 2^{-k}$ and the observation made after (4.8), the number of balls of radius 2κ covering S_{κ,k,α_k} is at most $M = c\alpha_k^{-2}\delta_k^2 2^{(d-2)k} e^{-\kappa 2^k}$. Let $\{B_i\}_{1\leq i\leq M_1}$, $M_1 \leq M$, denote these balls. Then

(4.10)
$$\begin{aligned}
\omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, S_{\kappa,k,\alpha_k}, p_2) &\leq \omega(B(p_1,\rho_1)\setminus S_{\kappa,k,\alpha_k}, S_{\kappa,k,\alpha_k}, p_2)\\
&\leq \omega(B(p_1,\rho_1)\setminus \overline{\cup B_i}, \partial(\cup B_i), p_2)\\
&= \sum_{i=1}^{M_1} \omega(B(p_1,\rho_1)\setminus \overline{\cup B_i}, \partial B_i, p_2)\\
&\leq \sum_{i=1}^{M_1} \omega(B(p_1,\rho_1)\setminus \overline{B_i}, \partial B_i, p_2)\\
&\leq c\alpha_k^{-2}\delta_k^2 2^{(d-2)k} e^{-\kappa 2^k}\left(\frac{\kappa}{\tau}\right)^{d-2}\\
&= \frac{c}{\tau^{d-2}}\alpha_k^{-2}\delta_k^2(\kappa 2^k)^{d-2}e^{-\kappa 2^k}\\
&\leq \frac{c}{\tau^{d-2}}\alpha_k^{-2}\delta_k^2(\kappa 2^k)^{-1},
\end{aligned}$$

where c is a constant which may vary from line to line. The first line of (4.10) follows from the rightmost inequality of Lemma 3.1, since $B(p_1,\rho_1)\setminus \overline{G}_\kappa$ has been relaced by $B(p_1,\rho_1)\setminus S_{\kappa,k,\alpha_k}$. The second line follows from the left-most inequality of Lemma 3.1, since both occurrences of S_{κ,k,α_k} have been replaced by the union of balls $\bigcup_{i=1}^{M_1} B_i$ of radius 2κ covering it. The third line follows from the subadditivity of harmonic measure in the second coordinate. The fourth line follows from the right-most inequality of Lemma 3.1, since $\bigcup_{i=1}^{M_1} B_i$ has been replaced by B_i in each term. To see the next line, write B_i as $B(q_i,\rho_i)$; use the explicit formula for the Poisson integral to estimate each term $\omega(B(p_1,\rho_1)\setminus \overline{B_i}, \partial B_i, p_2)$ by $\frac{\rho_i^{d-2}}{|p_2-q_i|^{d-2}}$, where $\rho_i = 2\kappa$ and $|p_2 - q_i| \geq \tau - 3\kappa > \tau/2$; and finally use the first line of (4.9) to estimate the number of terms.

Case (ii): $\kappa^{-1/2} \leq 2^k < \kappa^{-1}$. By (4.9) with $\eta = 2^{-k}$, the number of balls of radius $2\cdot 2^{-k}$ covering S_{κ,k,α_k} is at most $c\alpha_k^{-2}\delta_k^2 2^{(d-2)k}(\kappa 2^k)^4$. So as shown in Case (i),

(4.11)
$$\begin{aligned}
\omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, S_{\kappa,k,\alpha_k}, p_2) &\leq c\alpha_k^{-2}\delta_k^2 2^{(d-2)k}(\kappa 2^k)^4\left(\frac{1}{2^k\tau}\right)^{d-2}\\
&\leq \frac{c}{\tau^{d-2}}\alpha_k^{-2}\delta_k^2(\kappa 2^k)^2.
\end{aligned}$$

Case (iii): $2^k < \kappa^{-1/2}$. By (4.9) with $\eta = \sqrt{\kappa}$, the number of balls of radius $2\sqrt{\kappa}$ covering S_{κ,k,α_k} is at most $\alpha_k^{-2}\delta_k^2 2^{-2k}\kappa^{-d/2}(\kappa 2^k)^4$. So,

