Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 1, 141-150.

Equivariant Higher K-theory for Compact Lie Group Actions

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Introduction

The aim of this paper is to construct an equivariant higher K-theory for compact Lie group actions in a way analogous to the ones constructed in [2], [3] for finite groups and [6] for profinite groups.

In Section 1, we discuss the category $\mathcal{A}(G)$ of homogeneous spaces on which Mackey functors are defined. In Section 2 we define the higher K-groups $K_n^G(G/H, \mathcal{C})$, $n \geq 0$ for any exact category $\mathcal{C}$ and show that $K_n^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor and that $K_n^G(-, \mathcal{C})$ are $K_0^G(-, \mathcal{C})$-modules. In Section 3, we explore induction techniques in the style of [4] to show that $K_n^G(G/G, \underline{M}(\mathbb{C}))$ (all $n \geq 0$) are hyperelementary computable where $\underline{M}(\mathbb{C})$ is the category of finite dimensional vector spaces over the complex numbers.

In a final section, we briefly discuss possible generalisations of the foregoing to the category of G-spaces of G-homotopy type of G-CW complexes.

Notations and Conventions

Let $\mathcal{C}$ be an exact category in the sense of Quillen [9]. Then for all $n \geq 0$ we write $K_n(\mathcal{C})$ for the Quillen K-groups $\pi_{n+1}(BQ\mathcal{C})$. If $\mathcal{B}$ is any small category, we denote by $[\mathcal{B}, \mathcal{C}]$ the category of covariant functors from $\mathcal{B}$ to $\mathcal{C}$. Unless otherwise stated, G denotes a compact Lie group, and maps between G-spaces are G-equivariant. If H is a closed subgroup of G, we write $N_G(H)$ or just NH for the normaliser of H in G. Also we write $R(G)$ for the complex representation ring of G.

0138-4821/93 \$ 2.50

1. Mackey functors on the category $\mathcal{A}(G)$ of homogeneous spaces

1.1. Let G be a compact Lie group, X a G-space. The component category $\pi_0(G, X)$ is defined as follows: Objects of $\pi_0(G, X)$ are homotopy classes of maps $\alpha : G/H \to X$ where H is a closed subgroup of G. A morphism from $[\alpha] : G/H \to X$ to $[\beta] : G/K \to X$ is a G-map $\sigma : G/H \to G/K$ such that $\beta\sigma$ is G-homotopic to α.

Note that since $\mathrm{Hom}(G/H, X) \simeq X^H$ where $\varphi \to \varphi(eH)$, we could consider objects of $\pi_0(G, H)$ as pairs (H, c) where $c \in \pi_0(X^H) =$ the set of path components of X^H.

1.2. A G-ENR (Euclidean Neighbourhood Retract) is a G-space that is G-homeomorphic to a G-retract of some open G-subset of some G-module V. Let Z be a compact G-ENR, $f : Z \to X$ a G-map. For $\alpha : G/H \to X$ in $\pi_0(G, X)$, we identify α with the path component X^H_α into which G/H is mapped by α.

Put $Z(f, \alpha) = Z^H \cap f^{-1}(X^H_\alpha) :=$ subspace of Z^H mapped under f into X^H_α. The action of $N_\alpha H/H$ on Z^H induces an action of $N_\alpha H/H$ on $Z(f, \alpha)$ i.e. $Z(f, \alpha)$ is an $\mathrm{Aut}(\alpha)$-space (see [4]). Note that $\mathrm{Aut}(\alpha) = \{\sigma : G/H \to G/H | \alpha\sigma \simeq \alpha\}$ and $N_\alpha(H)_H$, the isotropy group of $\alpha \in \pi_0(X^H)$ is isomorphic to $\mathrm{Aut}(\alpha)$.

1.3. Let $\{Z_i\}$ be a collection of G-ENR, $f_i : Z_i \to X$, G-maps. Say that $f_i : Z_i \to X$ is equivalent to $f_j : Z_j \to X$ if and only if for each $\alpha : G/H \to X$ in $\pi_0(G, X)$, the Euler characteristic $\chi(Z(f_i, \alpha)/\mathrm{Aut}(\alpha)) = \chi(Z(f_j, \alpha)/\mathrm{Aut}(\alpha))$.

