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The Concept of Cutting Vectors for Vector Systems in the Euclidean Plane

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Abstract. The concept of cutting vectors is introduced to study the combinatorial structure of vector configurations in the plane. We give a full characterization of those n-tuples of integers which are realizable as cutting vector. Furthermore, 'simple' mappings are defined that transform cutting vectors into each other. Thus it becomes possible to consider all combinatorial types of vector systems.

0. Introduction

In the context of combinatorial and computational geometry often occur vector systems, e.g., as normal vectors of hyperplanes in the Euclidean or projective d-space.

Starting with Grünbaum [6], vector systems of hyperplanes became a basic subject. Edelsbrunner [4] and others developed algorithms for computational treatments. In recent years the theory of oriented matroids has been developed [2], [5], [1], [3]. Arrangements of oriented hyperplanes may be viewed as realizations of oriented matroids. In general, arrangements yield affine sign vector systems, which have been studied extensively by Karlander [8].

In [9] and [10] Linhart has shown that in E^2 interesting questions still are open, and that it is natural to use a 'geometric language'.

In this paper the concept of 'cutting vectors' is developed to code the combinatorial structure of systems of unit vectors in E^2 . In order to consider all possible combinatorial types of vector systems it is important to characterize the cutting vectors and to show how to 'move' in the system of cutting vectors (of a fixed number of components). This is done in Theorem 2.2 and Theorem 3.7.

In [7] we will use the concept of cutting vectors to derive results on the total weight of arrangements.

1. Basic notation

A finite collection of $n \geq 2$ unit vectors in the Euclidean plane E^2 is called a *vector system.* Let $V = \{v_1, \ldots, v_n\}$ be such a vector system.

The angle $\sphericalangle(v_i, v_j)$ is measured from v_i to v_j in the range $[0, 2\pi)$ in mathematically positive sense. We assume that the vectors are labelled according to mathematically positive sense, i.e., it holds that

$$0 < \sphericalangle(v_1, v_j) < \sphericalangle(v_1, v_{j'}) \Leftrightarrow 1 < j < j'. \tag{1.1}$$

For each $v_i \in V$ we determine the cardinality

$$k_i := \text{ card } \{v \in V : 0 < \sphericalangle(v_i, v) < \pi\}$$

and the set

$$V_i := \begin{cases} \emptyset & \text{if} \quad k_i = 0 \\ \{v_{i+1}, \ldots, v_{i+k_i}\} & \text{else.} \end{cases}$$

2. The concept of cutting vectors

Let $V = \{v_1, \ldots, v_n\}$ be a vector system, then there exists a vector $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ uniquely determined by the cardinalities k_i in the previous section. This vector is called the *cutting vector* of the vector system V. It characterizes the combinatorial structure of the vector system up to cyclic permutations, i.e., cyclic permutations of the cutting vector correspond to appropriate relabelings of the vectors and, consequently, preserve the combinatorial properties of the vector system.

Furthermore, we define the *cutting matrix* $C = (c_{ij})$ by

$$c_{ij} := \text{ sign } \det (v_i, v_j) \in \{+, -, 0\}.$$

This is essentially the same as the *chirotope* of the considered vector system [1]. It follows that

$$c_{ij} = \begin{cases} 0 & \text{for} \quad j = i & \\ + & \text{for} \quad j = i+1, \ldots, i+k_i & \text{if } k_i \neq 0 \\ 0 & \text{for} \quad j = i+k_i+1 & \text{if } v_{i+k_i+1} = -v_i \\ - & \text{for} \quad j = i+k_i+1 & \text{if } v_{i+k_i+1} \neq -v_i \wedge k_i \leq n-2 \\ - & \text{for} \quad j = i+k_i+2, \ldots, i+n-1 & \text{if } k_i \leq n-3 \end{cases}$$

(the expressions of the index j are taken modulo n).

Lemma 2.1. *Let $k = (k_1, \ldots, k_n)$ be the cutting vector of the vector system V, and γ be the number of pairs of oppositely directed vectors in V. Then*

$$c\sum_{i=1}^{n} k_i = \frac{1}{2}n(n-1) - \gamma. \tag{2.1}$$

Proof. Let $c_i := (c_{i1}, \dots, c_{in})$ be the i-th row of the matrix C and φ_i the number of '+' in c_i. Then $\varphi_i = k_i$.