(4.12)
$$\begin{aligned}\omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, S_{\kappa,k,\alpha_k}, p_2) &\leq c\alpha_k^{-2}\delta_k^2 2^{-2k}\kappa^{-d/2}(\kappa 2^k)^4\left(\frac{\sqrt{\kappa}}{\tau}\right)^{d-2}\\ &\leq c\alpha_k^{-2}\delta_k^2 2^{-2k}\kappa^{-d/2}(\kappa 2^k)^2\left(\frac{\sqrt{\kappa}}{\tau}\right)^{d-2}\\ &\leq \frac{c}{\tau^{d-2}}\alpha_k^{-2}\delta_k^2\kappa.\end{aligned}$$

When (4.10), (4.11), and (4.12) are combined, it follows from (4.8) and the definitions of α_k that

(4.13)
$$\begin{aligned}\omega(B(p_1,\rho_1)\setminus \overline{G}_\kappa, S_{\kappa,\alpha}, p_2) &\leq \frac{c}{\tau^{d-2}}\left\{\sum_{2^k\geq\kappa^{-1}}\alpha_k^{-2}\delta_k^2(2^k\kappa)^{-1}+\right.\\ &\left.\sum_{\kappa^{-1/2}\leq 2^k<\kappa^{-1}}\alpha_k^{-2}\delta_k^2(2^k\kappa)^2+\sum_{2^k<\kappa^{-1/2}}\alpha_k^{-2}\delta_k^2\kappa\right\}\\ &\leq \frac{c}{\tau^{d-2}}\alpha^{-2}\left\{\max_{2^k>\kappa^{-1/2}}\delta_k^2+\kappa|\log\kappa|^5\right\}.\end{aligned}$$

Choose
$$\beta_\kappa = \left\{\max_{2^k>\kappa^{-1/2}}\delta_k^2+\kappa|\log\kappa|^5\right\}^{1/2}.$$

Then by (4.7),
$$I_6 \leq \int_0^{\beta_\kappa} d\alpha + \frac{c}{\tau^{d-2}}\beta_\kappa^2\int_{\beta_\kappa}^\infty \alpha^{-2}\,d\alpha = \left(\frac{c}{\tau^{d-2}}+1\right)\beta_\kappa \to 0$$

as $\kappa\to 0$. This completes the proof.

5. Proof of Lemma 3.3

Let the average of H on the surface of $B(x,\rho)$ be denoted by
$$D_\rho H(x) = \frac{1}{\sigma_d\rho^{d-1}}\int_{\partial B(x,\rho)} H(z)\,d\sigma(z),$$

where σ is the surface measure, and $\sigma_d = dv_d$ is the surface area of the unit ball in $\mathbb{R}^d$. Then,

$$
\begin{aligned}
(5.1)\qquad A_\rho H(x) &= \frac{1}{v_d\rho^d}\int_{B(x,\rho)} H(z)\,dz \\
&= \frac{1}{v_d\rho^d}\int_0^\rho\int_{\partial B(x,\beta)} H(z)\,d\sigma(z)\,d\beta \\
&= \frac{\sigma_d}{v_d\rho^d}\int_0^\rho D_\beta H(x)\beta^{d-1}\,d\beta \\
&= \frac{d}{\rho^d}\int_0^\rho D_\beta H(x)\beta^{d-1}\,d\beta.
\end{aligned}
$$

For any $\eta > 0$ and $x \in B(p,r)$, by (3.11) there exist two sequences $\rho_{i,n} = \rho_{i,x,\eta,n} \downarrow 0$ (with $\rho_{i,n} < r - |x-p|$), $i = 1,2$, such that for all $n \geq 1$,

$$A_{\rho_{1,n}}S(x) - S(x) \geq -\eta\rho_{1,n}^2 \text{ and } A_{\rho_{2,n}}f_2(x) - f_2(x) \leq c_x\rho_{2,n}^2,$$

where c_x is a positive constant independent of $\rho_{2,n}$. Thus, by (5.1), the above inequalities imply for all $n \geq 1$,