Let $\mathcal{U}(G, X)$ be the set of equivalence classes $[f : Z \to X]$ where addition is given by $[f_0 : Z_0 \to X] + [f_1 : Z_1 \to X] = [f_0 + f_1 : Z_0 + Z_1 \to X]$; the identity element is $\phi \to X$; and the additive inverse of $[f : Z \to A]$ is $[f \circ p : Z \times A \to Z \to X]$, where A is a compact G-ENR with trivial G-action and $\chi(A) = -1$ (see [4]).

Then $\mathcal{U}(G, X)$ is the free Abelian group generated by $[\alpha]$, $\alpha \in \pi_0(G, X)$ i.e. $[f : Z \to X] = \Sigma n(\alpha)[\alpha]$, where $G/H \times E^n \subset Z$ is an open n-cell of Z, and the restriction of f to $G/H \times E^n$ defines an element $[\alpha]$ of $\mathcal{U}(G, X)$.

The cell is called an n-cell of type α. Let $n(\alpha) = \Sigma(-1)^i n(\alpha, i)$ where $n(\alpha, i) =$ number of i-cells of type α (see [4]).

If X is a point, write $\mathcal{U}(G)$ for $\mathcal{U}(G, X)$.

1.4. For a compact Lie group G, the category $\mathcal{A}(G)$ is defined as follows: $ob\mathcal{A}(G) :=$ homogeneous spaces G/H. The morphisms in $\mathcal{A}(G)(G/H, G/K)$ are the elements of the Abelian group $\mathcal{U}(G, G/H \times G/K)$ and have the form $\alpha : G/L \to G/H \times G/K$ which can be represented by diagram $\{G/H \overset{\alpha}{\leftarrow} G/L \overset{\beta}{\rightarrow} G/K\}$, so that $\mathcal{U}(G, G/H \times G/K) =$ free Abelian group on the equivalence classes of diagrams $G/H \overset{\alpha}{\leftarrow} G/L \overset{\beta}{\rightarrow} G/K$ where two such diagrams are equivalent if there exists an isomorphism $\sigma : G/L \to G/L'$ such that the diagram

$$\begin{array}{ccccc} & & G/L & & \\ & \nearrow & & \searrow & \\ G/H & & \downarrow \sigma & & G/K \\ & \searrow & & \nearrow & \\ & & G/L' & & \end{array}$$

commutes up to homotopy.

Composition of morphisms is given by a bilinear map

$$\mathcal{U}(G, G/H_1 \times G/H_2) \times \mathcal{U}(G, G/H_2 \times G/H_3) \to \mathcal{U}(G, G/H_1 \times G/H_3)$$

where the composition of $(\alpha, \beta_1) : A \to G/H_1 \times G/H_2$ and $(\beta_2, \gamma) : B \to G/H_2 \times G/H_3$ yields a G-map $(\alpha\bar{\alpha}, \gamma\bar{\gamma}) : C \to G/H_1 \times G/H_3$, where $\bar{\gamma}$, $\bar{\alpha}$ are maps $\bar{\gamma} : C \to B$ and $\bar{\alpha} : C \to A$, respectively.

1.5. Remarks. (i) Each morphism $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ is the composition of special types of morphisms

$$G/H \xleftarrow{\alpha} G/L \xrightarrow{\text{id}} G/L \quad \text{and} \quad G/L \xleftarrow{\text{id}} G/L \xrightarrow{\beta} G/K\,.$$

(ii) Let π_0 (or G) be the homotopy category of the orbit category or (G), that is, the objects of π_0 (or G) are the homogeneous G-spaces G/H and morphisms are homotopy classes $[G/L \to G/K]$ of G-maps $G/L \to G/K$. We have a covariant functor π_0 (or G) $\to \mathcal{A}(G)$ given by $[G/L \xrightarrow{\beta} G/K] \to (G/L \xleftarrow{\text{id}} G/L \xrightarrow{\beta} G/K)$ and a contravariant functor π_0 (or G) $\to \mathcal{A}(G)$ given by

$$[G/H \to G/L] \to (G/H \leftarrow G/L \xrightarrow{\text{id}} G/L)$$

(iii) Addition is defined in $\mathcal{A}(G)(G/H, G/K) = \mathcal{U}(G, G/H \times G/K)$ by

$$(G/H \leftarrow G/L \to G/K) + (G/H \leftarrow G/L' \to G/K)$$

$$= (G/H \leftarrow (G/L)\dot{\cup}(G/L') \to G/K)$$

where $(G/L)\dot{\cup}(G/L')$ is the topological sum of G/L and G/L'.