Let $c^i := (c_{1i}, \dots, c_{ni})^T$ be the i-th column of C and ψ_i the number of '+' in c^i. Then

$$\psi_i = \begin{cases} n - k_i - 2 & \text{if } v_i \text{ and } v_{i+k_i+1} \text{ are oppositely directed} \\ n - k_i - 1 & \text{otherwise.} \end{cases}$$

Consequently, $\sum_{i=1}^{n}(\varphi_i + \psi_i) = n(n-1) - 2\gamma$. On the other hand, if κ denotes the number of '+' in C, then $\kappa = \sum_{i=1}^{n} \varphi_i = \sum_{i=1}^{n} \psi_i$. Thus, we have $\kappa = \sum_{i=1}^{n} \varphi_i = \sum_{i=1}^{n} k_i$ and

$$\kappa = \frac{1}{2}\left(\sum_{i=1}^{n} \varphi_i + \sum_{i=1}^{n} \psi_i\right) = \frac{1}{2}\sum_{i=1}^{n}(\varphi_i + \psi_i) = \frac{1}{2}n(n-1) - \gamma. \qquad \square$$

Now we consider the question for the *realizability* as cutting vector:

> Given a vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. Under which conditions do there exist vector systems with cutting vector a?

We can give an answer to this question by a characterization of cutting vectors (again all indices are taken modulo n):

Theorem 2.2. *The vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ is realizable as cutting vector of vector systems if and only if the following system of inequalities is valid for $i = 1, \dots, n$:*

$$0 \le a_i \le n - 1, \tag{2.2}$$

$$a_i \le a_{i+1} + 1, \tag{2.3}$$

$$a_{i+a_i} \le n - a_i - 1, \tag{2.4}$$

$$a_{i+a_i+1} \ge n - a_i - 2, \tag{2.5}$$

$$a_{i+a_i+2} \ge n - a_i - 2. \tag{2.6}$$

Equality in (2.5) *holds if and only if the vectors v_i and v_{i+a_i+1} are oppositely directed.*

Proof.

(1) Let $a = (a_1, \dots, a_n)$ be the cutting vector of a vector system V.

(a) (2.2) follows from the definition of cutting vectors.

(b) (2.3) holds true for $a_i = 0$ and $a_i = 1$.
If $a_i \ge 2$, then from $c_{i,i+a_i} = +$ it follows that $c_{i+1,i+a_i} = +$. From

$$\sphericalangle(v_{i+1}, v_{i+a_i}) < \pi \;\Leftrightarrow\; v_{i+a_i} \in V_{i+1} \;\Leftrightarrow\; c_{i+1,i+a_i} = +$$

we get $a_{i+1} \ge a_i - 1$, which proves (2.3).

(c) (2.4) holds true for $a_i = 0$.
If $a_i \geq 1$, then from $c_{i,i+a_i} = +$ it follows that $c_{i+a_i,i} = -$. Consequently, $i + a_i + a_{i+a_i} + 1 \leq n + i$, which proves (2.4).

(d) (2.5) holds true for $a_i = n - 1$.
If $c_{i,i+a_i+1} = 0$, then it follows that $c_{i+a_i+1,i} = 0$. Consequently, if $a_i \neq n - 1$, then $i + a_i + 1 + a_{i+a_i+1} = n + i - 1$, and

$$a_{i+a_i+1} = n - a_i - 2.$$

If $c_{i,i+a_i+1} = -$, then it follows that $c_{i+a_i+1,i} = +$. Consequently, $i + a_i + 1 + a_{i+a_i+1} \geq n + i$, which proves (2.5).

(e) (2.6) holds true for $a_i = n - 1$ and $a_i = n - 2$.
If $a_i \leq n-3$, then $c_{i,i+a_i+2} = -$ and $c_{i+a_i+2,i} = +$. Consequently, $i+a_i+2+a_{i+a_i+2} \geq n + i$, which proves (2.6).