$$
\begin{aligned}
(5.2)\qquad \int_0^{\rho_{1,n}}[D_\beta S(x) - S(x) + a\eta\beta^2]\beta^{d-1}\,d\beta &\geq 0 \\
\int_0^{\rho_{2,n}}[D_\beta f_2(x) - f_2(x) - ac_x\beta^2]\beta^{d-1}\,d\beta &\leq 0
\end{aligned}
$$

where $a = \frac{d+2}{d}$. So there exist $\beta_n = \beta_{x,\eta,n} \leq \rho_{1,n}, \beta_n \downarrow 0$, and $r_n = r_{x,\eta,n} \leq \rho_{2,n}, r_n \downarrow 0$, such that

$$(5.3)\qquad D_{\beta_n}S(x) - S(x) \geq -a\eta\beta_n^2 \text{ and } D_{r_n}f_2(x) - f_2(x) \leq ac_xr_n^2.$$

Let $B(q,\rho_1) \subset B(p,r)$. We show, for $y \in B(q,\rho_1) \setminus \overline{W}$, that

$$
\begin{aligned}
(5.4)\qquad S(y) &\leq \int_{\partial B(q,\rho_1)} S(z)\omega(B(q,\rho_1),dz,y) \\
&= \frac{1}{\sigma_d\rho_1}\int_{\partial B(q,\rho_1)}\frac{\rho_1^2 - |q-y|^2}{|z-y|^d}S(z)\,d\sigma(z).
\end{aligned}
$$

If (5.4) holds, then for $q \in W$, by (5.3), there exists a decreasing sequence r_n of positive numbers going to 0 such that for each n

$$(5.5)\qquad f_2(q) - D_{r_n}f_2(q) \geq -ac_qr_n^2.$$

For any given $\epsilon > 0$, using upper semi-continuity of $\overline{U}$ at q, we have for large n,

$$\overline{U}(q) \geq \sup_{|y-q|\leq r_n}\overline{U}(y) - \epsilon.$$

Thus, for large n,

$$\overline{U}(q) \geq D_{r_n}\overline{U}(q) - \epsilon.$$

Consequently

$$(5.6)\qquad S(q) \geq D_{r_n}S(q) - ac_qr_n^2 - \epsilon.$$

Note that for each $r > 0$, by the mean value theorem, there exists a constant c such that

$$\left| \frac{1}{r^{d-2}} - \frac{r^2 - |q-y|^2}{|z-y|^d} \right| \le c \frac{|q-y|}{r^{d-1}},$$

if $|y-q| < \frac{1}{2}|z-q| = \frac{1}{2}r$. Therefore, for $|y-q| < \frac{1}{2}r$,

$$\begin{aligned} &\left| \frac{1}{\sigma_d r} \int_{\partial B(q,r)} \frac{r^2 - |q-y|^2}{|z-y|^d} S(z)\, d\sigma(z) - D_r S(q) \right| \\ &\le \frac{1}{\sigma_d r} \int_{\partial B(q,r)} \left| \frac{1}{r^{d-2}} - \frac{r^2 - |q-y|^2}{|z-y|^d} \right| |S(z)| d\sigma(z) \\ &\le c \frac{|q-y|}{r} D_r |S(q)|. \end{aligned} \tag{5.7}$$

Combining (5.4)–(5.7), we have for any given ϵ, for n large,

$$S(y) - S(q) \le a c_q r_n^2 + c \frac{|q-y|}{r_n} D_{r_n} |S(q)| + \epsilon,$$

if $|y-q| \le \frac{1}{2} r_n$ and $y \in B(q, r_n) \setminus \overline{W}$. Letting $y \to q$, then $n \to \infty$, and then $\epsilon \to 0$, we have

$$\limsup_{\substack{y \to q \\ y \in B(q,\rho_1) \setminus \overline{W}}} S(y) \le S(q).$$

Thus, S is upper semi-continuous at q since S is upper semi-continuous when restricted to $B(p,r) \cap \overline{W}$. Consequently, W must be the empty set. So S is upper semi-continuous in $B(p,r)$. Inequality (5.4) also implies that $S(q) \le A_\rho S(q)$ for all $B(q,\rho) \subset B(p,r)$. Thus S is subharmonic in $B(p,r)$ since it is also in L^1.