1.6. Let R be a commutative ring with identity. A Mackey functor M from $\mathcal{A}(G)$ to $\underline{R\text{-mod}}$ is a contravariant additive functor. Note that M is additive if

$$M : \mathcal{A}(G)(G/H, G/K) \to \underline{R\text{-mod}}(M(G/K), M(G/H))$$

is an Abelian group homomorphism.

1.7. Remarks. M comprises of two types of induced morphisms

(i) If $\alpha : G/H \to G/K$ is a G-map, regarded as an ordinary morphism $\alpha_! : G/H \xleftarrow{\text{id}} G/H \xrightarrow{\alpha} G/K$ of $\mathcal{A}(G)$, we have an induced morphism

$$M(\alpha_!) = M^*(\alpha) =: \alpha^* : M(G/K) \to M(G/H)$$

(ii) If α in (i) is induced from $H \subset K$, i.e. $\alpha(gH) = gK$, call α^* the restriction morphism.

(iii) If we consider α as a transfer morphism $\alpha^! : G/H \leftarrow G/K \to G/K$ in $\mathcal{A}(G)$, then we have

$$M(\alpha^!) =: M_*(\alpha) =: \alpha_* : M(G/H) \to M(G/K)$$

and call α_* the induced homomorphism associated to α.

1.8. Let M, N, L be Mackey functors on $\mathcal{A}(G)$. A pairing $M \times N \to L$ is a family of bilinear maps $M(S) \times N(S) \to L(S)$: $(x,y) \to x \cdot y$ $(S \in \mathcal{A}(G))$ such that for each G-map $f : G/H = S \to T = G/K$ we have

$$\begin{aligned} L^*f(x,y) &= (M^*fx) \cdot (N^*fy) && (x \in M(T),\ y \in N(T)); \\ x \cdot (N_*fy) &= L_*f((M^*fx) \cdot y) && (x \in M(T),\ y \in N(S)); \\ (M_*fx) \cdot y &= L_*f(x \cdot (N^*fy)) && (x \in N(S),\ y \in N(T)). \end{aligned}$$

A Green functor $V : \mathcal{A}(G) \to \underline{R\text{-mod}}$ is a Mackey functor together with a pairing $V \times V \to V$ such that for each object S, the map $V(S) \times V(S) \to V(S)$ turns $V(S)$ into an associative R-algebra such that f^* preserves units.

If V is a Green functor, a left V-module is a Mackey functor M together with a pairing $V \times M \to M$ such that $M(S)$ is a left $V(S)$-module for every $S \in \mathcal{A}(G)$.

1.9. Remarks. The Mackey functor as defined in 1.6 is equivalent to the earlier definitions in [2], [3], [7] defined for finite and profinite groups G as functors from the category $\hat{G}$ of G-sets to $\underline{R\text{-mod}}$. Observe that if $(M_*, M^*) = M$ is a Mackey functor (bifunctor) $\hat{G} \to \underline{R\text{-mod}}$ (see [7]), we can get $\tilde{M} : \mathcal{A}(G) \to \underline{R\text{-mod}}$ by putting $\tilde{M}(G/H) = M_*(G/H) = M^*(G/H)$ on objects while a morphism $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ in $\mathcal{A}(G)$ is mapped onto $M(G/H) \underset{M_*(\alpha)}{\longleftarrow} M(G/L) \underset{M^*(\beta)}{\longleftarrow} M(G/K)$ in $\underline{R\text{-mod}}$. Then M is compatible with composition of morphisms.

Conversely, let $\tilde{M} : \mathcal{A}(G) \to \underline{R\text{-mod}}$ be given and for $(\alpha, \beta) \in \mathcal{A}(G)(G/H, G/K)$, let $\tilde{M}^*(\alpha)$ and $\tilde{M}_*(\beta)$ be as defined in 1.7. Then, we can extend $\tilde{M}$ additively to finite G-sets to obtain Mackey functors as defined in [2], [3], [7].