(2) Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfy the conditions (2.2)–(2.6). We construct a vector system with cutting vector a.

(a) Let v_1 be an arbitrary unit vector in E^2. If $a_1 = 0$, then we set $V_1 := \emptyset$.
It is easy to find a second unit vector v_2 satisfying $\sphericalangle(v_1, v_2) \geq \pi$.
If $a_1 \neq 0$, then it is easy to find a set $V_1 = \{v_2, \ldots, v_{1+a_1}\}$ of unit vectors satisfying (1.1) and $0 < \sphericalangle(v_1, v) < \pi \ \forall v \in V_1$.

(b) Assume that

$$V_i = \begin{cases} \emptyset & \text{if} \quad a_i = 0 \\ \{v_{i+1}, \ldots, v_{i+a_i}\} & \text{else} \end{cases}$$

satisfies (1.1) and $0 < \sphericalangle(v_i, v) < \pi \ \forall\, v \in V_i$.
If $a_i \geq 2$, then from $\sphericalangle(v_i, v_{i+a_i}) < \pi$ it follows that

$$\sphericalangle(v_{i+1}, v_j) < \pi \quad \text{for } j = i+2, \ldots, i+a_i,$$

and we may define the set

$$\tilde{V}_{i+1} := \begin{cases} \emptyset & \text{if} \quad a_i < 2 \\ \{v_{i+2}, \ldots, v_{i+a_i}\} & \text{else.} \end{cases}$$

If $a_i = a_{i+1} + 1$, then we get $i + a_i = (i+1) + a_{i+1}$. Consequently, we can set

$$V_{i+1} := \begin{cases} \emptyset & \text{if} \quad a_{i+1} = 0 \\ \tilde{V}_{i+1} & \text{else.} \end{cases}$$

If $a_i < a_{i+1} + 1$, then we get $i + a_i < (i+1) + a_{i+1}$.

We consider the two cases:

(b_1) If $(i+1) + a_{i+1} < n$, then we find a set

$$V'_{i+1} = \begin{cases} \emptyset & \text{if} \quad a_{i+1} = 0 \\ \{v_{i+a_i+1}, \ldots, v_{i+1+a_{i+1}}\} & \text{else,} \end{cases}$$

of unit vectors satisfying

$$\sphericalangle(v_{i+1}, v_j) < \pi,\ \sphericalangle(v_i, v_j) \geq \pi \quad \text{for } v_j \in V'_{i+1}.$$

Then

$$V_{i+1} := \tilde{V}_{i+1} \cup V'_{i+1}$$

satisfies (1.1), $0 < \sphericalangle(v_{i+1}, v) < \pi \,\forall\, v \in V_{i+1}$, and this inequality fails for $v \in \{v_1, \dots, v_{i+1}\}$.

(b_2) If $(i+1) + a_{i+1} = n + \alpha$, $\alpha \geq 0$, then we find a set

$$V'_{i+1} = \begin{cases} \emptyset & \text{if} \quad a_{i+1} = 0 \\ \{v_{i+a_i+1}, \dots, v_n\} & \text{else,} \end{cases}$$

of unit vectors satisfying the analogous conditions like in (b_1). Furthermore, from (2.4) it follows that

$$\alpha + a_\alpha = (i+1) + a_{i+1} - n + a_{i+1+a_{i+1}} \leq i.$$

Consequently, $v_{i+1} \notin V_\alpha$. Now we define the set

$$V''_{i+1} := \begin{cases} \emptyset & \text{if} \quad \alpha = 0 \\ \{v_1, \dots, v_\alpha\} & \text{else.} \end{cases}$$

