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Reguli and Chains over Skew Fields

Andrea Blunck

Fachbereich Mathematik der TU Darmstadt
Schloßgartenstraße 7, D-64289 Darmstadt, Germany

Abstract. We give a geometric definition of a regulus in a not necessarily pappian projective space of arbitrary dimension, such that the geometry of all reguli is a certain chain geometry.

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The usual definitions of reguli ([5], [9]) and of chains ([1], [8]) only make sense over commutative fields: In a non-pappian projective space reguli do not exist. A chain geometry $\Sigma(K, R)$ is only defined if the field K is contained in the center of the ring R.

In this paper we generalize both notions to skew fields, i.e., not necessarily commutative fields.

In Section 1 we collect some preliminary results. Then, in Section 2, we define reguli in a geometric way that works also for non-pappian spaces, thus generalizing both the usual definition and a definition due to B. Segre for three-dimensional projective spaces. We derive some properties of these reguli, using mainly linear algebra.

We want to remark here that H. Havlicek in [6] introduces Segre manifolds over skew fields, and thus also obtains a notion of reguli. It is not clear so far whether his notion coincides with ours.

In Section 3 we introduce a chain geometry $\Sigma(K, R)$ associated to the set of all reguli in the not necessarily pappian projective space $\mathbb{P}(K, V)$. We know that $V \cong U \times U$ for some left vector space U over K. The ring R is the endomorphism ring of this vector space U. We prove that our reguli correspond to the chains of $\Sigma(K, R)$. Thus we generalize a theorem proved in [2] for the pappian case.

In Section 4 we use the description of the reguli as chains in order to investigate the intersections of all reguli through three given subspaces.

1. Preliminaries

The usual definition of a regulus in a projective space of arbitrary dimension is as follows (see [2], [5], [9]):

1.1. Definition. *Let $\mathbb{P}$ be a projective space. A set $\mathcal{R}$ of at least three pairwise complementary subspaces of $\mathbb{P}$ is called a regulus, if the following two conditions are satisfied:*

(R1) *Any line of $\mathbb{P}$, transversal to three elements of $\mathcal{R}$, is transversal to every element of $\mathcal{R}$.*

(R2) *Each point on a line transversal to $\mathcal{R}$ belongs to an element of $\mathcal{R}$.*

Here we call a line L *transversal* to a subspace W of $\mathbb{P}$, if L meets W in a unique point. Then we also say that W is transversal to L. Moreover, we call L transversal to a family of subspaces, if it is transversal to each member of the family. Analogously we use the formulation that a subspace W is transversal to a family of lines.

Note that a regulus can only exist if $\mathbb{P}$ is spanned by two isomorphic complementary subspaces. Hence $\mathbb{P} = \mathbb{P}(K, V)$ for some left vector space V over a not necessarily commutative field K, where $V \cong U \times U$ for another left vector space U over K. In the case that V is finite-dimensional this means that $\dim V$ must be even.

Moreover, apart from the trivial situation that $\dim V = 2$ (where the whole set of points of the projective line $\mathbb{P}(K, V)$ is a regulus), reguli exist only in pappian projective spaces, i.e., over commutative fields (see [5]):

1.2. Remark. Let $\dim V > 2$. Then the following statements hold:

(i) If K is not commutative, then there are no reguli in $\mathbb{P}(K, V)$.

(ii) If K is commutative, then any three pairwise complementary subspaces of $\mathbb{P}(K, V)$ are contained in exactly one regulus.

Let us consider the special case that V is four-dimensional over the commutative field K: Then the elements of a regulus in $\mathbb{P}(K, V)$ are lines. If G, H, L are pairwise complementary (i.e., pairwise skew) lines, then the unique regulus $\mathcal{R}$ containing them consists of all lines that meet the set $\mathcal{R}'$ of all lines transversal to G, H, L. The set $\mathcal{R}'$ is itself a regulus, called the regulus *opposite* to $\mathcal{R}$. In particular, any regulus in the three-dimensional pappian projective space $\mathbb{P}(K, V)$ can be described as the set of all lines that meet three fixed pairwise skew lines.

This observation was utilized by Segre in [10] in order to define reguli in such a way that they exist also in non-pappian spaces (see also [3]):

1.3. Definition. (B. Segre) *Let $\mathbb{P}$ be a three-dimensional projective space with line set $\mathcal{G}$. Let $T_0, T_1, T_2 \in \mathcal{G}$ be pairwise skew. Then the set*

$$\mathcal{R} = \{G \in \mathcal{G} \mid G \text{ is transversal to } T_0, T_1, T_2\}$$

is called a regulus.

Obviously the two definitions of a regulus coincide if the ground field K is commutative. Segre shows in [10] that only in the commutative case three pairwise skew lines belong to exactly one regulus. Otherwise there are many reguli through three given pairwise skew lines (compare Section 4 below).

In Section 2 we are going to generalize Segre's definition to higher dimensions. In Section 3 we shall show that our reguli are the chains of a chain geometry. This was already shown for the commutative case in [2], [7].

Before stating our definitions, we need some terminology. We recall the following notions and results from [2]:

Since reguli again shall consist of at least three pairwise complementary subspaces of $\mathbb{P}(K,V)$, we assume that $V = U \times U$ for some left vector space U.

We shall always identify a subspace of the vector space V with the associated projective subspace of $\mathbb{P}(K,V)$.

Obviously, $U \times \{0\}$, $\{0\} \times U$, and $\{(u,u) \mid u \in U\}$ are pairwise complementary subspaces. A subspace $W \leq V$ is part of a triple of pairwise complementary subspaces exactly if it belongs to the set

$$\mathcal{G} := \{W \leq V \mid W \cong V/W (\cong U)\}.$$

If $\dim V = 2n$, then $\mathcal{G}$ consists of all n-dimensional vector subspaces of V, and thus corresponds to the set of $(n-1)$-dimensional projective subspaces of $\mathbb{P}(K,V)$. In particular, if $n = 2$ these are the lines in $\mathbb{P}(K,V)$.

The reguli we are going to define are always subsets of the set $\mathcal{G}$.

Let $R = \mathrm{End}_K(U)$ be the ring of K-linear mappings $U \to U$. As usual, by R^* we denote the group of units of R. The *projective line* over R is the set

$$\mathbb{P}(R) := \left\{ R(\alpha,\beta) \leq R^2 \mid \exists \gamma, \delta \in R : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(R) \right\}.$$

Two elements $p = R(\alpha,\beta)$ and $q = R(\gamma,\delta)$ of $\mathbb{P}(R)$ are called *distant*, if the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belongs to $GL_2(R)$.

In [2] the following statements are shown:

1.4. Remark.

(i) Each $W \in \mathcal{G}$ has the form $U^{(\alpha,\beta)} := \{(u^\alpha, u^\beta) \mid u \in U\}$ for a suitable pair $(\alpha,\beta) \in R^2$ with $R(\alpha,\beta) \in \mathbb{P}(R)$.

(ii) The mapping $\Phi : \mathbb{P}(R) \to \mathcal{G} : R(\alpha,\beta) \mapsto U^{(\alpha,\beta)}$ is a well-defined bijection.

(iii) Two points $p, q \in \mathbb{P}(R)$ are distant exactly if their images $p^\Phi, q^\Phi \in \mathcal{G}$ are complementary.

In particular, we have $U \times \{0\} = U^{(1,0)}$, $\{0\} \times U = U^{(0,1)}$, and $\{(u,u) \mid u \in U\} = U^{(1,1)}$, where $1 \in R$ is the identity on U and 0 is the zero mapping.

By definition, the group $GL_2(R)$ operates transitively on $\mathbb{P}(R)$ and leaves the relation "distant" invariant. It even acts transitively on the set of triples of pairwise distant points.

