Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 35 (1994), No. 1, 125-129.

A Quantitative Steinitz' Theorem

Shmuel Onn[1] and **Bernd Sturmfels**[2]
Department of Operations Research, Technion
Haifa, 32000, Israel
Department of Mathematics, Cornell University
Ithaca, NY 14853, U.S.A.

Abstract. Any 3-dimensional convex polytope with n vertices can be realized in Euclidean 3-space with all coordinates of all vertices being integers of absolute value not exceeding n^{169n^3}.

1. Introduction

The following classical theorem of Ernst Steinitz [6] gives a characterization of combinatorial types of 3-dimensional convex polytopes (or, 3-polytopes, for short) in terms of their edge graphs.

Theorem 1.1 (Steinitz' Theorem). *A graph is isomorphic to the edge graph of a 3-polytope if and only if it is 3-connected and planar.*

The hard part of this theorem is the "if" direction; the usual proofs proceed by constructing the 3-polytope from its given graph. It is known (see [2] or [9]) that these constructions can be carried out over the field of *rational* numbers. This implies that all 3-polytopes admit an embedding with all coordinates of all vertices being integers. The following question then arises. Let $f(n)$ be the smallest integer such that any 3-polytope with n vertices can be embedded with integer coordinates of absolute value not exceeding $f(n)$. How fast does $f(n)$ grow with n? From Steinitz' proof one derives an upper bound on $f(n)$ which is doubly exponential in n. The purpose of this note is to provide the following *singly exponential* upper bound on $f(n)$.

Theorem 1.2. *Any 3-polytope with n vertices can be realized in Euclidean 3-space with all coordinates of all vertices being integers of absolute value not exceeding n^{169n^3}.*

[1] Supported in part by the Alexander von Humboldt Stiftung.
[2] Supported in part by the National Science Foundation and the David and Lucile Packard Foundation.

This is somewhat surprising: if a realization is required for all *order types* of point configurations rather than for *convex* polytopes, then a doubly exponential *lower bound* is available even in the plane [1]. We remark that no characterization of the combinatorial structure of polytopes in dimensions higher than 3 is known. Also, starting in dimension 8, there are polytopes which admit no rational embedding (see [9]), in which case the quantitative analysis would be restricted to the class of rational polytopes. An important question left open here is whether the rate of growth of $f(n)$ is indeed singly exponential (in which case a matching lower bound is desirable), or whether $f(n)$ can in fact be bounded from above by a polynomial in n.

2. Preparation

Let $G = (V, E)$ be a planar, 3-connected graph. Thus, V is the set of vertices of G and E is its set of edges; we shall use the short notation uv for an edge $\{u, v\} \in E$, where $u, v \in V$. A theorem of Whitney [8] asserts that the embedding of G into the plane (or the 2-sphere) is topologically unique. Thus, the set of *faces* of G is well defined: a face is a subset of V which induces a cycle bounding a face in the unique planar embedding of G. In particular, the combinatorial type of a 3-polytope P is determined by its edge graph G. We shall also make use of the *planar dual* G^* of G, which is again 3-connected and planar. The vertex set of G^* is the set of faces of G and uv is an edge of G^* whenever u and v are adjacent faces in the planar embedding of G. Planar duality is an involution, so that $(G^*)^* = G$.

For the proof of our result we need the following lemma from linear algebra. It can be deduced easily from Cramer's rule, Caratheodory's theorem [2] and Hadamard's bound [5].

Lemma 2.1. *Consider a system of linear equations and inequalities in n indeterminates*

$$\sum_{k=1}^{n} a_{i,k} x_k = b_i \qquad (i \in I), \qquad \sum_{k=1}^{n} a_{j,k} x_k \geq b_j \qquad (j \in J),$$

where all $a_{r,s}$ and b_r are integers of absolute value $\leq M$. If this system is solvable, then it has a solution $x_1 = \frac{p_1}{D}, \cdots, x_n = \frac{p_n}{D}$ with $D, p_1, \cdots, p_n$ integers of absolute value at most $\leq n^{n/2} M^n$.

We shall base our argument on the following well-known connection between Steinitz' theorem and rigidity theory.