Then

$$V_{i+1} := \tilde{V}_{i+1} \cup V'_{i+1} \cup V''_{i+1}$$

satisfies (1.1) and $0 < \sphericalangle(v_{i+1}, v) < \pi \,\forall\, v \in V_{i+1}$.
We still have to prove that the inequality fails for $v \in \{v_{\alpha+1}, \dots, v_{i+1}\}$, i.e., α is maximal or, equivalently, $v_{\alpha+1} \notin V_{i+1}$. From $(i+1) + a_{i+1} = n + \alpha$ it follows that

$$\alpha = (i+1) + a_{i+1} - n \leq (i+1) + (n-1) - n = i\,.$$

If $\alpha = i$, then $\sphericalangle(v_{i+1}, v_{\alpha+1}) = 0$.
If $\alpha < i$, then we consider two cases: If equality holds in (2.5), then from

$$V_{\alpha+1} = \{v_{\alpha+2}, \dots, v_{\alpha+1+a_{\alpha+1}}\}$$

and

$$\alpha + 1 + a_{\alpha+1} = (i+1) + a_{i+1} - n + 1 + a_{i+1+a_{i+1}+1} = i$$

it follows that $v_{i+1} \notin V_{\alpha+1}$, which is equivalent to $\sphericalangle(v_{\alpha+1}, v_{i+1}) \geq \pi$.
On the other hand, from

$$V_{i+1} = \{v_{i+2}, \dots, v_{i+1+a_{i+1}}\}$$

and $i + 1 + a_{i+1} + 1 = n + \alpha + 1$ it follows that

$$\sphericalangle(v_{i+1}, v_{\alpha+1}) \geq \pi.$$

The two inequalities yield $\sphericalangle(v_{i+1}, v_{\alpha+1}) = \pi$ and consequently $v_{\alpha+1} \notin V_{i+1}$. If equality does not hold in (2.5), then from $\alpha + 1 + a_{\alpha+1} \geq i + 1$ it follows that $v_{i+1} \in V_{\alpha+1}$, which is equivalent to $v_{\alpha+1} \notin V_{i+1}$. Together with $V_1, \dots, V_n$ we obtain a vector system $V = \{v_1, \dots, v_n\}$ with cutting vector a. □

Corollary 2.3. *Let $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ be realizable as cutting vector of a vector system V. Then there exists an index $j \in \{1, \dots, n\}$ such that*

$$a_j \leq a_{j+1}.$$

Proof. If we assume that $a_i = a_{i+1} + 1$, $i = 1, \dots, n$, then we get the contradiction $a_1 = a_1 + n$. □

Corollary 2.4. *Let $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ be realizable as cutting vector of a vector system V. If $a_j \leq a_{j+1}$ for some $j \in \{1, \dots, n\}$, then*

$$a_j \leq n - 2. \tag{2.7}$$

Proof. From (2.2) we know that $a_j \leq n - 1$. If $a_{j+1} < n - 1$, then it follows that $a_j < n - 1$. If we assume that $a_j = a_{j+1} = n - 1$, then from (2.4) it follows that

$$a_j = a_{j+1+a_{j+1}} \leq n - a_{j+1} - 1 = 0,$$

which is a contradiction. □

Corollary 2.5. *Let $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ be realizable as cutting vector of a vector system V,*

$$a_j \leq a_{j+1} \quad \textit{for some} \quad j \in \{1, \dots, n\},$$

and assume that v_j and v_{j+a_j+1} are not oppositely directed. Then

$$a_j + a_{j+a_j+1} = n - 1. \tag{2.8}$$

Proof. Setting $\alpha := a_{j+1} - a_j$, we get the following observations. From (2.5) we know that

$$a_j + a_{j+a_j+1} \geq n - 1;$$

on the other hand, from (2.3) it follows that

$$a_{j+a_j+1} = a_{j+a_{j+1}-\alpha+1} \leq a_{j+1+a_{j+1}} + \alpha,$$

and from (2.4) we get

$$a_{j+1+a_{j+1}} + \alpha \leq n - a_{j+1} - 1 + \alpha = n - a_j - 1.$$

Consequently, (2.8) holds. □

3. Φ_j-mappings

Now we deal with the question: How to get a cutting vector from another one?

We will define mappings which correspond to special rotations of the vectors in the vector system. On this way we describe the change of the combinatorial structure of the vector system step by step.