The group $\mathrm{Aut}_K(V)$ of all K-linear bijections $V \to V$ is isomorphic to $GL_2(R)$; each linear bijection $\psi \in \mathrm{Aut}_K(V)$ has the form $(u,w) \mapsto (u,w)^\psi = (u^\alpha + w^\gamma, u^\beta + w^\delta)$, where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a uniquely determined matrix in $GL_2(R)$.

This leads directly to the following:

1.5. Remark. The actions of $GL_2(R)$ on $\mathbb{P}(R)$ and of $\mathrm{Aut}_K(V)$ on $\mathcal{G}$ are equivalent via Φ. In particular, the group $\mathrm{Aut}_K(V)$ acts transitively on the set of triples of pairwise complementary subspaces of $\mathbb{P}(K,V)$.

In the sequel we shall always identify the element $\psi \in GL_2(R)$ with the associated linear bijection in $\mathrm{Aut}_K(V)$. Moreover, by abuse of notation, we shall even denote the projective collineation of $\mathbb{P}(K,V)$ induced by this linear bijection with ψ.

In Section 3 we need that the field K is embedded into the ring R as a subfield. This can only be done with respect to a fixed given basis. Assume that $(b_i)_{i\in I}$ is a basis of U. Then for $k \in K$ we let λ_k be the linear mapping $U \to U$ defined by $b_i^{\lambda_k} := kb_i$. The mapping $k \mapsto \lambda_k$ is an embedding of K into R. Note that things become easier when restricting to the finite-dimensional case because then one might use (scalar) matrices. However, one has to keep in mind that also in that case λ_k is not the left multiplication $u \mapsto ku$ if K is not commutative.

2. A new definition of a regulus

In this section we generalize Segre's definition of a regulus to higher dimensions. We give two definitions, one of them only makes sense in finite-dimensional projective spaces. Later we shall prove that both definitions coincide in this case.

Our first definition is a rather natural generalization of Segre's.

2.1. Definition. *Let V be of dimension $2n$ over K. Let $T_0, T_1, \dots, T_n$ be $n+1$ lines in $\mathbb{P}(K,V)$ that are in general position (i.e., any n of them span $\mathbb{P}(K,V)$). Then the set*

$$\mathcal{R} := \{W \in \mathcal{G} \mid W \text{ is transversal to } T_0, T_1, \dots, T_n\}$$

is called a regulus of type (1) *in* $\mathbb{P}(K,V)$.

We want to remark here that for commutative K this construction was already used by W. Burau in [4, IV, §33, 5] in the more general context of Segre manifolds. In case $\dim V = 2n$ a regulus as defined in 1.1 is a family of generators of a Segre manifold $S_{1,n-1}$ according to Burau's notation.

The following proposition collects some more or less immediate consequences of the definition.

Recall that a *frame* in an m-dimensional projective space $\mathbb{P}$ is a family of $m+2$ points such that any $m+1$ of them span $\mathbb{P}$ (and hence are a *basis* of $\mathbb{P}$).

2.2. Proposition. *Let $\mathcal{R}$ be the regulus of type* (1) *transversal to $T_0, \dots, T_n$.*

(i) *For $W \in \mathcal{R}$ and $i \in \{0, \dots, n\}$ let $Q_i := W \cap T_i$. Then $(Q_0, \dots, Q_n)$ is a frame of $\mathbb{P}(K,W)$.*

(ii) *Through each point Q on any of the lines T_i there is exactly one $W \in \mathcal{R}$.*

(iii) *Any two (different) elements W, W' of $\mathcal{R}$ are complementary.*

Proof. (i) is clear because the vector space W is n-dimensional and the T_i are in general position.

(ii): Without loss of generality consider $Q =: Q_0 \in T_0$. The subspace spanned by Q and $T_2, \dots, T_n$ meets the line T_1 in exactly one point, Q_1 say. By induction one obtains points

$Q_i \in T_i$ $(1 \le i < n)$ lying in the subspace spanned by $Q_0, \dots, Q_{i-1}$ and $T_{i+1}, \dots, T_n$. The points $Q_0, \dots, Q_{n-1}$ span an $(n-1)$-dimensional projective subspace W that meets T_n and thus belongs to $\mathcal{R}$. This $W \in \mathcal{R}$ is uniquely determined by Q_0 because the induction process does not depend on the order of the lines T_i.

(iii) is a consequence of (i) and (ii): Let $Q_i = T_i \cap W$ and $Q_i' = T_i \cap W'$. By (1) we have $W = \bigoplus_{i=1}^n Q_i$ and $W' = \bigoplus_{i=1}^n Q_i'$. From (2) we know that $T_i = Q_i \oplus Q_i'$, and hence $V = \bigoplus_{i=1}^n T_i = \bigoplus_{i=1}^n (Q_i \oplus Q_i') = (\bigoplus_{i=1}^n Q_i) \oplus (\bigoplus_{i=1}^n Q_i') = W \oplus W'$. □

Now we come to our second definition. It is valid for projective spaces $\mathbb{P}(K, V)$ of arbitrary dimension. Recall that we always assume that $V = U \times U$ for some vector space U.

2.3. Definition. *Let $V = U \times U$ be a left vector space over K. Let $(T_i)_{i \in I}$ be a minimal family of lines spanning $\mathbb{P}(K, V)$. Moreover, let H_1, H_2 be two hyperplanes in $\mathbb{P}(K, V)$ such that each line T_i $(i \in I)$ meets $H_1 \cup H_2$ in exactly two points (not belonging to $H_1 \cap H_2$). We say that a subspace W of $\mathbb{P}(K, V)$ is regular with respect to $((T_i)_{i \in I}, H_1, H_2)$, if*

(i) *W is transversal to $(T_i)_{i \in I}$ and spanned by the points $(W \cap T_i)_{i \in I}$, and*
(ii) *$W \subseteq H_1$ or $W \subseteq H_2$ or $W \cap (H_1 \cup H_2) \subseteq H_1 \cap H_2$.*

The set

$$\mathcal{R} = \{W \le V \mid W \text{ is regular w.r.t. } ((T_i)_{i \in I}, H_1, H_2)\}$$

is called a regulus of type (2) *in $\mathbb{P}(K, V)$.*

The first two cases of condition 2.3(ii) give two special elements of the regulus $\mathcal{R}$.

2.4. Lemma. *Let $((T_i)_{i \in I}, H_1, H_2)$ and $\mathcal{R}$ be as in 2.3. For $k \in \{1, 2\}$ we denote the subspace of $\mathbb{P}(K, V)$ spanned by the points $(T_i \cap H_k)_{i \in I}$ by X_k. Then the following statements hold:*

(i) *The subspaces X_1 and X_2 belong to the regulus $\mathcal{R}$. They are isomorphic and complementary, hence they belong to the set $\mathcal{G}$ as well.*
(ii) *If a subspace $W \in \mathcal{R}$ contains one of the points $T_i \cap H_k$ $(i \in I,\ k \in \{1, 2\})$, then $W = X_k$.*

Proof. (i): Obviously, X_k is regular w.r.t. $((T_i)_{i \in I}, H_1, H_2)$ and so belongs to $\mathcal{R}$. The assumptions on $((T_i)_{i \in I}, H_1, H_2)$ yield that $V = X_1 \oplus X_2$. Moreover, $X_1 \cong X_2$ via $T_i \cap H_1 \mapsto T_i \cap H_2$. So $X_1, X_2 \in \mathcal{G}$.

(ii): Assume that $Q_i := T_i \cap H_k$ belongs to $W \in \mathcal{R}$. If $W \subseteq H_k$, then of course $W = X_k$. Otherwise 2.3(ii) implies $Q_i \in H_1 \cap H_2$, which is not allowed for a point of T_i. □

We also have some direct consequences of the second definition that are analogous to those of 2.2 above.