Theorem 2.2 (Maxwell's Theorem). *Let $G = (V, E)$ be a 3-connected planar graph containing a triangular face $F = \{v_1, v_2, v_3\}$, and let $f : F \longrightarrow \mathbb{R}^2$ be an embedding of F as a triangle in the plane. Let λ be a positive stress on G, i.e., an assignment of a positive real number $\lambda(e)$ to each edge e. Then the following hold:*

1. *The map f can be uniquely extended to an embedding $f : V \longrightarrow \mathbb{R}^2$ of the graph G, which is an equilibrium with respect to the stress λ, i.e., an embedding satisfying*

$$\sum_{\{u:\ uv \in E\}} \lambda(uv)(f(u) - f(v)) = 0 \qquad \text{for all } v \in V \setminus F.$$

2. *Any equilibrium f with respect to a positive stress on G can be factored as $f = \pi \circ g$, where $g : V \longrightarrow \mathbb{R}^3$ is an embedding such that the 3-polytope* $\mathrm{conv}\,(g(V))$ *has the edge graph G, and π denotes the orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : \ (x, y, z) \mapsto (x, y)$.*

The origins of Theorem 2.2 can be traced back to Maxwell [4]. The first complete proof was published by Whiteley in [7]. Another self-contained proof and computational information on stresses and equilibria can be found in [3]; the first part, as stated above, follows from assertion 2.1.3 in this reference, and the second part from assertions 2.1.4, 2.3.1, 3.3.5 and 3.6.1 in [3].

3. A singly exponential upper bound

Lemma 3.1 *For any 3-connected, planar graph $G = (V, E)$ containing a triangular face $F = \{v_1, v_2, v_3\}$, there exists an embedding $g : V \to \mathbb{R}^3$ such that G is the edge graph of* $\mathrm{conv}\,(g(V))$ *and the coordinates of each vertex $g(v)$ are integers of absolute value $\leq n^{7n^2}$, where n is the number of vertices of V.*

Proof. Given $G = (V, E)$ as above, we embed the triangle $F = \{v_1, v_2, v_3\}$ by putting

$$f'(v_1) = (x_1, y_1) = (0, 0), \qquad f'(v_2) = (x_2, y_2) = (1, 0), \qquad f'(v_3) = (x_3, y_3) = (0, 1),$$

and we consider the stress given by $\lambda(e) = 1$ for all edges e. Write $V = \{v_1, \cdots, v_n\}$ and $f'(v_i) = (x_i, y_i)$ with x_i, y_i indeterminates for $i \geq 4$. The first assertion of Theorem 2.2 guarantees the existence of an equilibrium f' with respect to λ, i.e., a solution to the following system of linear equations in these $2(n-3)$ indeterminates:

$$\sum_{\{v_j :\ v_j v_i \in E\}} (x_j - x_i) = 0, \qquad \sum_{\{v_j :\ v_j v_i \in E\}} (y_j - y_i) = 0 \qquad\qquad (i = 4, \cdots, n).$$

Each coefficient in this linear system is an integer of absolute value at most n. Lemma 2.1 implies that there is a solution $x_4 = \frac{p_4}{D}, \cdots, x_n = \frac{p_n}{D}, y_4 = \frac{q_4}{D}, \cdots, y_n = \frac{q_n}{D}$ with $D, p_4, \cdots, p_n, q_4, \cdots, q_n$ integers of absolute value less than $2^n n^{3n}$. But then $f = D \cdot f'$ is an equilibrium with respect to the same stress λ, which has all coordinates integral and of absolute value $\leq 2^n n^{3n}$.

We now proceed to find a map $g : V \longrightarrow \mathbb{R}^3$, the existence of which is guaranteed by the second assertion of Theorem 2.2, such that $f = \pi \circ g$ and $\mathrm{conv}(g(V))$ has the edge graph G. Write $g(v_i) = (x_i, y_i, z_i)$, where $(x_i, y_i) = f(v_i)$ and z_i is an indeterminate.

Given any four vertices $v_i, v_j, v_k, v_l \in V$, we shall use the bracket notation

$$[v_i, v_j, v_k, v_l] \quad := \quad det \begin{bmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

for the corresponding determinant. Note that $[v_i, v_j, v_k, v_l]$ is a linear form in z_i, z_j, z_k, z_l.