It only remains to prove (5.4). Let $\{X_t\}_{t \ge 0}$ be the standard Brownian motion starting from a fixed point $y \in B(q,\rho_1) \setminus \overline{W}$ in the probability space $(\Omega, \mathcal{F}, P^y)$. Define

$$T = \inf\{t \ge 0 : X_t \in \partial B(q, \rho_1)\}$$

to be the exit time of X_t from $B(q,\rho_1)$. Then by (4.1), inequality (5.4) is equivalent to

$$S(y) \le E^y[S(X_T)]. \tag{5.8}$$

We first show that for any stopping time $S \le T$,

$$P^y(X_S \in W) = 0. \tag{5.9}$$

This is implied by

$$P^y(X_S \in \overline{W}_\varepsilon) = 0 \tag{5.10}$$

as $W \subset \bigcup_{\varepsilon>0} W_\varepsilon$. Let R be the hitting time of X_t with $\partial(B(q,\rho_1)\setminus \overline{W}_\varepsilon)$:

$$R = \inf\{t \geq 0 : X_t \in \partial(B(q,\rho_1)\setminus \overline{W}_\varepsilon)\}.$$

Then $R \leq T$. Since $y \in B(q,\rho_1)\setminus \overline{W} \subset B(q,\rho_1)\setminus \overline{W}_\varepsilon$ and, by assumption,

$$\begin{aligned} 0 &= \omega(B(q,\rho_1)\setminus \overline{W}_\varepsilon, \partial(B(q,\rho_1)\cap \overline{W}_\varepsilon), y) \\ &= P^y(X_R \in \partial(B(q,\rho_1)\cap \overline{W}_\varepsilon)) \\ &= P^y(X_R \in \overline{W}_\varepsilon), \end{aligned} \tag{5.11}$$

we see that

$$P^y(X_R \in \overline{W}_\varepsilon) = 0. \tag{5.12}$$

Next, by definition,

$$\{R < T\} \subset \{X_R \in \partial W_\varepsilon\}.$$

So

$$P^y(R<T) \leq P^y(X_R \in \partial W_\varepsilon) = 0. \tag{5.13}$$

Thus, by (5.11) and (5.13) we have

$$\begin{aligned} P^y(X_T \in \overline{W}_\varepsilon) &= P^y(X_T \in \overline{W}_\varepsilon, R=T) + P^y(X_T \in \overline{W}_\varepsilon, R<T) \\ &\leq P^y(X_R \in \overline{W}_\varepsilon) + P^y(R<T) \\ &= 0. \end{aligned} \tag{5.14}$$

To show (5.10) for a general stopping time S, note that for any $\tau > 0$, there exists an open set G such that $\overline{W_\varepsilon} \subset G$ and

$$\omega(B(q,\rho_1)\setminus \overline{G}, \partial(B(q,\rho_1)\cap G), y) < \tau. \tag{5.15}$$

Define a function u on $\overline{B}(q,\rho_1)$ as follows:

$$u(x) = \begin{cases} \omega(B(q,\rho_1)\setminus \overline{G}, \partial(B(q,\rho_1)\cap G), x) & \text{on } x \in B(q,\rho_1)\setminus \overline{G}, \\ 1 & \text{on } \overline{B(q,\rho_1)\cap G}, \\ 0 & \text{on } \partial B(q,\rho_1)\setminus \partial(B(q,\rho_1)\cap G). \end{cases}$$

Then u is superharmonic on $B(q,\rho_1)$. Let $\tilde{r}_n$ be an increasing sequence going up to ρ_1 and $y \in B(q,\tilde{r}_1)$. Denote T_n to be the exit time of X_t from $B(q,\tilde{r}_n)$. Clearly T_n is increasing and convergent to T. Since Brownian motion is continuous, we have

$$\{X_S \in \overline{W}_\varepsilon, S<T\} \subset \bigcup_{n\geq 1}\{X_{S\wedge T_n} \in \overline{W}_\varepsilon, S<T\},$$

where $S\wedge T_n = \min\{S, T_n\}$. So (5.10) is implied by the following:

$$P^y(X_{S\wedge T_n} \in \overline{W}_\varepsilon) = 0, \text{ for each } n, \tag{5.16}$$

since by (5.14)

$$\begin{aligned} P^y(X_S \in \overline{W}_\varepsilon) &= P^y(X_S \in \overline{W}_\varepsilon, S = T) + P^y(X_S \in \overline{W}_\varepsilon, S < T) \\ &\leq P^y(X_T \in \overline{W}_\varepsilon) + P^y(X_S \in \overline{W}_\varepsilon, S < T) \\ &\leq \lim_{n\to\infty} P^y(X_{S\wedge T_n} \in \overline{W}_\varepsilon, S < T). \end{aligned}$$