2.5. Proposition. *Let $\mathcal{R}$ the regulus of type* (2) *regular w.r.t. $((T_i)_{i \in I}, H_1, H_2)$.*

(i) *For $W \in \mathcal{R}$ and $i \in I$ let $Q_i := W \cap T_i$. Then $(Q_i)_{i \in I}$ is a basis of W.*
(ii) *Through each point Q on any of the lines T_i there is exactly one $W \in \mathcal{R}$.*
(iii) *Any two elements W, W' of $\mathcal{R}$ are complementary. Moreover, they are isomorphic and thus belong to $\mathcal{G}$.*

Proof. (i) follows directly from the definition.

(ii): Consider a point $Q \in T_i$ for a fixed $i \in I$. By 2.4(ii) the only $W \in \mathcal{R}$ through $T_i \cap H_k$ is X_k ($k \in \{1,2\}$). So without loss of generality we may assume that Q does not belong to $H_1 \cup H_2$. Hence $H := (H_1 \cap H_2) \oplus Q$ is a hyperplane in V, different from H_1 and H_2. Each line T_j, $j \in I$, meets H in a unique point, because otherwise T_j would intersect $H_1 \cup H_2$ in at most one point of $H_1 \cap H_2$, which is not allowed by 2.3. Let $Q_j := T_j \cap H$ (in particular, $Q_i = Q$). The subspace $W := \bigoplus_{j\in I} Q_j$ belongs to $\mathcal{R}$ because $W \subseteq H$ and $H \cap H_k = H_1 \cap H_2$ (for $k = 1, 2$), which implies $W \cap (H_1 \cup H_2) \subseteq H_1 \cap H_2$.

It remains to prove that $W \in \mathcal{R}$ is uniquely determined by this construction. Assume that $W' \in \mathcal{R}$ contains Q. Let $Q'_j := W' \cap T_j$. Since $W' = \bigoplus_{j\in I} Q'_j$, we have to show that $Q'_j = Q_j = H \cap T_j$ holds for every $j \in I$. Suppose that this is not the case. Then there is a $j \in I$ with $Q'_j \notin H$, and thus $V = H \oplus Q'_j = (H_1 \cap H_2) \oplus Q \oplus Q'_j$. In particular, the line $L := Q \oplus Q'_j$ meets each of the hyperplanes H_1, H_2 in a point not contained in $H_1 \cap H_2$. On the other hand, L is entirely contained in W', a contradiction to $W' \cap (H_1 \cup H_2) \subseteq H_1 \cap H_2$ (which holds since $Q \in W' \setminus (H_1 \cup H_2)$).

(iii) can be shown similarly to 2.2(iii). □

Note that so far it is not clear that the reguli defined above coincide with those of 1.1 in the commutative case. This will be shown at the end of this section.

By 2.2 and 2.5 any two elements of a regulus of type (1) and of type (2), respectively, are complementary. Moreover, we know that each regulus is a subset of $\mathcal{G}$ and consists of at least three elements.

Now we consider three arbitrary pairwise complementary subspaces W_1, W_2, W_3 in $\mathbb{P}(K, V)$. We want to show that there exist reguli of type (1) (if $\dim V < \infty$) and of type (2) containing W_1, W_2, W_3.

The direct decompositions $V = W_1 \oplus W_2 = W_1 \oplus W_3 = W_2 \oplus W_3$ give rise to a linear bijection $\pi : W_1 \to W_2$, uniquely determined by W_3, namely $w_1 = w_2 + w_3 \mapsto w_2$. We denote this linear mapping and also the induced projectivity with center W_3 by $\pi(W_1, W_2, W_3)$. It will be used in the subsequent propositions in order to describe the reguli through W_1, W_2, W_3.

2.6. Proposition. *Let* $\dim V = 2n$, *and let* $W_1, W_2, W_3 \in \mathcal{G}$ *be pairwise complementary. Then there is at least one regulus of type* (1) *containing* W_1, W_2, W_3.
Every regulus of type (1) *through* W_1, W_2, W_3 *can be obtained as follows:*
Let $(P_0, P_1, \dots, P_n)$ *be a frame in* W_1, *and let* $\pi = \pi(W_1, W_2, W_3)$. *Then* $(P_0^\pi, P_1^\pi, \dots, P_n^\pi)$ *is a frame in* W_2, *and the lines* $T_0 = P_0 \oplus P_0^\pi, \dots, T_n = P_n \oplus P_n^\pi$ *are in general position. The regulus* $\mathcal{R}$ *with transversals* $T_0, \dots, T_n$ *contains* W_1, W_2, W_3.

Proof. Let $(P_0, P_1, \dots, P_n)$ be a frame in W_1 such that $P_i = Kb_i$, $b_i = c_i + d_i \in W_1$, $c_i \in W_2, d_i \in W_3$ (for $i \in \{0, \dots, n\}$). Then T_i meets W_2 in the point $Kc_i = P_i^\pi$ and W_3 in the point Kd_i. So the construction yields a regulus through W_1, W_2, W_3.

Every regulus of type (1) through W_1, W_2, W_3 can be obtained in this way, because for given transversals $T_0, \dots, T_n$ the points $P_i = W_1 \cap T_i$ ($i \in \{0, \dots, n\}$) build a frame of W_1, and $P'_i = W_2 \cap T_i$ equals P_i^π since T_i is contained in the subspace spanned by P_i and the center W_3 of π. □

A similar statement holds for the reguli of type (2). Before stating it we need the definition of a *fundamental figure* of a projective space $\mathbb{P}$. This is a pair $((P_i)_{i\in I}, H)$ consisting of a basis $(P_i)_{i\in I}$ of $\mathbb{P}$ and hyperplane H in $\mathbb{P}$ such that no point P_i $(i \in I)$ belongs to H.

The notion of a fundamental figure in a projective space is due to H. Brauner [3]. It serves as a substitute for the notion of a frame (called *fundamental set* by Brauner) in the infinite-dimensional case. In particular, the group of projective collineations of a projective space $\mathbb{P}$ acts transitively on the set of fundamental figures in $\mathbb{P}$, and it acts sharply transitively exactly if $\mathbb{P}$ is pappian (see [3, Satz 4.6]).

2.7. Proposition. *Let $W_1, W_2, W_3 \in \mathcal{G}$ be pairwise complementary. Then there is at least one regulus of type* (2) *containing W_1, W_2, W_3.*

Every regulus of type (2) *through W_1, W_2, W_3 satisfying $W_1 \subseteq H_1$, $W_2 \subseteq H_2$ (with H_1, H_2 as in 2.3) can be obtained as follows:*

Let $((P_i)_{i\in I}, H)$ be a fundamental figure in W_1, and let $\pi = \pi(W_1, W_2, W_3)$. Then $((P_i^\pi)_{i\in I}, H^\pi)$ is a fundamental figure in W_2. The family $(T_i = P_i \oplus P_i^\pi)_{i\in I}$ is a minimal family of lines spanning $\mathbb{P}(K, V)$, and the union of the two hyperplanes $H_1 = W_1 \oplus H^\pi$, $H_2 = H \oplus W_2$ is met by each line T_i in exactly two points. The regulus regular w.r.t. $((T_i)_{i\in I}, H_1, H_2)$ contains W_1, W_2, W_3 and satisfies $W_1 \subseteq H_1, W_2 \subseteq H_2$.