We consider the mapping $\Phi_j : \mathbb{N}^n \longrightarrow \mathbb{N}^n$, where

$$a' = \Phi_j(a),\ j \in \{1,\dots,n\} \text{ is given by}$$

$$a'_i = \begin{cases} \left\{ \begin{array}{lll} a_i + 1 & \text{for} & i = j \\ a_i & \text{else} & \end{array} \right\} & \text{if} \quad a_j + a_{j+a_j+1} = n-2 \\ \\ \left\{ \begin{array}{lll} a_i - 1 & \text{for} & i = j + a_j + 1 \\ a_i & \text{else} & \end{array} \right\} & \text{if} \quad a_j + a_{j+a_j+1} \neq n-2. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $k = (k_1,\dots,k_n)$ be the cutting vector of the vector system V. Then $k' = \Phi_j(k)$ is realizable as cutting vector of a vector system if the following condition holds:*

$$k_j \leq k_{j+1}. \tag{3.2}$$

Proof. Using Corollary 2.4 and Corollary 2.5 it is easy to check that $k' = \Phi_j(k)$ fulfils the conditions of Theorem 2.2. □

In the following for given cutting vectors $k = (k_1,\dots,k_n)$ and $k' = (k'_1,\dots,k'_n)$ we consider the vector $\kappa = (\kappa_1,\dots,\kappa_n)$, where

$$\kappa_i := k'_i - k_i,\ i = 1,\dots,n.$$

Corollary 3.2. *Let $k = (k_1,\dots,k_n)$ and $k' = (k'_1,\dots,k'_n)$ be the cutting vectors of vector systems V and V', respectively. If $\kappa_l > 0$ for some $l \in \{1,\dots,n\}$, then there exists an index $j \in \{1,\dots,n\}$ such that*

$$k_j \leq k_{j+1} \ \wedge \ \kappa_j > 0.$$

Proof. If $k_l \leq k_{l+1}$, then the assertions holds for $j = l$.
If we assume that

$$k_i = k_{i+1} + 1 \ \text{ for } i = \begin{cases} l,\dots,j-1 & \text{if} \quad l < j \\ l,\dots,n+j-1 & \text{if} \quad l > j \end{cases} \wedge \ k_j \leq k_{j+1}$$

(from Corollary 2.3 we know that such an index j exists), then it follows that

$$k_l = \begin{cases} k_j + (j-l) & \text{if} \quad l < j \\ k_j + (n+j-l) & \text{if} \quad l > j. \end{cases}$$

On the other hand, from (2.3) it follows that

$$k'_l \leq \begin{cases} k'_j + (j-l) & \text{if} \quad l < j \\ k'_j + (n+j-l) & \text{if} \quad l > j. \end{cases}$$

Thus we obtain

$$\kappa_j = k'_j - k_j \geq k'_l - k_l = \kappa_l > 0.$$

□

Corollary 3.3. *Let* $k = (k_1, \ldots, k_n)$ *and* $k' = (k'_1, \ldots, k'_n)$ *be the cutting vectors of vector systems* V *and* V'*, respectively. If* $\kappa_j > 0$ *for some* $j \in \{1, \ldots, n\}$*, then*

$$\kappa_{j+k_j+1} \leq \begin{cases} 0 & \textit{if} \quad k_j + k_{j+k_j+1} & = n-2 \\ -1 & \textit{if} \quad k_j + k_{j+k_j+1} & \geq n-1. \end{cases}$$

Proof.

(1) If $k_j + k_{j+k_j+1} = n-2$, then

$$k'_j + k'_{j+k_j+1} = k_j + \kappa_j + k_{j+k_j+1} + \kappa_{j+k_j+1} = (n-2) + \kappa_j + \kappa_{j+k_j+1}.$$

(2) If $k_j + k_{j+k_j+1} \geq n-1$, then

$$k'_j + k'_{j+k_j+1} \geq (n-1) + \kappa_j + \kappa_{j+k_j+1}.$$

(3) Furthermore, from Theorem 2.2 we have

$$k'_j + k'_{j+k_j+1} = k'_j + k'_{j+k'_j-\kappa_j+1} \leq (n-1) + (\kappa_j - 1).$$

□

Corollary 3.4. *Let* $k = (k_1, \ldots, k_n)$ *and* $k' = (k'_1, \ldots, k'_n)$ *be the cutting vectors of vector systems* V *and* V'*, respectively. If* $\kappa_j < 0$ *for some* $j \in \{1, \ldots, n\}$*, then*

$$\kappa_{j+k_j+1} \geq \begin{cases} 1 & \textit{if} \quad k_j + k_{j+k_j+1} & = n-2 \\ 0 & \textit{if} \quad k_j + k_{j+k_j+1} & = n-1. \end{cases}$$

Proof.