1.10. Universal example of a Green functor. Define $\bar{V}(G/H) := \mathcal{U}(G, G/H)$, $\mathcal{U}(H, G/G) := \bar{V}(H)$. Now, consider $\mathcal{U}(G, G/H) = \mathcal{U}(G, G/G \times G/H)$, as a morphism set in $\mathcal{A}(G)$. Then, the composition of morphisms

$$\mathcal{U}(G, G/G \times G/K) \times \mathcal{U}(G, G/H \times G/K) \to \mathcal{U}(G, G/G \times G/H)$$

defines an action of $\bar{V}$ on morphisms.

1.11. Theorem. [4] *$\bar{V} : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ is a Green functor, and any Mackey functor $M : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ is a $\bar{V}$-module.*

2. An equivariant higher K-theory for G-actions

2.1. Let G be a compact Lie group, X a G-space. We can regard X as a category $\underline{X}$ as follows. The objects of $\underline{X}$ are elements of X and for x, $x' \in X$, $\underline{X}(x, x') = \{g \in G | gx = x'\}$.

2.2. Let X be a G-space, $\mathcal{C}$ an exact category in the sense of Quillen [9]. i.e. $\mathcal{C}$ is an additive category embeddable as a full subcategory of an Abelian category $\mathcal{O}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of exact sequences

$$0 \to M' \to M \to M'' \to 0 \tag{I}$$

such that

(i) $\mathcal{E}$ is the class of sequences (I) in the $\mathcal{C}$ that are exact in $\mathcal{O}$.

(ii) $\mathcal{C}$ is closed under extensions in $\mathcal{O}$ that is, if (I) is an exact sequence in $\mathcal{O}$ and $M', M'' \in \mathcal{C}$ then $M \in \mathcal{C}$.

Let $[\underline{X}, \mathcal{C}]$ be the category of functors $\underline{X} \to \mathcal{C}$. Then $[\underline{X}, \mathcal{C}]$ is an exact category where a sequence $0 \to \zeta' \to \zeta \to \zeta'' \to 0$ is exact in $[\underline{X}, \mathcal{C}]$ if and only if

$$0 \to \zeta'(x) \to \zeta(x) \to \zeta''(x) \to 0$$

is exact in $\mathcal{C}$. In particular for $X = G/H$ in $\mathcal{A}(G)$, $[\underline{G/H}, \mathcal{C}]$ is an exact category.

2.3. Example. The most important example of $[\underline{G/H}, \mathcal{C}]$ is when $\mathcal{C}$ is the category $\underline{M}(\mathbb{C})$ of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. Here, the category $[\underline{G/H}, \underline{M}(\mathbb{C})]$ can be identified with the category of G-vector bundles on the compact G-space G/H where for any $\zeta \in [G/H, \underline{M}(\mathbb{C})]$. $x \in G/H$, $\zeta(x) \in \underline{M}(\mathbb{C})$ is the fibre $\hat{\zeta}_x$ of the vector bundle $\hat{\zeta}$ associated with ζ. Indeed, $\hat{\zeta}$ is completely determined by $\zeta_{\bar{e}}$ where $\bar{e} = eH$ (see [10]).

2.4. Definition. *For $X = G/H$ and all $n \geq 0$, define $K_n^G(X, \mathcal{C})$ as the n^{th} algebraic K-group of the exact category $[\underline{X}, \mathcal{C}]$ with respect to fibre-wise exact sequence introduced in* 2.2.

2.5. Theorem. (i) *For all $n \geq 0$, $K_n^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor,*
(ii) $K_0^{\mathcal{C}}(-, \mathcal{C}) : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ *is a Green functor and $K_n^G(-, \mathcal{C})$ is a $K_0^G(-, \mathcal{C})$-module for all $n \geq 0$.*

Before proving 2.5, we first briefly discuss pairings and module structures on higher K-theory of exact categories.

2.6. Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be three exact categories and $\mathcal{E}_1 \times \mathcal{E}_2$ the product category. An exact pairing $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E} : (M_1, M_2) \to M_1 \circ M_2$ is a covariant functor from $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}$ such that $\mathcal{E}_1 \times \mathcal{E}_2((M_1, M_2), (M_1', M_2')) = \mathcal{E}_1(M_1, M_1') \times \mathcal{E}_2(M_2, M_2') \to \mathcal{E}(M_1 \circ M_2,\ M_1' \circ M_2')$ is bi-additive and bi-exact, that is, for a fixed M_2, the functor $\mathcal{E}_1 \to \mathcal{E}$ given by $M_1 \to M_1 \circ M_2$ is additive and exact and for fixed M_1, the functor $\mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ is additive and exact. It follows from [12] that such a pairing gives rise to a K-theoretic product $K_i(\mathcal{E}_1) \times K_j(\mathcal{E}_2) \to K_{i+j}(\mathcal{E})$ and in particular to natural pairing $K_0(\mathcal{E}_1) \circ K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$ which could be defined as follows.