Proof. Let $((P_i)_{i\in I}, H)$ be a fundamental figure in W_1. As in the proof of 2.6 one can see that every line $T_i = P_i \oplus P_i^\pi$ meets W_k $(k \in \{1, 2, 3\})$. The points of intersection span W_k since $(P_i)_{i\in I}$ is a basis of W_1. Moreover, we have $W_1 \subseteq H_1$ and $W_2 \subseteq H_2$. Next we show that also W_3 satisfies 2.3(ii): Consider a point Kw_3 in $W_3 \cap H_1$ with $w_3 = w_1 + w_2$ for unique $w_1 \in W_1, w_2 \in W_2$, $w_1 \neq 0 \neq w_2$. By definition of π we have $-w_2 = w_1^\pi$. From $w_1 - w_1^\pi = w_3 \in H_1 = W_1 \oplus H^\pi$ we conclude $w_1^\pi \in H^\pi$, hence we have $w_1 \in H$ and $w_3 \in H \oplus H^\pi = H_1 \cap H_2$. Similarly, a point in $W_3 \cap H_2$ must belong to $H_1 \cap H_2$. So $W_3 \cap (H_1 \cup H_2) \subseteq H_1 \cap H_2$. Altogether, the regulus regular w.r.t. $((T_i)_{i\in I}, H_1, H_2)$ contains W_1, W_2, W_3 and satisfies $W_1 \subseteq H_1, W_2 \subseteq H_2$.

Every regulus $\mathcal{R}$ of type (2) through W_1, W_2, W_3 with $W_1 \subseteq H_1, W_2 \subseteq H_2$ can be obtained in this way: Let $\mathcal{R}$ be regular w.r.t. $((T_i)_{i\in I}, H_1, H_2)$. Then $((P_i)_{i\in I}, H)$ with $P_i = W_1 \cap T_i$, $H = W_1 \cap H_2$ is a fundamental figure in W_1: First of all, by definition of the regulus, $(P_i)_{i\in I}$ is a basis of W_1. Moreover, no P_i belongs to H because $P_i \in H_1 \setminus (H_1 \cap H_2)$. This also implies that H is a hyperplane in W_1. As in the proof of 2.6 one sees that $T_i = P_i \oplus P_i^\pi$. In addition, the hyperplane $H \oplus W_2 = (W_1 \cap H_2) \oplus (W_2 \cap H_2)$ of V is contained in the hyperplane H_2, so $H \oplus W_2 = H_2$. Analogously, $H_1 = W_1 \oplus H^\pi$. So $\mathcal{R}$ equals the regulus obtained from the fundamental figure $((P_i)_{i\in I}, H)$. □

Our next aim is to show that the reguli of type (1) and type (2) coincide in the finite-dimensional case. In order to make calculations easier, we want to consider reguli containing the three pairwise complementary subspaces $U^{(1,0)} = U \times \{0\}$, $U^{(0,1)} = \{0\} \times U$, and $U^{(1,1)} = \{(u, u) \mid u \in U\}$.

This is possible without loss of generality because of the next lemma.

2.8. Lemma. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be the sets of all reguli of type* (1) *and* (2), *respectively, in $\mathbb{P}(K, V)$. Then the following statements hold:*

(i) *Every collineation of $\mathbb{P}(K, V)$ acts on the sets $\mathfrak{R}_1$ and $\mathfrak{R}_2$, i.e., maps reguli to reguli (of the same type).*

(ii) *The group* $\mathrm{Aut}_K(V)$ *acts transitively on both sets* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$*, i.e., any regulus can be mapped to any other regulus (of the same type) by a projective collineation.*

Proof. (i) is clear by the definitions of the reguli.

(ii): By 2.6 and 2.7 each regulus is determined by three pairwise complementary subspaces W_1, W_2, W_3 and a frame (or a fundamental figure) in W_1. By 1.5, the group $\mathrm{Aut}_K(V)$ acts transitively on the set of triples of pairwise complementary subspaces. So without loss of generality we may assume $W_1 = U^{(1,0)}$, $W_2 = U^{(0,1)}$, and $W_3 = U^{(1,1)}$. Now let $\mathcal{F}, \mathcal{F}'$ be two frames (or two fundamental figures) in W_1. Then there is a projective collineation of the projective space $\mathbb{P}(K, W_1)$ mapping $\mathcal{F}$ to $\mathcal{F}'$, and hence there is a linear bijection $\alpha \in R = \mathrm{End}_K(U)$ such that $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in GL_2(R) \cong \mathrm{Aut}_K(V)$ maps $\mathcal{F}$ to $\mathcal{F}'$ (and fixes W_1, W_2, W_3). □

We need two more lemmas in order to arrive at our desired result that $\mathfrak{R}_1 = \mathfrak{R}_2$ in the finite-dimensional case.

2.9. Lemma. *Let* $(b_1, \ldots, b_n)$ *be a basis of* U*, and let* $T_i = K(b_i, 0) \oplus K(0, b_i)$ *for* $i \in \{1, \ldots, n\}$*. Then* $W \in \mathcal{G}$ *is transversal to* $(T_1, \ldots, T_n)$ *exactly if*
$W = \left\{ (\sum_{i=1}^{n} x_i k_i b_i, \sum_{i=1}^{n} x_i l_i b_i) \;\middle|\; x_1, \ldots, x_n \in K \right\}$ *for fixed pairs* $(k_i, l_i) \in K^2 \setminus \{(0,0)\}$.

Proof. This is clear because the intersection point of W with T_i is some $K(k_i b_i, l_i b_i)$ with $(k_i, l_i) \neq (0,0)$. The linearly independent vectors $(k_1 b_1, l_1 b_1), \ldots, (k_n b_n, l_n b_n)$ span the n-dimensional vector space W. □

In the formulation of the next lemma we make use of the embedding $k \mapsto \lambda_k$ of K into $R = \mathrm{End}_K(U)$ introduced at the end of Section 1.

2.10. Lemma. *Let* $(b_1, \ldots, b_n)$ *be a basis of* U*, let* $T_i = K(b_i, 0) \oplus K(0, b_i)$ *for* $i \in \{1, \ldots, n\}$*, and let* $W \in \mathcal{G}$ *be transversal to* $(T_1, \ldots, T_n)$*. Moreover, let* $b_0 = \sum_{i=1}^{n} b_i$*,* $T_0 = K(b_0, 0) \oplus K(0, b_0)$*, and* $H = \{\sum_{i=1}^{n} x_i b_i \mid x_1, \ldots, x_n \in K, \sum_{i=1}^{n} x_i = 0\}$*,* $H_1 = U \times H$*,* $H_2 = H \times U$*. Then the following statements are equivalent:*

(i) $W = U^{(1,0)}$ *or* $W = U^{(\lambda_k, 1)} = \{(u^{\lambda_k}, u) \mid u \in U\}$ *for some* $k \in K$.

(ii) W *is transversal to* T_0.

(iii) $W \subseteq H_1$ *or* $W \subseteq H_2$ *or* $W \cap (H_1 \cup H_2) \subseteq H_1 \cap H_2$.

Proof. One can easily check that (i) implies (ii) and (iii).

Now we consider the implications (ii) ⇒ (i) and (iii) ⇒ (i). Since W is transversal to $(T_1, \ldots, T_n)$, we know from 2.9 that $W = \left\{ (\sum_{i=1}^{n} x_i k_i b_i, \sum_{i=1}^{n} x_i l_i b_i) \mid x_1, \ldots, x_n \in K \right\}$ for fixed pairs $(k_i, l_i) \in K^2 \setminus \{(0,0)\}$. What we have to show is that the equations $k_1 = \ldots = k_n$ and $l_1 = \ldots = l_n$ hold.

First let (ii) be satisfied. Then W contains a point $K(kb_0, lb_0)$ with suitable $(k,l) \neq (0,0)$. Since $((k_1 b_1, l_1 b_1), \ldots, (k_n b_n, l_n b_n))$ is a basis of W, we conclude $k = k_1 = \ldots = k_n$ and $l = l_1 = \ldots = l_n$, as desired.

