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Still another Proof of the Existence of Sylow–p–Subgroups

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The classical inductive proof of the existence of Sylow–p–subgroups became obsolete when H. Wielandt published his ingenious non–inductive proof in 1958 (cf. [4], see also [3] for a more structural interpretation of Wielandt's approach). In this note we want to point out still another simple, though inductive proof for the existence of Sylow–p–subgroups which is based on the essentially trivial observation that for every non–trivial subgroup $1 \neq U \leq G$ of a finite group G the set $X(U) := \{f : G/U \to G \mid f(gU) \in gU \text{ for all } g \in G\}$ of all systems of coset representatives of U in G is a fixed point free G–set of cardinality $|X(U)| = |U|^{(G:U)}$ and hence, by orbit decomposition, there exist proper subgroups $V_i < G$ of $G\,(i = 1, \ldots, n_U)$ with

$$|U|^{(G:U)} = \sum_i (G : V_i).$$

Indeed, assume that there exists some finite group G and some prime p for which no subgroup $U \leq G$ of order $|G|_p := \max(p^\alpha \mid p^\alpha \text{ divides } |G|)$ exists and assume that among all such pairs the pair (G, p) is chosen so that G has minimal order.

Then this order can't be a prime power, in particular it is not a prime and so G has at least one proper, non–trivial subgroup $1 \neq U < G$. Since $|U| < |G|$, there exists a subgroup $V \leq U$ with $|V| = |U|_p$, hence $|U|_p \neq |G|_p$, that is p divides $(G : U)$ for all proper subgroups U. Therefore, by the above observation, p divides $|U|$ for every non–trivial subgroup $1 < U < G$, so a proper subgroup U of G cannot contain a non–trivial Sylow–q–subgroup for every prime $q \neq p$ and hence, in view of the minimality of $|G|$, we must have $|U|_q = 1$ for every prime $q \neq p$, that is, every proper subgroup of G must be a p–group. But this implies that $|G|_{p'} := |G|/|G|_p$ divides $(G : V)$ for every proper subgroup $V < G$ of G and so, using the above observation once more, it must divide the p–power $|U|^{(G:U)}$ for every proper, non–trivial subgroup $1 \neq U < G$, a final contradiction.

The argument, we have actually used in this proof, can be put in the following more abstract form:

Lemma: *Assume that for some $n \in \mathbf{N}$ a subset P of the set*

$$P(n) := \{d \in \mathbf{N} \mid d \text{ divides } n\}$$

of divisors of n *is given such that the following holds:*

- $1, n \in P$,
- *for every* $d \in P$ *which is not a prime there exists a non–empty subset*

$$P_d \subseteq P \cap P(d) \setminus \{1, d\}$$

such that for every $t \in P_d$ *the power* $t^{d/t}$ *can be written in the form*

$$(*) \qquad t^{d/t} = \sum_i d/t_i$$

for appropriate $t_1, t_2, \ldots \in P_d$.

Then for every $d \in P$ *and every prime* p *the* p*-part* d_p *of* d *is also in* P.

Proof: Otherwise, there exists a minimal $d \in P$ for which there exists some prime p with $d_p \notin P$, in particular $1 \neq d_p \neq d$. It follows that P_d is non–empty and that $t_p \in P$ for every $t \in P_d$. Hence, p must divide the quotient d/t for every $t \in P_d$ and therefore, in view of $(*)$, it must divide every $t \in P_d$. So, using the minimality of d once more, this time with respect to primes $q \neq p$, every $t \in P_d$ must be a p-power, so $1 \neq d_{p'} := d/d_p$ must divide every quotient $d/t\,(t \in P_d)$ and therefore, using $(*)$ once more, it must divide the numbers $t^{d/t}$ which we have just recognized as p-powers, — the final contradiction. □

Since the above observation ensures that for every finite group G of order n the set $P_G := \{|U| \mid U \leq G\}$ satisfies the condition of the Lemma (for $d \in P_G$ define $P_d := \bigcup_{U \leq G, |U|=d} P_U \setminus \{1, d\}$), the existence of Sylow-p-subgroups can now be derived as its corollary.

From a (permutation–)group theoretic point of view it might be more interesting to study the orbit structure of $X(U)$ more explicitly, noting that it plays a crucial rôle in multiplicative transfer (cf. for example [1, Theorem 5, p. 222]) and related considerations (cf. [2, §5]).

References

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