Any object $M_1 \in \mathcal{E}$ induces an exact functor $M_1 : \mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ and hence a map $K_n(M_1) : K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$. If $M_1' \to M_1 \to M_1''$ is an exact sequence in $\mathcal{E}$, then we have an exact sequence of exact functors $M_1^{'*} \to M_1^* \to M_1^{''*}$ from $\mathcal{E}_2$ to $\mathcal{E}$ such that for each object $M_2 \in \mathcal{E}_2$ the sequence $M_1'(M_2) \to M_1^*(M_2) \to M_1^{''*}(M_2)$ is exact in $\mathcal{E}$ and hence by a result of Quillen [9] induces a relation $K_n(M_1^{'*}) + K_n(M_1^{''*}) = K_n(M_1^*)$. So the map $M_1 \to K_n(M_1) \in \mathrm{Hom}(K_n(\mathcal{E}_2),\ K_n(\mathcal{E}))$ induces a homomorphism $K_0(\mathcal{E}_1) \to \mathrm{Hom}(K_n(\mathcal{E}), K_n(\mathcal{E}))$ and hence a pairing $K_0(\mathcal{E}_1) \times K_n(\mathcal{E}) \to K_n(\mathcal{E})$.

If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and the pairing $\mathcal{E} \times \mathcal{E}$ is naturally associative (and commutative), then the associated pairing $K_0(\mathcal{E}) \times K_0(\mathcal{E}) \to K_0(\mathcal{E})$ turns $K_0(\mathcal{E})$ into an associative (and commutative) ring which may not contain the identity.

Now, suppose that there is a pairing $\mathcal{E} \circ \mathcal{E}_1 \to \mathcal{E}_1$ which is naturally associative with respect to the pairing $\mathcal{E} \circ \mathcal{E} \to \mathcal{E}$, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a $K_0(\mathcal{E})$-module which may or may not be unitary. However, if $\mathcal{E}$ contains a unit i.e. an object E such that $E \circ M = M \circ \mathcal{E}$ are naturally isomorphic to M for each $\mathcal{E}$-object M, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a unitary $K_0(\mathcal{E})$-module.

Proof of 2.5. (i) It is clear from the definition of $K_n^G(G/H, \mathcal{C})$ that for any $G/H \in \mathcal{A}(G)$, $K_n^G(G/H, \mathcal{C}) \in \underline{\mathbb{Z}\text{-mod}}$. Now suppose that $(G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K) \in \mathcal{A}(G)(G/H, G/K)$, then $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ goes to

$$K_n^G(G/H, \mathcal{C}) \longleftarrow K_n^G(G/L, \mathcal{C}) \longleftarrow K_n^G(G/K, \mathcal{C})$$

in $\underline{R\text{-mod}}(K_n^G(G/K, \mathcal{C})$. If we write K_n^G for $K_n^G(-, \mathcal{C})$, then

$$K_n^G(G/H \leftarrow (G/L)\dot{\cup}(G/L') \to G/K)$$

$$= K_n^G(G/H \leftarrow G/L \to G/K) + K_n^G(G/H \leftarrow G/L' \to G/K) \ .$$

Hence $K_n^G(-, \mathcal{C})$ is a Mackey functor.

(ii) From the discussion in 2.6, it is clear that if we put $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} = [G/H, \mathcal{C}]$, then

$$K_0^G(G/H, \mathcal{C}) \times K_0^G(G/H, \mathcal{C}) \to K_0^G(G/H, \mathcal{C})$$

turns $K_0^G(G/H, \mathcal{C})$ into a commutative ring with identity. Also,

$$K_0^G(G/H, \mathcal{C}) \times K_n^G(G/H, \mathcal{C}) \to K_n^G(G/H, \mathcal{C})$$

turns

$$K_n^G(G/H, \mathcal{C}) \quad \text{into} \quad K_0^G(G/H, \mathcal{C})\text{-modules}$$

Hence the result.