Now assume that (iii) holds. If $W \subseteq H_1$, then $W = U \times \{0\}$. Next we show that otherwise $l_i \neq 0$ holds for each $i \in \{1, \dots, n\}$: If $l_i = 0$, then the point $K(k_i b_i, l_i b_i) = K(k_i b_i, 0)$ belongs to $H_1 = U \times H$, and from (iii) and $b_i \notin H$ we conclude that $k_i = 0$, a contradiction.

So without loss of generality we may assume $l_1 = \dots = l_n = 1$. For $i, j \in \{1, \dots, n\}$ with $i \neq j$ we consider the point $K(k_i b_i - k_j b_j, b_i - b_j)$ in W, which again belongs to H_1, and thus we have $k_i b_i - k_j b_j \in H$ by (3), which means $k_i = k_j$. □

Using the notation from above, we now consider the points $P_i = K(b_i, 0)$ (for $i \in \{0, \dots, n\}$) and the hyperplane $H \times \{0\}$ in $U \times \{0\}$. Obviously $(P_1, \dots, P_n)$ is a basis of the projective space corresponding to $U \times \{0\}$. Moreover, $(P_0, \dots, P_n)$ is a frame and $((P_1, \dots, P_n), H \times \{0\})$ is a fundamental figure. Hence Propositions 2.6 and 2.7 together with Lemmas 2.8 and 2.10 imply the following:

2.11. Corollary. *Let V be finite-dimensional. Then each regulus of type* (1) *in $\mathbb{P}(K, V)$ is also a regulus of type* (2) *and vice versa.*

So we do not have to distinguish between both types of reguli any more (note that if $\dim V = \infty$ we only had one type of reguli from the beginning). In the sequel we shall simply speak of reguli.

We know already (from 2.6, 2.7) that through any three pairwise complementary subspaces W_1, W_2, W_3 there is at least one regulus. Our next aim is to show that this regulus is uniquely determined exactly if the ground field K is commutative. In order to prove this, we need again several lemmas.

According to Proposition 2.7 one can construct a (unique) regulus from the given data (W_1, W_2, W_3) and $\mathcal{F}$, where $\mathcal{F}$ is a fundamental figure in W_1 (note that W_3 is involved because it gives rise to the projectivity $\pi(W_1, W_2, W_3)$). We call this regulus $\mathcal{R}(W_1, W_2, W_3, \mathcal{F})$. It is clear that $\mathcal{R}(W_1, W_2, W_3, \mathcal{F})$ contains W_1, W_2, W_3. It is not yet clear, however, that all reguli containing W_1, W_2, W_3 look like this (with a suitable fundamental figure $\mathcal{F}$). We shall show that in fact this is true (no matter whether K is commutative or not).

2.12. Lemma. *Let $\mathcal{R} = \mathcal{R}(W_1, W_2, W_3, \mathcal{F})$ and let $\pi = \pi(W_1, W_2, W_3)$. Then*

$$\mathcal{R} = \mathcal{R}(W_2, W_1, W_3, \mathcal{F}^\pi) = \mathcal{R}(W_1, W_3, W_2, \mathcal{F}).$$

Proof. We use the notation of 2.7. So $\mathcal{R} = \{W \in \mathcal{G} \mid W \text{ is regular w.r.t. } ((T_i)_{i \in I}, H_1, H_2)\}$. The first equation of our assertion is clear, because obviously $\mathcal{R}(W_2, W_1, W_3, \mathcal{F}^\pi)$ consists of all $W \in G$ that are regular w.r.t. $((T_i)_{i \in I}, H_2, H_1)$.

It remains to show that $\mathcal{R} = \mathcal{R}(W_1, W_3, W_2, \mathcal{F})$. Let $\tilde{\pi} = \pi(W_1, W_3, W_2)$. Let $\tilde{H}_1 = W_1 \oplus H^{\tilde{\pi}}$ and $\tilde{H}_2 = H \oplus W_3$. An easy calculation (using that each $w_1 \in W_1$ has the form $w_1 = w_1^\pi + w_1^{\tilde{\pi}}$) shows that $\tilde{H}_1 = H_1$ and $\tilde{H}_1 \cap \tilde{H}_2 = H_1 \cap H_2$.

We know that the regulus $\tilde{\mathcal{R}} = \mathcal{R}(W_1, W_3, W_2, \mathcal{F})$ consists of all $W \in \mathcal{G}$ regular w.r.t. $((T_i)_{i \in I}, H_1, \tilde{H}_2)$. Of course, W_1, W_2, W_3 belong to $\tilde{\mathcal{R}}$. In particular, the subspace $\tilde{X}_1$ spanned by all points $T_i \cap \tilde{H}_1$ $(i \in I)$ equals W_1, and, analogously, $\tilde{X}_2$ equals W_3.

Consider $W \in \mathcal{R}$, $W \neq W_1, W_2, W_3$. We have to show that W is regular w.r.t. $((T_i)_{i\in I}, H_1, \tilde{H}_2)$. The only thing that is not yet clear is that $W \cap \tilde{H}_2 \subseteq H_1 \cap \tilde{H}_2$. The subspace $W' = W \cap \tilde{H}_2$ is a hyperplane in W (since otherwise $W = \tilde{X}_2 = W_3$ by 2.4(ii) because all T_i meet W). Moreover, W' contains $W'' = W \cap H_1 \cap \tilde{H}_2 = W \cap H_1 \cap H_2$, which equals $W \cap H_2$ because W is regular w.r.t. $((T_i)_{i\in I}, H_1, H_2)$. But $W'' = W \cap H_2$ also is a hyperplane in W and hence must coincide with W'. □

2.13. Lemma. *Let $\mathcal{R} = \mathcal{R}(W_1, W_2, W_3, \mathcal{F})$, and let $\tilde{\mathcal{R}} = \mathcal{R}(W_1, W_2, \tilde{W}_3, \mathcal{F})$. Moreover, let $\pi = \pi(W_1, W_2, W_3)$ and $\tilde{\pi} = \pi(W_1, W_2, \tilde{W}_3)$. Then the following statements are equivalent:*

(i) $\mathcal{F}^{\pi} = \mathcal{F}^{\tilde{\pi}}$,
(ii) $\mathcal{R} = \tilde{\mathcal{R}}$,
(iii) $\tilde{W}_3 \in \mathcal{R}$.

Proof. (i) ⇒ (ii) is clear because W_3 and $\tilde{W}_3$ are only needed in the construction of the reguli in order to determine $\mathcal{F}^{\pi}$ and $\mathcal{F}^{\tilde{\pi}}$, respectively.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i): Assume that $\tilde{W}_3 \in \mathcal{R}$, and let $\mathcal{F} = ((P_i)_{i\in I}, H)$. Since every $T_i = P_i \oplus P_i^{\pi}$ meets $\tilde{W}_3$ we have $P_i^{\pi} = P_i^{\tilde{\pi}}$. Consider $h \in H$. We show that $h^{\tilde{\pi}} \in H^{\pi}$. Then $H^{\tilde{\pi}}$ is a hyperplane in W_2 contained in the hyperplane H^{π} and hence equal to H^{π}. By definition of $\tilde{\pi}$ there is a $w \in \tilde{W}_3$ with $w = h - h^{\tilde{\pi}}$. So $w \in \tilde{W}_3 \cap (H \oplus W_2) = \tilde{W}_3 \cap H_2$, which, by assumption, is contained in $H_1 \cap H_2 = H \oplus H^{\pi}$. This implies $h^{\tilde{\pi}} \in H^{\pi}$, because $h^{\tilde{\pi}}$ is unique in W_2. □

Now we come to the announced statement that the reguli through W_1, W_2, W_2 only depend on the choice of a fundamental figure $\mathcal{F}$ in W_1.