(1) If $k_j + k_{j+k_j+1} = n-2$, then

$$k'_j + k'_{j+k_j+1} = (n-2) + \kappa_j + \kappa_{j+k_j+1}.$$

(2) If $k_j + k_{j+k_j+1} = n-1$, then

$$k'_j + k'_{j+k_j+1} = (n-1) + \kappa_j + \kappa_{j+k_j+1}.$$

(3) Furthermore, from Theorem 2.2 we have

$$k'_j + k'_{j+k_j+1} = k'_j + k'_{j+k'_j-\kappa_j+1} \geq (n-1) + \kappa_j\,.$$

□

Corollary 3.5. *Let $k = (k_1, \ldots, k_n)$ and $k' = (k'_1, \ldots, k'_n)$ be the cutting vectors of vector systems V and V', respectively. Then*

$$\kappa_{j+k_j} \geq \begin{cases} 0 & if \quad \kappa_j = -1 \\ 1 & if \quad \kappa_j < -1. \end{cases}$$

Proof.

(1) If $\kappa_j = -1$, then from (2.5) it follows that

$$k'_j + k'_{j+k_j} = k'_j + k'_{j+k'_j-\kappa_j} \geq n - 2.$$

(2) If $\kappa_j < -1$, then from (2.3) and (2.5) it follows that

$$k'_j + k'_{j+k_j} = k'_j + k'_{j+k'_j-\kappa_j} \geq n - (-\kappa_j).$$

(3) Furthermore, from (2.4) we have

$$k'_j + k'_{j+k_j} = k_j + k_{j+k_j} + \kappa_j + \kappa_{j+k_j} \leq (n-1) + \kappa_j + \kappa_{j+k_j}. \qquad \square$$

From now on we use a further distinction. The vector system V is called *non-degenerate* if and only if there are no two oppositely directed vectors in V. Otherwise V is said to be *degenerate*.

Corollary 3.6. *Let $k = (k_1, \ldots, k_n)$ and $k' = (k'_1, \ldots, k'_n)$ be the cutting vectors of vector systems V and V', respectively. If V' is non-degenerate and*

$$\kappa_l \leq 0 \quad \forall l \in \{1, \ldots, n\},$$

then $k = k'$.

Proof. Assume that $\kappa_j < 0$ for some $j \in \{1, \ldots, n\}$.

(1) If $\kappa_j < -1$, then $\kappa_{j+k_j} \geq 1$, by Corollary 3.5, is a contradiction.

(2) If $\kappa_j = -1$, then $\kappa_{j+k_j} \geq 0$ by Corollary 3.5.

  (a) If $\kappa_{j+k_j} > 0$, we have a contradiction.

  (b) If $\kappa_{j+k_j} = 0$, then from (2.4) we get

$$\begin{aligned} k'_j + k'_{j+k'_j+1} &= k'_j + k'_{j+k_j} = k_j - 1 + k_{j+k_j} \\ &\leq (n-1) - 1, \end{aligned}$$

  which is a contradiction. $\square$

In the following we make use of

$$\Phi_{j_1}\Phi_{j_2}(k) := \Phi_{j_1}(\Phi_{j_2}(k)) \quad \text{and} \quad \Psi_j(k) := \Phi_j\Phi_j(k).$$

If $k' = \Psi_j(k)$, then from the definition of Φ_j (see (3.1)) it follows that

$$k'_i = \begin{cases} k_i + 1 & \text{for} \quad i = j \\ k_i - 1 & \text{for} \quad i = j + k_j + 1 \\ k_i & \text{else.} \end{cases}$$