2.7. Examples. (i) In general $[G/H, \mathcal{C}]$ = category of H-representations in $\mathcal{C}$. Hence $[G/G, \mathcal{C}]$ = category of G representations in $\mathcal{C}$. If $\mathcal{C} = \underline{M}(\mathbb{C})$, the category of finite dimensional vector spaces over the complex numbers $\mathbb{C}$, $K_0^G(G/G, \underline{M}(\mathbb{C}))$ is the complex representation ring denoted by $R_{\mathbb{C}}(G)$ or simply $R(G)$ in the literature.

(ii) If $\mathcal{C} = \underline{M}(R)$:= category of finitely generated R-modules, where R is a Noetherian ring compatible with the topological structure of G, then $K_n^G(G/H, \underline{M}(R) \simeq G_n(RH)$.

(iii) If $\mathcal{C} = \underline{P}(R)$ = category of finitely generated projective R-modules, we have

$$K_n^G(G/H, \underline{P}(R)) = G_n(H, R) \text{ and}$$

when R is regular, $G_n(R, H) \simeq G_n(RH)$.

3. Induction theory for equivariant higher K-functors

In this section, we discuss the induction properties of the equivariant K-functors constructed in Section 2 leading to the proof of Theorem 3.10 below.

3.1. Definition. *Let G be a compact Lie group. A finite family $\Sigma = (G/H)_{j\in J}$ is called an inductive system. Such a system yields two homomorphisms $p(\Sigma)$ (induction map) and $i(\Sigma)$ (restriction maps) defined by*

$$\begin{array}{rcl} p(\Sigma) : \underset{j\in J}{\oplus} M(G/H_j) & \to & M(G/G) \\ (x_j|_{j\in J}) & \mapsto & \underset{j\in J}{\Sigma} p(H_j)_*(x_j) \\ i(\Sigma) : M(G/G) & \to & \underset{j\in J}{\oplus} M(G/H_j) \\ x & \mapsto & (p(H_j)^* x | j \in J) \end{array}$$

Note that $p(H)$ denotes the unique morphism $G/H \to G/G$. Σ is said to be projective if $p(\Sigma)$ is surjective and Σ is said to be injective if $i(\Sigma)$ is injective. Note that the identity [id] *of $\mathcal{U}(G, G/K \times G/H)$ has the form* (see 1.3)

$$[\mathrm{id}] = \Sigma_\alpha n_\alpha [\alpha : G/L_\alpha \to G/K \times G/H]. \qquad \text{(I)}$$

3.2. Let $S(K,H)$ be the set of α over which the summation (I) is taken and let $\alpha = (\alpha(1), \alpha(2))$ be the component of α, where $\alpha(1) : G/L_\alpha \to G/K$; $\alpha(2) : G/L_\alpha \to G/H$.

Define induction map

$$p(\Sigma, G/H) : \oplus_{j\in J}(\oplus_{\alpha\in S(H_j,H)} M(G/L_\alpha)) \to M(G/H)$$

by

$$(x(j,\alpha)) \mapsto (\Sigma_{j\in J}(\Sigma_{\alpha\in S(H_j,H)} n_\alpha \alpha(2)_* x(j,\alpha))$$

and restriction maps

$$i(\Sigma, G/H) : M(G/H) \to \oplus_{j\in J}(\oplus_{\alpha\in S(H_j,H)} M(G/L_\alpha)$$

by

$$x \to (\alpha(2)^* x | (\alpha \in (S(H_j, H) \;\; j \in J) \; .$$

3.3. Theorem. *Let $M = K_n^G(-, \mathcal{C})$, $V = K_0^G(-, \mathcal{C})$ be respectively Mackey and Green functors $\mathcal{A}(G) \to \mathbb{Z}$-mod defined in* 2.5. *If Σ is projective for V, then for each homogeneous space G/H, the induction map $p(\Sigma, G/H)$ is split surjective and the restriction map $i(\Sigma, G/H)$ is split injective.*

Proof. Since $p(\Sigma)$ is surjective for V, there exist elements $x_j \in V(G/H_j)$ such that $\Sigma(H_j)_* x_j = 1$. Define a homomorphism

$$q(\Sigma, G/H) : M(G/H) \to \oplus_j(\oplus_\alpha M(G/L_\alpha))$$

by

$$q(\Sigma, G/H)x = \alpha(1)^* x_j \cdot \alpha(2)^* x \text{ such that } \alpha \in S(H_j, h),$$

