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Initially Koszul Algebras

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Introduction

In this paper we study initially Koszul algebras. Let $R = K[X_1, \dots X_n]/I$ be a homogeneous K-algebra where K is a field and $I \subset (X_1, \dots, X_n)^2$ is graded ideal with respect to the standard grading $\deg(X_i) = 1$. Such an algebra R is Koszul, if its residue class field has an R-free linear resolution. So far, Koszul algebras have been discussed in several contexts. In [9] Fröberg gives a survey on this subject.

An effective method to show that an algebra is Koszul has been introduced by Conca, Trung and Valla [6]. They have defined Koszul filtrations, that is a family F of ideals generated by linear forms with the following properties: The ideal (0) and the maximal homogeneous ideal $\mathfrak{m}$ of R belong to F, and for every $I \in F$, $I \neq (0)$, there exists $J \in F$ such that $J \subset I$, I/J is cyclic and $J \colon I \in F$. It is easy to see that an algebra which admits a Koszul filtration is Koszul (see [6]).

We call a K-algebra R initially Koszul (i-Koszul for short) with respect to a sequence $x_1, \dots, x_n \in R_1$, if the flag $F = \{(x_1, \dots, x_i) \colon i = 0, \dots, n\}$ is a Koszul filtration of R. Conca, Rossi and Valla have proved that i-Koszulness implies a quadratic Gröbner basis with respect to the reverse lexicographic order on $K[X_1, \dots, X_n]$ induced by $X_1 < \dots < X_n$ (see [5]).

In the first section of the article we give a condition on $\mathrm{in}(I)$ which characterizes i-Koszulness of R with respect to $X_1 + I, \dots, X_n + I$. Using this criterion we show that i-Koszulness is preserved under tensor products over K. Moreover, if R is i-Koszul and the defining ideal I is generated by monomials of degree 2, then the d-th Veronese subring of R is again i-Koszul.

In Section 3 we study algebras for which generic flags are Koszul filtrations. We will see that this is equivalent to the property that I has a 2-linear resolution. Furthermore we

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discuss algebras which are i-Koszul with respect to any K-basis of R_1. We call such algebras universally initially Koszul (u-i-Koszul for short). In case that K is algebraically closed and $\operatorname{char}(K) \neq 2$ we classify all u-i-Koszul algebras, showing that $I = (0)$ or $I = (X_1, \dots, X_n)^2$ or $I = (g^2)$ for some linear form g.

In the last section we study homogeneous semigroup rings. Let $G = \{\alpha_1, \dots, \alpha_k\}$ be a minimal system of generators of an affine semigroup in $\mathbb{N}^n$. We say a semigroup ring R is i-Koszul, if R is i-Koszul with respect to the semigroup generators $X^{\alpha_1}, \dots, X^{\alpha_k}$. R is said to be u-i-Koszul, if R is i-Koszul with respect to all permutations of the semigroup generators.

We consider natural shellability of the divisor poset Σ of R which is closely related to Λ-shellability in [1]. Let $\lambda : G \to \Lambda$ be a map which totally orders the generators. For any semigroup element α the lexicographic order on Λ^r gives a linear order $>$ on the maximal chains of the interval $[1, \alpha]$. R is said to be naturally shellable, if for each semigroup element α the interval $[1, \alpha]$ is shellable with order $>$. Using a lemma of Hibi we show that an i-Koszul semigroup ring is natural shellable. We also show that a u-i-Koszul semigroup is already a polynomial ring.

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1. Notation and definitions and background

In this paper $S = K[X_1, \dots, X_n]$ denotes always the polynomial ring and $\mathfrak{m} = (X_1, \dots, X_n)$ the graded maximal ideal of S. We set $R = S/I$ where $I \subset \mathfrak{m}^2$ is a homogeneous ideal. We recall from [6] the following definition:

Definition 1.1. *Let R be a homogeneous K-algebra. A family F of ideals of R is called a Koszul filtration* of R, if:

(a) *Every ideal $J \in F$ is generated by linear forms,*

(b) *The ideal (0) and the maximal homogeneous ideal of R belong to F and*

(c) *For every $J \in F$, $J \neq 0$, there exists $L \in F$ such that $L \subset J$, J/L is cyclic and $L : J \in F$.*

The following is noted in [6].