2.14. Theorem. *Let $\mathcal{R}$ be a regulus containing the pairwise complementary subspaces $W_1, W_2, W_3 \in \mathcal{G}$. Then $\mathcal{R} = \mathcal{R}(W_1, W_2, W_3, \mathcal{F})$ for a suitable fundamental figure $\mathcal{F}$ in W_1.*

Proof. We know that $\mathcal{R} = \mathcal{R}(W_1', W_2', W_3', \mathcal{F}')$ for $W_1', W_2', W_3' \in \mathcal{R}$ and a fundamental figure $\mathcal{F}'$ in W_1'. First we want to substitute W_1' by W_1 and $\mathcal{F}'$ by a fundamental figure $\mathcal{F}$ in W_1. We assume that $W_1 \neq W_1', W_2', W_3'$ (otherwise an application of 2.12 leads to the desired result). Because of $W_1 \in \mathcal{R}$ we know from 2.13 that $\mathcal{R} = \mathcal{R}(W_1', W_2', W_1, \mathcal{F}')$. Applying 2.12 yields $\mathcal{R} = \mathcal{R}(W_1, W_1', W_2', \mathcal{F})$ for a suitable $\mathcal{F}$. A similar procedure then substitutes W_2' by W_2 and W_3' by W_3. □

Now we turn to the commutative case. Recall that a projective collineation of a projective space $\mathbb{P}(K, W)$ is uniquely determined by its action on a fundamental figure exactly if K is commutative.

2.15. Theorem. *Let $W_1, W_2, W_3 \in \mathcal{G}$ be pairwise complementary. Then the equation $\mathcal{R}(W_1, W_2, W_3, \mathcal{F}) = \mathcal{R}(W_1, W_2, W_3, \mathcal{F}')$ holds for any two fundamental figures $\mathcal{F}, \mathcal{F}'$ of W_1 if, and only if, K is commutative.*

Proof. Let $W \in \mathcal{G}$ be a subspace complementary to W_1 and W_2. Moreover, let $\pi = \pi(W_1, W_2, W_3)$ and $\tilde{\pi} = \pi(W_1, W_2, W)$. By 2.13 we know that $W \in \mathcal{R} = \mathcal{R}(W_1, W_2, W_3, \mathcal{F})$

is equivalent to $\mathcal{F}^\pi = \mathcal{F}^{\tilde{\pi}}$. Analogously, $W \in \mathcal{R}' = \mathcal{R}(W_1, W_2, W_3, \mathcal{F}')$ is equivalent to $\mathcal{F}'^\pi = \mathcal{F}'^{\tilde{\pi}}$. Now the condition $\mathcal{F}^\pi = \mathcal{F}^{\tilde{\pi}}$ implies $\pi = \tilde{\pi}$, and hence also $\mathcal{F}'^\pi = \mathcal{F}'^{\tilde{\pi}}$ for every $\mathcal{F}'$, exactly if K is commutative. This yields the assertion. □

So in the commutative case a regulus only depends on three of its elements. It also is a regulus in the usual sense (of Definition 1.1):

2.16. Corollary. *Let K be commutative, and let $W_1, W_2, W_3 \in \mathcal{G}$ be pairwise complementary. Then there is exactly one regulus containing W_1, W_2, W_3. This regulus coincides with the unique regulus in the sense of Definition 1.1 that contains W_1, W_2, W_3.*

Proof. The first statement is clear because of the preceding theorems.

By 2.5(iii) any two elements of a regulus are complementary. It remains to show that the regulus $\mathcal{R}$ through W_1, W_2, W_3 satisfies conditions (R1) and (R2) of 1.1.

Since $\mathcal{R}$ is independent of the choice of the fundamental figure in W_1, according to 2.7 any line T meeting W_1, W_2, W_3 can be chosen as one of the transversals T_i that determine $\mathcal{R}$. So (R1) is satisfied, and, moreover, 2.5(ii) yields (R2). □

3. Reguli and chains

In this section we are going to interpret the reguli from above as so-called *chains*. First we recall what a chain geometry is. Note that our definition here is broader than the usual one that can be found, e.g., in [1] or [8].

As mentioned in Section 1, the set $\mathcal{G}$ of all subspaces of $\mathbb{P}(K, V)$ that possess an isomorphic complement can be identified with the projective line $\mathbb{P}(R)$ over the ring $R = \mathrm{End}_K(U)$ via $\Phi : R(\alpha, \beta) \mapsto U^{(\alpha,\beta)}$.

Moreover, after fixing a basis $(b_i)_{i \in I}$ of U, one obtains an embedding $k \mapsto \lambda_k$ of the field K into R. Hence the projective line over K may be embedded into $\mathbb{P}(R)$ as well, via $K(k, l) \mapsto R(\lambda_k, \lambda_l)$.

The image of $\mathbb{P}(K)$ under this embedding will be denoted by $\mathcal{C}_0$. It is our *standard chain*. All the other chains in $\mathbb{P}(R)$ are images of $\mathcal{C}_0$ under the group $GL_2(R)$. So the *chain set* $\mathfrak{C}(K, R)$ is the orbit $\mathcal{C}_o^{GL_2(R)} = \{\mathcal{C}_0^\gamma \mid \gamma \in GL_2(R)\}$.

The incidence structure $\Sigma(K, R) := (\mathbb{P}(R), \mathfrak{C}(K, R))$ is called the *chain geometry* over (K, R). One can easily see that two points of $\mathbb{P}(R)$ are joined by a chain exactly if they are distant (in the sense of Section 1).

Up to now, chain geometries $\Sigma(K, R)$ have mainly been investigated in the case that R is a K-algebra, i.e., the field K is contained in the center of R. Of course then K has to be commutative. In that case the chain geometry $\Sigma(K, R)$ satisfies the extra condition that any three pairwise distant points are joined by a unique chain. See [1] and [8] for more results in this case.

In [2, Thm. 2.8] it is shown that if the ground field K of the projective space $\mathbb{P}(K, V)$ is commutative, then the incidence structure $(\mathcal{G}, \mathfrak{R})$, where $\mathfrak{R}$ is the set of all reguli (in the sense of Definition 1.1), is isomorphic to the chain geometry $\Sigma(K, R)$ over the ring $R = \mathrm{End}_K(U)$, which in this case is a K-algebra. The isomorphism $\Sigma(K, R) \to (\mathcal{G}, \mathfrak{R})$ is our bijection Φ.

We want to show now that this result can be carried over to the non-commutative case, using our new definition of a regulus.

Let $\mathfrak{R}$ be the set of all reguli (according to Definitions 2.1 and 2.3) in $\mathbb{P}(K,V)$, where K need not be commutative. As before, we assume $V = U \times U$. For the rest of this section, we fix a basis $(b_i)_{i\in I}$ of U and embed K into $R = \mathrm{End}_K(U)$ with respect to this basis.

In order to show that the incidences structure $(\mathcal{G}, \mathfrak{R})$ is isomorphic to $\Sigma(K,R)$, we shall proceed as follows: Our crucial result will say that the image of the standard chain $\mathcal{C}_0$ under Φ is a *standard regulus* $\mathcal{R}_0 = \mathcal{R}(U^{(1,0)}, U^{(0,1)}, U^{(1,1)}, \mathcal{F}_0)$, with respect to some *standard fundamental figure* $\mathcal{F}_0$. The announced isomorphism theorem then follows easily by using 1.5 and 2.8.

We begin with the definition of the standard fundamental figure in $U^{(1,0)} = U \times \{0\}$. We assume that the index set I of our basis $(b_i)_{i\in I}$ of U contains the element 0. For $i \in I_0 := I \setminus \{0\}$ we define $a_i := b_i - b_0$. Then $(a_i)_{i\in I_0}$ is a basis of a hyperplane H in U, such that no b_i $(i \in I)$ belongs to H. For $i \in I$ let $P_i := K(b_i, 0)$. Moreover, let $H_0 := H \times \{0\}$. Then $\mathcal{F}_0 := ((P_i)_{i\in I}, H_0)$ is a fundamental figure in $U \times \{0\}$.