Theorem 3.7. *Let $k, k' \in \mathbb{N}^n$ be realizable as cutting vectors of vector systems V and V', respectively. Then they may be transformed into each other by a finite number of Φ_j-mappings. Each intermediate vector is realizable as cutting vector of a vector system.*

Proof. We give an inductive construction:

(1) Setting $\mathcal{L} := \{1, \ldots, n\}$, $\kappa_l^{(0)} := k'_l - k_l$, $l = 1, \ldots, n$, we consider

$$\mathcal{L}_+^{(0)} := \{l \in \mathcal{L} : \kappa_l^{(0)} > 0\}.$$

(2) If $\mathcal{L}_+^{(i)} \neq \emptyset$, then, according to Corollary 3.2, it follows that there exists $j \in \mathcal{L}_+^{(i)}$ such that $k_j^{(i)} \leq k_{j+1}^{(i)}$ and $\kappa_j^{(i)} := k'_j - k_j^{(i)} > 0$.

(a) If $k_j^{(i)} + k_{j+k_j^{(i)}+1}^{(i)} = n - 2$, then $k^{(i+1)} := \Phi_j(k^{(i)})$ is realizable as cutting vector of a vector system, and

$$\kappa_l^{(i+1)} := k'_l - k_l^{(i+1)} = \begin{cases} \kappa_l^{(i)} - 1 & \text{if} \quad l = j \\ \kappa_l^{(i)} & \text{else.} \end{cases}$$

(b) If $k_j^{(i)} + k_{j+k_j^{(i)}+1}^{(i)} = n - 1$, then $k^{(i+1)} := \Psi_j(k^{(i)})$ is realizable as cutting vector of a vector system, and

$$\kappa_l^{(i+1)} := k'_l - k_l^{(i+1)} = \begin{cases} \kappa_l^{(i)} - 1 & \text{if} \quad l = j \\ \kappa_l^{(i)} + 1 & \text{if} \quad l = j + k_j^{(i)} + 1 \\ \kappa_l^{(i)} & \text{else.} \end{cases}$$

From Corollary 3.3 it follows that

$$\kappa_{j+k_j^{(i)}+1}^{(i+1)} = \kappa_{j+k_j^{(i)}+1}^{(i)} + 1 \leq 0.$$

Setting $\mathcal{L}_+^{(i+1)} := \{l \in \mathcal{L} : \kappa_l^{(i+1)} > 0\}$, in both the cases (a) and (b) we have $\mathcal{L}_+^{(i+1)} \subseteq \mathcal{L}_+^{(i)}$. If $\mathcal{L}_+^{(i+1)} \neq \emptyset$, the procedure of this part works again.

(3) If $\mathcal{L}_+^{(i)} = \emptyset$ and V' is non-degenerate, from Corollary 3.6 it follows that $k' = k^{(i)}$. Thus, we have to consider only the case where $\mathcal{L}_+^{(i)} = \emptyset$ and V' is degenerate. If $\kappa_j^{(i)} < 0$ for some $j \in \mathcal{L}$, then by Corollary 3.4 we obtain

$$\kappa_{j+k_j^{(i)}+1}^{(i)} \geq \begin{cases} 1 & \text{if} \quad k_j^{(i)} + k_{j+k_j^{(i)}+1}^{(i)} = n - 2 \\ 0 & \text{if} \quad k_j^{(i)} + k_{j+k_j^{(i)}+1}^{(i)} = n - 1. \end{cases}$$

Only the second case is relevant (notice $\mathcal{L}_+^{(i)} = \emptyset$). Thus, we may assume

$$k_j^{(i)} + k_{j+k_j^{(i)}+1}^{(i)} = n - 1 \qquad (*_1)$$

and

$$\kappa_{j+k_j^{(i)}+1}^{(i)} := k'_{j+k_j^{(i)}+1} - k_{j+k_j^{(i)}+1}^{(i)} = 0. \qquad (*_2)$$