Proposition 1.2. *Let F be a Koszul filtration of R. Then $\operatorname{Tor}_i^R(R/J, K)_j = 0$ for $i \neq j$ and for all $J \in F$. In particular, the homogeneous maximal ideal of R has a system of generators $x_1, \dots, x_n$ such that all ideals $(x_1, \dots, x_j)$ with $j = 1, \dots, n$ have a linear R-free resolution and R is Koszul.*

Definition 1.3. *Let $x_1, \dots, x_n \in R_1$. We call R initially Koszul (i-Koszul for short) with respect to $x_1, \dots, x_n$, if $F = \{(x_1, \dots, x_i) : i = 0, \dots, n\}$ is a Koszul filtration.*

In order to simplify notation we say that $R = S/I$ is i-Koszul, if R is initially Koszul with respect to $X_1 + I, \dots, X_n + I$. Koszul filtrations as in 1.3 which are generated by a flag of linear subspaces of R, first considered in [4], are called Gröbner flags. The reason for this naming is the following result.

Theorem 1.4. [Conca, Rossi, Valla] *Let $R = K[X_1, \dots, X_n]/I$ be i-Koszul. Then I has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $X_1 < X_2 < \dots < X_n$.*

By 1.2 any Koszul filtration of R contains a flag. Thus i-Koszulness is equivalent to the existence of a Koszul filtration which is as smallest as possible.

2. Characterization of i-Koszulness

In this section $>$ denotes the reverse lexicographic order induced by $X_1 < X_2 < \dots < X_n$. The following result, which was shown independently in [5], characterizes i-Koszulness in terms of initial ideals.

Theorem 2.1. *The following statements are equivalent:*
(a) *$R = K[X_1, \dots, X_n]/I$ is i-Koszul.*
(b) *$R' = K[X_1, \dots, X_n]/\mathrm{in}_>(I)$ is i-Koszul.*
(c) *I has a quadratic Gröbner basis with respect to $>$ and if $X_iX_j \in \mathrm{in}_>(I)$ for some $i < j$, then $X_iX_k \in \mathrm{in}_>(I)$ for all $i \le k < j$.*

For the proof of 2.1 we need the following property concerning the chosen term order $>$.

Lemma 2.2. *Let $I \subset S$ be a graded ideal and set $\bar{S} = K[X_2, \dots, X_n]$. Let $\sigma : S \to \bar{S}$ be the K-algebra homomorphism with $X_1 \mapsto 0$, $X_i \mapsto X_i$ for $i > 1$. Suppose that $g_1, \dots, g_t$ is a Gröbner basis of I such that $X_1 \nmid \mathrm{in}(g_i)$ for $i = 1, \dots, r$ and $X_1 \mid \mathrm{in}(g_i)$ for $i = r+1, \dots, t$. Then $\sigma(g_1), \dots, \sigma(g_r)$ is a Gröbner basis of $\bar{I} = (\sigma(f)\colon f \in I)$. In particular, it holds $\overline{\mathrm{in}(I)} = \mathrm{in}(\bar{I})$.*

Proof. We use Buchberger's criterion. Since $>$ is the reverse lexicographic order we have for any $f \in S$: If $X_1 \mid \mathrm{in}(f)$, then $X_1 \mid f$ (see [7] 15.4). Thus we get $S(\sigma(g_i), \sigma(g_j)) = \sigma(S(g_i, g_j))$ for all $i, j \in \{1, \dots, r\}$, $i \neq j$ and the assertion follows immediately. □

We return now to 2.1.

Proof. We prove the equivalence of (a) and (b) by induction on n. The case $n = 1$ is trivial. Let $x_i = X_i + I$ and $x_i' = X_i + \mathrm{in}(I)$ for $i = 1, \dots, n$. Note that R is i-Koszul, if and only if
(i) R/x_1R is i-Koszul and
(ii) $0 : x_1 = (x_1, \dots, x_k)$ for some k.

Using $\mathrm{in}(X_1 + I) = (X_1) + \mathrm{in}(I)$ ([7] 15.12) and 2.2 we see that (i) is equivalent to $R'/x_1'R'$ being i-Koszul. Since $\mathrm{in}(I : X_1) = \mathrm{in}(I) : X_1$ (see [7] 15.12) we get $0 : x_1' = (x_1', \dots, x_k')$ if and only if (ii) holds. This proves the equivalence of (a) and (b). For the equivalence of (b) and (c) we need □

Proposition 2.3. *Let $R = S/I$ where $I = (m_1, \dots, m_r)$ is generated by monomials of degree 2. Then the following statements are equivalent:*
(a) *R is i-Koszul.*
(b) *If $X_iX_j \in I$ for some $j > i$, then $X_iX_k \in I$ for all $i \le k < j$.*

Proof. Let $x_k = X_k + I$ for $k = 1, \dots, n$ and $J_i = (x_1, \dots, x_i)$ for $i = 0, \dots, n$.