$j \in J$. Then $p(\Sigma, G/H)q(\Sigma, G/H)$

$$\begin{aligned} &= \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_*(\alpha(1)^* x_j \cdot \alpha(2)^* x)) \\ &= \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_* \alpha(1)^* x_j)x \\ &= \Sigma p(H)^* p(H_j)_* x_j) \cdot x \\ &= \Sigma p(H)^*(\Sigma p(H_j)_* x_j)x \\ &= p(H)^*(1)x = 1 \cdot x = x \ . \end{aligned}$$

So, $p(\Sigma, G/H)q(\Sigma, G/H)$ is the identity. Hence $q(\Sigma, G/H)$ is a splitting for $p(\Sigma, G/H)$. We can also define a splitting $j(\Sigma, G/H)$ for $i(\Sigma, G/H)$ by $j(\Sigma, G/H) : \oplus_j(\oplus_\alpha(MG/L_\alpha)) \to M(G|H)$ where

$$x(j, \alpha) \mapsto \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_* \alpha(1)^* x_j \circ x(j, \alpha)) \ .$$

3.4. Remarks. As will be seen below (3.6), $V = K_0^G(-, \mathcal{C})$ has "defect sets" $D(V)$ and so $\Sigma = \{G/H | H \in D(V)\}$ is projective for V. It would then mean that an induction theorem for $K_0^G(-, \mathcal{C})$ implies a similar theorem for $K_n^G(-, \mathcal{C})$.

3.5. Definition. *A finite set E of conjugacy classes (H) is an induction set for a Green functor V if $\oplus V(G/H) \to V(G/G)$ given by*

$$(x(H)) \to \Sigma p(H)_* x(H) \text{ is surjective} \ .$$

Define $E \leq F$ iff for each $(H) \in E$, there exists $(K) \in F$ such that $(H) \leq (K)$ i.e. H is subconjugate to K. Then $\leq$ is a partial ordering on induction sets.

3.6. Lemma. *Every Green functor V possesses a minimal induction set $D(V)$, called the defect set of V.*

For proof see [4]. Hence $K_0^G(-, \mathcal{C})$ has defect sets.

3.7. Let V be a Green functor and M be a V-module. Define homomorphisms

$$p_1, p_2 : \bigoplus_{i,j \in J} \bigoplus_{\alpha \in S(i,j)} M(G/L_\alpha) \to \bigoplus_{k \in J} M(G/H_i)$$

by

$$p_2(x(i, j, \alpha)) = \sum_{i \in J} \sum_{\alpha \in S(i,j)} \eta_\alpha \alpha(2)_* x(i, j, \alpha)$$

and

$$p_1(x(i, j, \alpha)) = \sum_{i \in J} \sum_{\alpha \in S(i,j)} \eta_\alpha \alpha(1)_* x(i, j, \alpha)$$

where $S(i, j) = S(H_i, H_j)$ and $\alpha \in S(i, j)$ is in the decomposition of $[id] \in \mathcal{U}(G, G/H_i \times G/H_j)$.

3.8. Theorem. *Let $M = K_n^G(-, \mathcal{C})$. Then there exists an exact sequence*

$$\oplus_{i,j \in J}(\oplus_{\alpha \in S(i,j)} M(G/L_\alpha) \stackrel{p_2 - p_1}{\longrightarrow} \oplus_{k \in J} M(G/H_k) \stackrel{p}{\longrightarrow} M(G/G) \to 0.$$

Proof. We have seen in 3.3 that p is surjective through the construction of a splitting homomorphism q such that pq=identity. We now construct a homomorphism q_1 such that $(p_2 - p_1)q_1 + qp = \text{id}$ from which exactness follows.

Since p_2 is defined as $\oplus_{k\in J}p(\Sigma G|H_k)$ we define $q_1 = \oplus_{k\in J}q(G/H_k)$ and obtain as in the proof of 3.3 that $p_2q_1 =$ identity. One can also show that $p_1q_1 = qp$. Hence the result.

3.9. Definition. *A subgroup C of G is said to be cyclic if powers of a generator of C are dense in G. If p is a rational prime, then a subgroup K of G is called p-hyperelementary if there exists an exact sequence $1 \mapsto C \to K \to P \to 1$ where P is a finite p-group and C a cyclic group such that the order of C/C_0 is prime to p. Here C_0 is the component of the identity in C. It is called hyperelementary if it is p-hyperelementary for some p.*

Let $\mathcal{H}$ be the set of hyperelementary subgroups of G. We now have the following result which is the goal of this section and typifies results that can be obtained.