The next statement is our decisive theorem. In the finite-dimensional case it is an immediate consequence of 2.10.

3.1. Theorem. *The image $\mathcal{C}_0^\Phi$ of the standard chain $\mathcal{C}_0$ is a regulus, namely $\mathcal{C}_0^\Phi = \mathcal{R}_0 := \mathcal{R}(U^{(1,0)}, U^{(0,1)}, U^{(1,1)}, \mathcal{F}_0)$.*

Proof. We recall that the regulus $\mathcal{R}_0 = \mathcal{R}(U^{(1,0)}, U^{(0,1)}, U^{(1,1)}, \mathcal{F}_0)$ consists of all subspaces $W \in \mathcal{G}$ regular w.r.t. $((T_i)_{i\in I}, H_1, H_2)$, where $T_i = K(b_i, 0) \oplus K(0, b_i)$, $H_1 = U \times H$ and $H_2 = H \times U$.

First we show that Φ maps each element of $\mathcal{C}_0$ into $\mathcal{R}_0$. This is clear for $R(1,0)$ and $R(0,1)$, because their images are $U^{(1,0)}$ and $U^{(0,1)}$, respectively. So it remains to consider $R(\lambda_k, 1)$ for $k \in K^* = K \setminus \{0\}$. Its image is $U^{(\lambda_k,1)} = \{(u^{\lambda_k}, u) \mid u \in U\}$. This subspace satisfies condition 2.3(i) because it contains exactly one point of each line T_i, namely, $K(kb_i, b_i) = K(b_i^{\lambda_k}, b_i)$, and it is spanned by these points. Moreover, $U^{(\lambda_k,1)}$ satisfies 2.3(ii): We have $U^{(\lambda_k,1)} \cap H_1 = \{(h^{\lambda_k}, h) \mid h \in H\}$. Since H is invariant under λ_k by construction, this means $U^{(\lambda_k,1)} \cap H_1 \subseteq H \times H = H_1 \cap H_2$. Analogously one sees that $U^{(\lambda_k,1)} \cap H_2 \subseteq H_1 \cap H_2$, because $U^{(\lambda_k,1)} = U^{(1,\lambda_k^{-1})}$.

Now we consider an arbitrary $W \in \mathcal{R}_0 \setminus \{U^{(1,0)}\}$. Then $W = U^{(\alpha,\beta)} = R(\alpha,\beta)^\Phi$ for some $R(\alpha,\beta) \in \mathbb{P}(R)$. By 2.5(iii) we know that W is complementary to $U^{(1,0)}$ and so we may assume that $\beta = 1$. Hence we have to prove that $\alpha = \lambda_k$ for some $k \in K$.

Since $W = U^{(\alpha,1)}$ is transversal to $(T_i)_{i\in I}$, for each $i \in I$ there are elements $u_i \in U \setminus \{0\}$ and $(k_i, l_i) \in K^2 \setminus \{(0,0)\}$ such that $u_i^\alpha = k_i b_i$ and $u_i = l_i b_i$. Because of $u_i \neq 0$ we may even assume that all l_i are equal to 1. So we have $b_i^\alpha = k_i b_i$ $(i \in I)$, which determines α uniquely.

It remains to show that $k_i = k_j$ holds for all pairs $(i,j) \in I \times I$. This will be done by using the fact that W fulfils 2.3(ii). The subspaces X_1 and X_2 of 2.4 are $U \times \{0\}$ and $\{0\} \times U$, respectively. So obviously $W \neq X_1$, and if $W = X_2$ our assertion is shown. Otherwise 2.3(ii) means (by 2.4) that the intersection $W \cap (H_1 \cup H_2) = \{(u^\alpha, u) \mid u \in H$ or $u^\alpha \in H\}$ is contained in $H_1 \cap H_2 = H \times H$. This implies $H^\alpha \subseteq H$. Now let $i \in I_0$.

Then $a_i = b_i - b_0 \in H$, and hence $a_i^\alpha = k_i b_i - k_0 b_0 \in H$. Since $(a_i)_{i \in I_0}$ is a basis of H and $(b_i)_{i \in I}$ is a basis of U, this means $k_i = k_0$, as desired. □

Now we know that the chain $\mathcal{C}_0$ is mapped to a regulus. This can easily be carried over to the other chains.

3.2. Corollary. *Let $\mathcal{C} \in \mathfrak{C}(K, R)$. Then $\mathcal{C}^\Phi$ is a regulus.*

Proof. By definition $\mathcal{C} = \mathcal{C}_0^\gamma$ for some $\gamma \in GL_2(R) \cong \mathrm{Aut}_K(V)$. Thus by 3.1 we have $\mathcal{C}^\Phi = \mathcal{C}_0^{\gamma\Phi} = \mathcal{C}_0^{\Phi\gamma} = \mathcal{R}_o^\gamma$, where $\gamma \in \mathrm{Aut}_K(V)$ by 2.8(i) maps reguli to reguli. This yields the assertion. □

By 3.1 we also know that the standard regulus $\mathcal{R}_0 = \mathcal{R}(U^{(1,0)}, U^{(0,1)}, U^{(1,1)}, \mathcal{F}_0)$ is the image of a chain. This holds for every other regulus as well, because of the action of $\mathrm{Aut}_K(V) \cong GL_2(R)$ on the set $\mathfrak{R}$.

3.3. Corollary. *Let $\mathcal{R} \in \mathfrak{R}$ be any regulus. Then $\mathcal{R} = \mathcal{C}^\Phi$ for some $\mathcal{C} \in \mathfrak{C}(K, R)$. Hence $\mathfrak{C}(K, R)^\Phi = \mathfrak{R}$.*

Proof. By 2.8(ii) there is a $\gamma \in \mathrm{Aut}_K(V) \cong GL_2(R)$ mapping the standard regulus $\mathcal{R}_0$ to $\mathcal{R}$. So by 3.1 we have $\mathcal{R} = \mathcal{R}_0^\gamma = \mathcal{C}_0^{\Phi\gamma} = \mathcal{C}_0^{\gamma\Phi}$, where $\mathcal{C}_0^\gamma$ is a chain. □

Altogether, we now have completed the proof of the isomorphism theorem generalizing [2, Thm. 2.8]:

3.4. Main Theorem. *The incidence structure $(\mathcal{G}, \mathfrak{R})$ of all reguli in $\mathbb{P}(K, V)$ is isomorphic to the chain geometry $\Sigma(K, R)$.*

After having established Theorem 3.4 we can make use of the algebraic description of $(\mathcal{G}, \mathfrak{R})$ as the chain geometry $\Sigma(K, R)$ for our investigation of the geometric properties of the reguli in $\mathbb{P}(K, V)$.

The first result of 2.16, e.g., now follows directly from the fact that in a chain geometry $\Sigma(K, R)$ over a K-algebra R any three pairwise distant points are joined by a unique chain (note that of course $R = \mathrm{End}_K(U)$ is a K-algebra if K is commutative). Moreover, also the second part of 2.16 now is immediately clear, namely, that in the commutative case the old and the new reguli coincide, because in [2] we showed the same theorem as 3.4 for the old reguli (and we used the same isomorphism Φ).

4. Intersections of reguli

In this section we want to determine the intersection $\mathcal{I} = \mathcal{I}(W_1, W_2, W_3)$ of all reguli containing three given pairwise complementary subspaces W_1, W_2, W_3 of $\mathbb{P}(K, V)$. If K is commutative this is the unique regulus through W_1, W_2, W_3. Otherwise the center $Z = Z(K)$ of the field K comes into play.