Let $F_1, \cdots, F_m$ be all the faces of G. Fix three distinct vertices u_i, v_i, w_i in each face F_i. Suppose that $g : V \longrightarrow \mathbb{R}^3$ is an embedding such that $f = \pi \circ g$ and $P = \mathrm{conv}(g(V))$ has the edge graph G. Consider a face F_i. For any vertex v, the sign $sign[u_i, v_i, w_i, v] \in \{-1, 0, +1\}$ is 0 if $g(v)$ lies on the affine plane H_i in $\mathbb{R}^3$ spanned by $g(u_i), g(v_i), g(w_i)$, is -1 if v lies strictly on one side of H_i, and is $+1$ if v lies strictly on the other side of H_i. But H_i is the affine plane supporting P at its face F_i, so all vertices not in F_i lie on the same side of H_i. Therefore $sign[u_i, v_i, w_i, v] = 0$ if $v \in F_i$, and $sign[u_i, v_i, w_i, v] = \epsilon_i$ if $v \notin F_i$, where ϵ_i is either -1 or $+1$. Moreover, multiplying $z_1, \cdots, z_n$ by a large positive number if necessary, we may assume that the following inequality holds for all i and all $v \notin F_i$:

$$\epsilon_i \cdot [u_i, v_i, w_i, v] \geq 1.$$

We conclude that the desired embedding corresponds to any solution of the following system of linear equations and inequalities in the indeterminates $z_1, \cdots, z_n$:

$$[u_i, v_i, w_i, v] = 0 \quad (v \in F_i), \qquad \epsilon_i \cdot [u_i, v_i, w_i, v] \geq 1 \quad (v \notin F_i) \qquad (i = 1, \cdots, m).$$

By the construction of f, the x, y coordinates of all vertices are integers of absolute value not exceeding $2^n n^{3n}$, and so each coefficient in this system, which is a 3×3 minor of some bracket $[u_i, v_i, w_i, v]$, has absolute value not exceeding

$$3!\,(2^n n^{3n})^2 \;=\; 6 \cdot 4^n n^{6n}.$$

Letting $z_1, \cdots, z_n$ be a solution to this system of bounded size as guaranteed by Lemma 2.1, and multiplying $z_1, \cdots, z_n$ by their common denominator if necessary, we obtain a solution yielding an embedding g of V with all coordinates integers of absolute value less than n^{7n^2}. □

Proof of Theorem 1.2. For any planar, 3-connected graph $G = (V, E)$ we must construct an embedding $g : V \to \mathbb{R}^3$ such that G is the edge graph of $P = \mathrm{conv}(g(V))$ and the vertices of P have integral coordinates of the asserted size. Let n be the number of vertices and m the number of faces of G. It is known (see [2], p. 236) that G contains either a triangular face or a trivalent vertex. In the former case, Theorem 1.2 follows from Lemma 3.1. Otherwise G has a trivalent vertex, so its planar dual G^* has a triangular face; by Lemma 3.1, the polar polytope P^* with graph G^* has an embedding g^* with integer coordinates of absolute value at most

$$m^{7m^2} \;<\; (2n)^{28n^2}.$$

Here we have used the easy inequality $m \leq 2n - 4$.

Translating P^* if necessary (and at most doubling the size of its coordinates), we may assume that P^* has the origin in its interior. We recover a polytope P with edge graph G as

$$P = \{v \in \mathbb{R}^3 : \; \langle g^*(u_i), v\rangle \leq 1 \qquad i = 1, \cdots, m\},$$

where $u_1, \cdots, u_m$ are the vertices of G^*. Each vertex v_i of the polytope P is obtained as the unique solution to a system of linear equations

$$\langle g^*(u_{i_1}), v_i\rangle = \langle g^*(u_{i_2}), v_i\rangle = \langle g^*(u_{i_3}), v_i\rangle = 1$$

for some $1 \leq i_1, i_2, i_3 \leq m$, and so, by Lemma 2.1, can be written as $v_i = (\frac{x_i}{D_i}, \frac{y_i}{D_i}, \frac{z_i}{D_i})$, with x_i, y_i, z_i, D_i integers of absolute value not exceeding

$$3!\,(2 \cdot (2n)^{28n^2})^3 \;<\; n^{169n^2}.$$

Multiplying this solution by $\Pi_{i=1}^{n} D_i$, we get the desired embedding of V with all coordinates of all vertices being integers of absolute value not exceeding $(n^{169n^2})^n \;=\; n^{169n^3}$. $\square$

Remark. The proofs of Lemma 3.1 and Theorem 1.2 are constructive and provide an algorithm that takes as an input either a rational embedding of a 3-polytope or a 3-connected planar graph G (whose faces can be listed by applying an appropriate planarity-testing algorithm to G), and outputs an integral embedding of the polytope which satisfies the bound in Theorem 1.2. Such an algorithm could be used as a preprocessing step to improve robustness of computational geometry algorithms that act on 3-polytopes.

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Received 25.3.94