Furthermore, from (2.4) we know that

$$k_j^{(i)} + k_{j+k_j^{(i)}}^{(i)} \leq n - 1. \qquad (*_3)$$

(a) If $\kappa_j^{(i)} = -1$, from (2.5) it follows that

$$k'_j + k'_{j+k_j^{(i)}} = k'_j + k'_{j+k'_j+1} \geq n - 2.$$

Thus, together with $(*_3)$ we obtain

$$\kappa_{j+k_j^{(i)}}^{(i)} = k'_{j+k_j^{(i)}} - k_{j+k_j^{(i)}}^{(i)} \geq (n-2) - k'_j + k_j^{(i)} - (n-1) = 0.$$

Again, only the case $\kappa_{j+k_j^{(i)}}^{(i)} = 0$ is relevant, and by $(*_3)$ it follows that

$$k'_j + k'_{j+k'_j+1} = k_j^{(i)} - 1 + k'_{j+k_j^{(i)}} = k_j^{(i)} + k_{j+k_j^{(i)}}^{(i)} - 1 \leq n - 2.$$

Together with (2.5) we obtain equality. Consequently,

$$k_j^{(i)} + k_{j+k_j^{(i)}}^{(i)} = n - 1. \qquad (*_4)$$

From $(*_1)$ and $(*_4)$ we get

$$k_{j+k_j^{(i)}}^{(i)} = k_{j+k_j^{(i)}+1}^{(i)}.$$

Thus, the conditions of Theorem 3.1 are satisfied and, consequently,

$$k^{(i+1)} := \Phi_{j+k_j^{(i)}}\left(k^{(i)}\right)$$

is realizable as cutting vector of a vector system. Furthermore, from

$$k_{j+k_j^{(i)}}^{(i)} + k_{j+k_j^{(i)}+k_{j+k_j^{(i)}}^{(i)}+1}^{(i)} = k_{j+k_j^{(i)}+1}^{(i)} + k_j^{(i)} = n - 1$$

we obtain (see (3.1))

$$\kappa_j^{(i+1)} = \kappa_j^{(i)} + 1 = 0.$$

(b) The case $\kappa_j^{(i)} < -1$ is not relevant, since from Theorem 2.2 it follows that

$$k'_j + k'_{j+k_j^{(i)}} = k'_j + k'_{j+k'_j-\kappa_j^{(i)}} \geq n + \kappa_j^{(i)}$$

and, together with $(*_3)$,

$$\kappa^{(i)}_{j+k_j^{(i)}} = k'_{j+k_j^{(i)}} - k^{(i)}_{j+k_j^{(i)}} \geq (n + \kappa_j^{(i)}) - k'_j + k_j^{(i)} - (n-1) = 1.$$

If $\kappa_j^{(i+1)} < 0$ for some $j \in \mathcal{L}$, then the procedure of this part works again.

(4) The procedure of part (2) and (3) stops at $i = \tilde{i}$ if $\kappa_l^{(\tilde{i})} = 0$, $l = 1, \ldots, n$. □

Example 3.8. We consider

$$\begin{aligned} k &= (5,6,5,5,5,4,3,3,4,4) \\ k' &= (7,6,5,5,4,3,2,1,4,7). \end{aligned}$$

The vectors k and k' satisfy the conditions of Theorem 2.2. We obtain

$$\begin{aligned} k^{(1)} &= \Phi_1(k) &&= (6,6,5,5,5,4,3,3,4,4) \\ k^{(2)} &= \Psi_1(k^{(1)}) &&= (7,6,5,5,5,4,3,2,4,4) \\ k^{(3)} &= \Psi_{10}(k^{(2)}) &&= (7,6,5,5,4,4,3,2,4,5) \\ k^{(4)} &= \Psi_{10}(k^{(3)}) &&= (7,6,5,5,4,3,3,2,4,6) \\ k^{(5)} &= \Psi_{10}(k^{(4)}) &&= (7,6,5,5,4,3,2,2,4,7) \\ k' &= \Phi_{10}(k^{(5)}) &&= (7,6,5,5,4,3,2,1,4,7). \end{aligned}$$

Thus $k' = \Phi(k)$ holds, where $\Phi := \Phi_{10}\Psi_{10}\Psi_{10}\Psi_{10}\Psi_1\Phi_1$.

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