By 2.14 we know that $\mathcal{I} = \bigcap\{\mathcal{R}(W_1, W_2, W_3, \mathcal{F}) \mid \mathcal{F}$ fundamental figure in $W_1\}$. We could use this description and some linear algebra for the explicit determination of $\mathcal{I}$. However, we prefer taking the chain-geometric point of view.

But before, we consider Segre's case of the reguli in the three-dimensional projective space $\mathbb{P}(K, K^4)$. Here both the transversals and the elements of a regulus are lines. One can easily see that in this case our intersection $\mathcal{I} = \mathcal{I}(W_1, W_2, W_3)$ is the line set $\mathcal{I} = \{L \in \mathcal{G} \mid L$ transversal to $\tilde{\mathcal{R}}\}$, where $\tilde{\mathcal{R}}$ is the regulus determined by the transversal lines W_1, W_2, W_3. Segre proves in [10] that this set $\mathcal{I}$ is a regulus over the center $Z = Z(K)$.

In order to make this more explicit, we introduce some terminology that will also be used later for stating our generalization of Segre's result.

Let V be a vector space (of arbitrary dimension) over K, and let $Z = Z(K)$. We call $v \in V$ a *Z-vector with respect to the basis* $(c_j)_{j\in J}$ of V if all coordinates of v (w.r.t. this basis) belong to Z. A subspace $W \leq V$ is a *Z-subspace w.r.t.* $(c_j)_{j\in J}$ if it possesses a basis of Z-vectors w.r.t. $(c_j)_{j\in J}$. The lattice of all Z-subspaces of V w.r.t. some fixed basis gives rise to a projective space over Z, contained in $\mathbb{P}(K, V)$ and with the same dimension as $\mathbb{P}(K, V)$. Such a projective space will be called a *projective Z-subspace* of $\mathbb{P}(K, V)$.

So Segre's result reads as follows: The intersection $\mathcal{I}$ is a regulus in some projective Z-subspace of $\mathbb{P}(K, K^4)$ (with respect to a suitable basis of the vector space K^4).

We want to show that a similar result holds in higher dimensions. We use the notation of the preceding sections. Instead of $\mathcal{I}$ we first consider the corresponding intersection of chains.

4.1. Lemma. *Let $\hat{\mathcal{I}}_0$ be the intersection of all chains containing the points $R(1,0)$, $R(0,1)$, $R(1,1)$. Then $\hat{\mathcal{I}}_0 = \{R(\lambda_z, 1) \mid z \in Z\} \cup \{R(1,0)\}$, i.e., $\hat{\mathcal{I}}_0$ is the projective line $\mathbb{P}(Z)$ embedded into $\mathbb{P}(R)$ with respect to $k \mapsto \lambda_k$.*

Proof. Since the standard chain $\mathcal{C}_0$ is one of the chains under consideration, each element of $\hat{\mathcal{I}}_0 \setminus \{R(1,0), R(0,1)\}$ has the form $R(\lambda_k, 1)$ for some $k \in K^*$. Let $\mathcal{C} = \mathcal{C}_0^\gamma$ $(\gamma \in GL_2(R))$ be another chain through $R(1,0), R(0,1), R(1,1)$. We may assume that γ fixes these three points (because the subgroup $GL_2(K)$ of $GL_2(R)$ acts triply transitively on $\mathcal{C}_0$). So $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in R^*$. A point $p = R(\lambda_k, 1) \in \mathcal{C}_0$ (with $k \in K^*$) also belongs to $\mathcal{C} = \mathcal{C}_0^\gamma$ exactly if $p = R(\lambda_l, 1)^\gamma = R(\lambda_l \alpha, \alpha)$ for an element $l \in K^*$. So p belongs to the intersection $\hat{\mathcal{I}}_0$ if, and only if, for all $\alpha \in R^*$ there is an $l \in K^*$ such that $\alpha\lambda_k = \lambda_l \alpha$.

In case $k \in Z$ this condition is fulfilled because then λ_k is the mapping $u \mapsto ku$ and hence commutes with all $\alpha \in R$.

In order to show that on the other hand the condition above implies $k \in Z$ we consider special mappings α: For $s \in K$ let $\alpha_s \in R^*$ be defined by $b_i^{\alpha_s} = b_i$ $(i \in I_0)$, $b_0^{\alpha_s} = b_0 + sb_j$, where $j \in I_0$ is some fixed index. Let $\alpha_s\lambda_k = \lambda_l\alpha_s$ be satisfied for some $l \in K^*$. Then for $i \in I_0$ we have $kb_i = b_i^{\alpha_s\lambda_k} = b_i^{\lambda_l\alpha_s} = lb_i$, and so $l = k$. Moreover, $kb_0 + skb_j = b_0^{\alpha_s\lambda_k} = b_0^{\lambda_l\alpha_s} = b_0^{\lambda_k\alpha_s} = kb_0 + ksb_j$, and hence $sk = ks$ holds. Since $s \in K$ was chosen arbitrarily, this means $k \in Z$. □

The result of this lemma now can be carried over (using the action of $GL_2(R)$) to the intersection of the chains through any other three pairwise distant points.

Our formulation of the subsequent proposition uses the chain geometry $\Sigma(Z, R)$, i.e., the incidence structure $(\mathbb{P}(R), \mathfrak{C}(Z, R))$. By 4.1 the elements of $\mathfrak{C}(Z, R)$ are the images of $\hat{\mathcal{I}}_0$ under $GL_2(R)$.

4.2. Proposition. *Let $R = \mathrm{End}_K(U)$ and $Z = Z(K)$. Let $p, q, r \in \mathbb{P}(R)$ be pairwise distant. Then the intersection $\hat{\mathcal{I}} = \hat{\mathcal{I}}(p, q, r)$ of all chains through p, q, r is a chain of the* *chain geometry* $\Sigma(Z, R)$.

Of course this means that $\hat{\mathcal{I}}(p, q, r)$ is the unique chain in $\Sigma(Z, R)$ containing p, q, r. Recall that this chain is uniquely determined because Z is contained in the center of R (after identifying $z \in Z$ with $\lambda_z \in R$) and hence R is a Z-algebra.

Intersections of chains as above are considered by Benz in [1] for chain geometries $\Sigma(K, L)$ where both K and L are skew fields. Benz calls them *traces* ("Fährten"); he proves a theorem similar to 4.2 for these traces ([1, IV 2.3.3]).

Using 3.4 one can now derive from 4.1 the following generalization of Segre's result:

4.3. Proposition. *Let* $W_1, W_2, W_3 \in \mathcal{G}$ *be pairwise complementary. Then the intersection* $\mathcal{I} = \mathcal{I}(W_1, W_2, W_3)$ *of all reguli containing* W_1, W_2, W_3 *is a regulus in a projective* Z*-subspace of* $\mathbb{P}(K, V)$.

Proof. We may assume that $W_1 = U^{(1,0)}$, $W_2 = U^{(0,1)}$, $W_3 = U^{(1,1)}$, because the group $GL_2(R) \cong \mathrm{Aut}_K(V)$ maps Z-subspaces to Z-subspaces (as it permutes the bases of V). By 3.4 the intersection $\mathcal{I}_0$ of all reguli through these three subspaces is $\hat{\mathcal{I}}_0^{\Phi}$, with $\hat{\mathcal{I}}_0$ defined as in 4.1. So by 4.1 we have $\mathcal{I}_0 = \{U^{(\lambda_z,1)} \mid z \in Z\} \cup \{U^{(1,0)}\}$. Obviously (compare 3.1) this is the standard regulus in the projective Z-subspace of $\mathbb{P}(K, V)$ with respect to the basis $((b_i, 0))_{i \in I}, ((0, b_i))_{i \in I}$ of $V = U \times U$. □

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