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On Lifting Prime Ideals

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It is one important consequence of classical elimination theory that an irreducible polynomial $p(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ in several variables $X_1, \ldots, X_n$ over an algebraically closed field K remains irreducible over every field extension L of K. Later, it was discovered that the same result is also a consequence of the valuation extension theorem, that is, of the fact that for any valuation ring R in a field F and for any field extension L of F there exists a valuation ring S in L with $S \cap F = R$. Indeed, if $T \subseteq L$ is a transcendence base of L over K, if $F := K(T)$, if

$$R = \{\frac{g}{f} \mid g, f \in K[T], f(0, \ldots, 0) \neq 0\},$$

and if S is a valuation ring in L with $S \cap F = R$, then the residue class field $\overline{S}$ of S must be an algebraic extension of the residue class field $\overline{R}$ of R and, since $\overline{R} = K$ and K was algebraically closed, $\overline{S}$ must also coincide with K. Since moreover every factorization $p = p_1 \cdot p_2$ of p in $L[X_1, \ldots, X_n]$ could be readjusted to yield a factorization in $S[X_1, \ldots, X_n]$, it would also lead to a factorization over $\overline{S}[X_1, \ldots, X_n] = K[X_1, \ldots, X_n]$, so it is impossible.

Combined with the fact that C. Chevalley provided an extremely simple proof for the valuation extension theorem (which will be recalled below) this observation led to the gradual elimination of elimination theory in the second volume of van der Waerdens "Algebra", regretted by many nowadays. Yet, in spite of the well justified resurgence of elimination theory in the last decade, the valuation theoretic approach to the above irreducibility result is still the fastest known today. It may therefore be of some interest that a slight modification of Chevalley's trick yields also the famous Cohen-Seidenberg prime ideal extension theorem for integral ring extensions, a basic result, used over and over again in commutative algebra and number theory.

The following lemma and its proof present this slight modification:

Lemma: *Let A be a subring of a commutative ring B with $1 \in B$ and let $\mathbf{p}$ denote a given, fixed prime ideal in A. Then for all $x, y \in B$ with $a := x \cdot y \in A$ there exists either a prime*

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ideal $\mathbf{p}_x \subseteq A[x]$ *with* $\mathbf{p}_x \cap A = \mathbf{p}$ *or there exists a prime ideal* $\mathbf{p}_y \subseteq A[y]$ *with* $\mathbf{p}_y \cap A = \mathbf{p}$ *(or both).*

Proof: Otherwise, the multiplicatively closed set $A \setminus \mathbf{p}$ intersects

$$\mathbf{p}[x] := \{\sum_{i=0}^{n} p_i x^i \mid n \in \mathbf{N}_0; p_i \in \mathbf{p} \text{ for } i = 0, 1, \ldots, n\}$$

as well as

$$\mathbf{p}[y] := \{\sum_{j=0}^{m} q_j y^j \mid m \in \mathbf{N}_0; q_j \in \mathbf{p} \text{ for } j = 0, 1, \ldots, m\}.$$

Choose $\sum_{i=0}^{n} p_i x^i =: s \in \mathbf{p}[x] \cap A \setminus \mathbf{p}$ and $\sum_{j=0}^{m} q_j y^j =: t \in \mathbf{p}[y] \cap A \setminus \mathbf{p}$ so that n and m are minimal and assume without loss of generality $p_0 = q_0 = 0$ and $m \leq n$. Then

$$\begin{aligned} A \setminus \mathbf{p} \ni t \cdot s &= \sum_{i=1}^{n-1} (t \cdot p_i) x^i + t \cdot p_n x^n = \\ &= \sum_{i=1}^{n-1} (t \cdot p_i) x^i + \sum_{j=1}^{m} q_j y^j \cdot p_n x^n = \\ &= \sum_{i=1}^{n-1} (t \cdot p_i) x^i + \sum_{j=1}^{m} (a^j \cdot q_j \cdot p_n) x^{n-j} = \\ &= \sum_{i=1}^{n-1} p_i' x^i \end{aligned}$$

with

$$p_i' := t p_i + a^{n-i} q_{n-i} p_n \in \mathbf{p}$$

if $n - m \leq i \leq n - 1$ and

$$p_i' := t p_i \in \mathbf{p}$$

if $1 \leq i < n - m$, a contradiction. □

Remark: This proof is an almost exact copy of Chevalley's trick except that he used it only in case $x \cdot y = 1$ and did not consider the more general situation $x \cdot y \in A$.

The remarkable fact we want to point out in this note is that this slight modification immediately yields not only the valuation extension theorem (which, of course, is no surprise since Chevalley introduced his trick precisely for this purpose), but it also yields the Cohen-Seidenberg prime ideal extension theorem or, more precisely, it implies a common generalization of both, that is, the following

Theorem: *For A, B and $\mathbf{p}$ as above, consider a maximal subring A' of B, containing A and containing a prime ideal $\mathbf{p}'$ with $A \cap \mathbf{p}' = \mathbf{p}$ (which exists due to Zorn's Lemma). Then $x, y \in B$ and $x \cdot y \in A'$ implies $x \in A'$ or $y \in A'$, in particular A' is integrally closed in B.*

Proof: It is clear that $x, y \in B$ and $x \cdot y \in A'$ implies $x \in A'$ or $y \in A'$ in view of the above lemma and the maximality of A'. Now assume $b \in B$ is integral over A', that is, $b^n + a_1\, b^{n-1} + \ldots + a_n = 0$ for some $n \in \mathbf{N}$ and $a_1, \ldots, a_n \in A'$. Assume furthermore that n has been chosen to be minimal so that $a_1, \ldots, a_n \in A'$ with the above property exist. Since

$$b(b^{n-1} + a_1\, b^{n-2} + \ldots + a_{n-1}) = -a_n \in A'$$

and since $b^{n-1} + a\, b^{n-2} + \ldots + a_{n-1} \notin A'$ by the minimality of n, we get $b \in A'$. So A' is indeed integrally closed in B. $\square$

If B is a field, A' is necessarily a valuation subring of B, so the above theorem implies indeed the valuation extension theorem. It implies also that A' must be equal to B, if B is an integral extension of A, that is, it implies simultaneously the Cohen-Seidenberg prime ideal extension theorem.