For a superharmonic function u and for each $n \geq 1$, there exists a sequence of increasing superharmonic functions $\{u_j\}$ such that $u_j \in C^2$ and

$$\lim_{j\to\infty} u_j = u \text{ on } \overline{B}(q, \tilde{r}_n) \tag{5.17}$$

(see, for example, Theorem 4.20 of [H]). Applying Itô's formula to $u_j(X_{S\wedge T_n})$, we have

$$E^y[u_j(X_{S\wedge T_n})] \leq u_j(y).$$

Let j go to infinity and apply (5.17) to see that

$$E^y[u(X_{S\wedge T_n})] \leq u(y).$$

Consequently

$$\tau \geq u(y) \geq E^y[u(X_{S\wedge T_n})] \geq P^y(X_{S\wedge T_n} \in \overline{B(q, \rho_1) \cap G}) \geq P^y(X_{S\wedge T_n} \in \overline{W}_\varepsilon).$$

Letting $\tau \to 0$ proves (5.16).

As a consequence of (5.9), since S is upper semi-continuous on $\overline{B}(p, r)\backslash W$, we have almost everywhere with respect to the probability measure P^y

$$\limsup_{n\to\infty} S(X_{S_n}) \leq S(X_{S_\infty}) \text{ if } S_n \uparrow S_\infty \leq T. \tag{5.18}$$

Let $\eta > 0$. Then by (5.3), for any $y \in B(q, \rho_1)$, there exists $0 < \beta = \beta_{y,\eta} < \rho_1 - |y - q|$, such that

$$S(y) - D_\beta S(y) \leq a\eta\beta^2. \tag{5.19}$$

Consider a family of stopping times

$$\mathfrak{S} = \{S \leq T : S(y) - E^y S(X_S) \leq a\eta E^y |y - X_S|^2\}.$$

Define

$$S_0 = \inf\{t \geq 0 : |X_t - y| \geq \beta_{y,\eta}\}.$$

Then X_{S_0} is uniformly distributed on $\partial B(y, \beta_{y,\eta})$. So by (5.19), we have

$$S(y) - E^y S(X_{S_0}) \leq a\eta E^y |y - X_{S_0}|^2.$$

Thus $S_0 \in \mathfrak{S}$ and hence $\mathfrak{S}$ is not empty.

For a sequence of increasing stopping times S_n in $\mathfrak{S}$, let $S_\infty = \lim_{n\geq 1} S_n$. Then by (5.18) and Fatou's lemma, we have

$$S(y) - E^y S(X_{S_\infty}) \leq a\eta E^y |y - X_{S_\infty}|^2.$$

So $S_\infty \in \mathfrak{S}$. Thus by an argument given in Halmoe [Ha] on page 121[1], there exists $S^* \in \mathfrak{S}$ such that S^* is a maximum of $\mathfrak{S}$. We show that $S^* = T$ almost everywhere with respect to P^y. In fact, if $S^* < T$ with positive P^y probability, then

$$S_1^* = \inf\{T \geq t \geq S^* : |X_t - X_{S^*}| \geq \beta_{X_{S^*},\eta}\},$$

where for $x \in \partial B(q, \rho_1)$, we define $\beta_{x,\eta}$ to be 0. Then, clearly, $S_1^* \geq S^*$ with strict inequality on $\{S^* < T\}$. On the other hand, conditional on X_{S^*}, $X_{S_1^*}$ is uniformly distributed on the surface of $B(X_{S^*}, \beta_{X_{S^*},\eta})$, if $S^* < S_1^*$. So by (5.3)

$$\begin{aligned} S(X_{S^*}) - E^{X_{S^*}} S(X_{S_1^*}) &= S(X_{S^*}) - D_{\beta_{X_{S^*},\eta}} S(X_{S^*}) \\ &\leq a\eta |\beta_{X_{S^*},\eta}|^2 \\ &= a\eta E^{X_{S^*}} |X_{S^*} - X_{S_1^*}|^2. \end{aligned} \tag{5.20}$$