3.10. Theorem. *Let $M = K_n^G(-, \underline{M}(\mathbb{C}))$. Then $\oplus_{\mathcal{H}}M(G/H) \to M(G/G)$ is surjective (i.e. M satisfies hyper-elementary induction) i.e. $M(G/G)$ can be computed in terms of p-hyperelementary subgroups of G).*

Proof. It suffices to show that if $V = K_0^G(-, \underline{M}(\mathbb{C}))$ then $\underset{H\in\mathcal{H}}{\oplus} V(G/H) \to V(G/G)(I)$ is surjective since $K_n^G(-, \underline{M}(\mathbb{C})$ is a V-module.

Now it is clear from example 2.7 that $K_0^G(G/G, \underline{M}(\mathbb{C})$ is the complex representation ring $R(G)$. Moreover, it is known (see [11]) that $R(G)$ is generated as an Abelian group by modules induced from hyperelementary subgroups of G. This is equivalent to the surjectivity of $\underset{H\in\mathcal{H}}{\oplus} V(G/H) \to V(G/G)$.

4. Remarks on possible generalizations

4.1. Let $\mathcal{B}$ be a category with finite sums, a final object and finite pull-backs (and hence finite products).

A Mackey functor $M : \mathcal{B} \to \mathbb{Z}$-mod is a bifunctor $M = (M_*, M^*)$, M_* covariant, M^* contravariant such that $M(X) = M_*(X) = M^*(X)$ for all $X \in \mathcal{B}$ and
(i) For any pullback diagram

$$\begin{array}{ccc} A' & \xrightarrow{p_2} & A_2 \\ \downarrow p_1 & & \downarrow f_2 \\ A_1 & \xrightarrow{f_1} & A \end{array} \quad \text{in } \mathcal{B}$$

the diagram

$$\begin{array}{ccc} M(A') & \xrightarrow{p_{2*}} & M(A_i) \\ \uparrow p_1^* & & \uparrow f_2^* \\ M(A_i) & \xrightarrow{f_{2*}} & M(A) \end{array}$$

(ii) M^* transforms finite coproducts in $\mathcal{B}$ over finite products in $\underline{\mathbb{Z}\text{-mod}}$.

4.2. Example. Now suppose that G is a compact Lie group. Let $\mathcal{B}$ be the category of G-spaces of the G-homotopy type of G-CW-complexes (e.g. G-ENR spaces, see [4] or [8]). Then $\mathcal{B}$ is a category with finite coproducts (topological sums), final object and finite pullbacks (fibred products) (see [1]). Hence a Mackey functor is defined on $\mathcal{B}$ along the lines of 4.1.

Hence, in a way analogous to what was done in [2] or [3] we could define for $X, Y \in \mathcal{B}$, the notion of Y-exact sequences in the exact category $[\underline{X}, \mathcal{C}]$ (where $\mathcal{C}$ is an exact category) and obtain $K_n^G(X, \mathcal{C}, Y)$ as the n^{th} algebraic K-group of $[\underline{X}, \mathcal{C}]$ with respect to Y-exact sequences.

We could also have the notion of an element $\zeta \in [\underline{X}, \mathcal{C}]$ being Y projective and obtain a full subcategory $[\underline{X}, \mathcal{C}]_Y$ of Y-projective functors in $[\underline{X}, \mathcal{C}]$ so that we could obtain $P_n^G(X, \mathcal{C}, Y)$ as the n^{th} algebraic K-group of $[X, \mathcal{C}]_Y$ with respect to split exact sequences and then show that $K_n^G(-, \mathcal{C}, Y)$, $P_n^G(-, \mathcal{C}, Y)$: $\mathcal{B} \to \mathbb{Z}$-mod are Mackey functors and that $K_0^G(-, \mathcal{C}, Y) : \mathcal{B} \to \mathbb{Z}$-mod is a Green functor and $K_n^G(-, \mathcal{C}, Y)$, $P_n^G(-, \mathcal{C}, Y)$ are $K_0^G(-, \mathcal{C}, T)$-modules in a way analogous to what was done in [2], [3].

It is hoped to explore these possibilities for further results in a future paper.

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Received April 3, 1998