Hence, by (5.20), the strong Markovian property, orthogonality between $X_{S_1^*} - X_{S^*}$ and $X_{S^*} - y$, and $S^* \in \mathfrak{S}$, we have

$$\begin{aligned} S(y) - E^y S(X_{S_1^*}) &= S(y) - E^y S(X_{S^*}) + E^y[S(X_{S^*}) - S(X_{S_1^*})] \\ &\leq a\eta E^y |y - X_{S^*}|^2 + E^y[S(X_{S^*}) - S(X_{S_1^*}), S^* < S_1^*] \\ &= a\eta E^y |y - X_{S^*}|^2 + E^y[S(X_{S^*}) - E^{X_{S^*}} S(X_{S_1^*}), S^* < S_1^*] \\ &\leq a\eta E^y |y - X_{S^*}|^2 + a\eta E^y[E^{X_{S^*}} |X_{S^*} - X_{S_1^*}|^2, S^* < S_1^*] \\ &= a\eta E^y |y - X_{S^*}|^2 + a\eta E^y |X_{S^*} - X_{S_1^*}|^2 \\ &= a\eta E^y |y - X_{S_1^*}|^2. \end{aligned}$$

Thus $S_1^* \in \mathfrak{S}$. This contradicts the maximality of S^* in $\mathfrak{S}$ since $S_1^* \geq S^*$ and $S_1^* \neq S^*$. Thus we have shown that $T \in \mathfrak{S}$. So

$$\begin{aligned} S(y) - \int_{\partial B(q,\rho_1)} S(z)\, \omega(B(q, \rho_1), dz, y) &= S(y) - E^y S(X_T) \\ &\leq a\eta E^y |X_T - y|^2 \leq 4a\eta\rho_1^2. \end{aligned}$$

This implies (5.4) by letting $\eta \to 0$. We have finished the proof.

6. Proof of Theorem 1.8

Without loss of generality, we assume that $q = 0$, the origin. As in the proof of Theorem 1.4, we have that $\tilde{\Delta}^* f_2(x) = \tilde{\Delta}_* f_2(x)$ almost everywhere in $\mathbb{T}^d \setminus \{0\}$, and that $\tilde{\Delta}^* f_2(x)$ is in $L^1(\mathbb{T}^d \setminus B(0, r))$ for any $r > 0$. Consequently, $\tilde{\Delta}^* f_2(x) = \tilde{\Delta}_* f_2(x)$ almost everywhere in $\mathbb{T}^d$.

[1] See [A] for the details.

As in Section 2, let $B(x) = \min\{f^*(x), \tilde{\Delta}^* f_2(x)\}$. Then by Lemma 2.1, on $\mathbb{T}^d \setminus \{0\}$, $f_*(x) \le B(x) \le f^*(x)$ and $\tilde{\Delta}_* f_2(x) \le B(x) \le \tilde{\Delta}^* f_2(x)$. Thus, $B(x) \in L^1(\mathbb{T}^d)$. Consequently, when we proceed as in Section 2, there exists a function $S^*(x)$, which is harmonic on $\mathbb{R}^d \setminus M$ and almost everywhere equals

$$S(x) = f_2(x) + (2\pi)^{-d} \int_{T^d} G(x-y)B(y)\,dy - b_0|x|^2/(2d),$$

where $M = \{2\mu\pi : \mu \in \mathbb{Z}^d\}$. The rest of the proof is identical to that of Theorem 2 of Shapiro ([Sh, p.479]).

Acknowledgment. We are grateful to Jean Bourgain, Robert Kaufman, and Victor Shapiro for helpful conversations.

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(Received March 17, 1997)

(Revised July 21, 1999)