(a) implies (b). If $X_iX_j \in I$ with $i < j$, then $x_ix_j = 0$ and so $x_j \in J_{i-1} : J_i$. Since R is i-Koszul it follows $J_{i-1} : J_i = J_l$ for some $l \geq i-1$. But then for each $i \leq k < j$ we get $x_ix_k \in J_{i-1}$. Therefore $X_iX_k - X_lX_s \in I$ for some $l \leq i-1$ and some s. Since I is a monomial ideal this implies $X_iX_k \in I$.

(b) implies (a). We have to show that $J_{i-1} : J_i = (x_1, \dots, x_{k(i)})$ for each $i = 1, \dots, n$. Let $u \in J_{i-1} : (x_i)$, $u \neq 0$. Since I is a monomial ideal we may assume that u is a monomial. It is clear that $J_{i-1} \subset J_{i-1} : J_i$. So we assume $u \notin J_{i-1}$. It follows that $ux_i = 0$. There are $k \leq l$ such that $X_kX_l \in I$ and $X_kX_l \mid uX_i$. If $i \neq k$ and $i \neq l$, we have $u = 0$ which is a contradiction. Since $u \notin J_{i-1}$ it follows that $i = k$ and $u \in (x_l)$. Condition (b) implies that $(x_i, \dots, x_l) \subset J_{i-1} : x_i$ which yields the assertion. □

3. Applications and examples

In this chapter we use the criterion of Section 2 to show that certain algebras are i-Koszul. First we need some notation.

Definition 3.1. *Let $m \in S$ be a monomial. We write* $\max(m)$ *for the largest index i such that $X_i \mid m$. A set M of monomials is called (combinatorially) stable, if for every $m \in M$ and $j < \max(m)$ the monomial $(X_j/X_{\max(m)})m \in M$.*

With 2.3 we get immediately

Corollary 3.2. *Let I be generated by monomials of degree 2 and $G(I)$ the set of minimal generators. If $G(I)$ is stable, then $R = S/I$ is i-Koszul.*

Moreover we observe that i-Koszulness is compatible with tensor products over K.

Proposition 3.3. *If $R = K[X_1, \dots, X_n]/I$ and $R' = K[Y_1, \dots Y_m]/J$ are i-Koszul algebras, then $T = R \otimes_K R'$ is also i-Koszul.*

Proof. By 2.1 there are Gröbner bases $f_1, \dots, f_k$ of I and $g_1, \dots, g_l$ of J, such that $\text{in}(I)$ and $\text{in}(J)$ satisfy condition 2.1(c). It is $T = K[X_1, \dots, X_n, Y_1, \dots, Y_m]/Q$ with $Q = IT + JT$. We take the reverse lexicographic order on $K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ induced by $X_1 < \dots < X_n < Y_1 < \dots < Y_n$. It follows immediately from the Buchberger criterion that $f_1, \dots, f_k, g_1, \dots, g_l$ form a Gröbner basis of Q. Thus condition (b) of 2.3 is satisfied for $\text{in}(Q)$ and by 2.1 we get the assertion. □

Theorem 3.4. *Let I be generated by monomials. If $R = S/I$ is i-Koszul, then the d-th Veronese subring $R^{(d)}$ is i-Koszul for all $d > 0$.*

Proof. We first consider the case $R = S$. Let M be the set of all monomials of degree d in S. We order the elements of M such that $m_1 >_{lex} m_2 >_{lex} \dots >_{lex} m_t$. Writing $S^{(d)} \cong K[Y_1, \dots, Y_t]/J$ each monomial m_l can be identified with a residue class $y_l = Y_l + J$. Thus we define $J_l := (m_1, \dots, m_l)$ for $l = 0, \dots, t$. We have to show that for every $l = 1, \dots, t$ the ideal $J_{l-1} \colon J_l$ is generated by an initial sequence of the m_i's. We set

$$M_l = \{m \in M \colon X_r | m \text{ for some } r \leq l\}$$

for $l = 1, \ldots, t$ and $M_0 = \emptyset$. The elements of each M_l form an initial sequence $m_1, m_2, \ldots, m_{i_l}$. We claim that

$$J_{l-1} \colon (m_l) = (M_{\max(m_l)-1})$$

which yields the assertion. In case $l = 1$ there is nothing to prove, thus we may assume $l > 1$. Let $s = \max(m_l) - 1$. We write $m_l = X_{i_1} \cdots X_{i_d}$ with $i_1 \leq \ldots \leq i_d = s + 1$. Let $u \in J_{l-1} \colon (m_l)$. We may assume that u is a monomial. Then we have that $um_l = wm_r$ for some monomial w and $r \in \{1, \ldots, l-1\}$. We write $m_r = X_{j_1} \cdots X_{j_d}$ with $j_1 \leq \ldots \leq j_d$. Since $m_r >_{lex} m_l$, there exists $q \in \{1, \ldots, d\}$ such that $j_m = i_m$ for all $m < q$ and $j_q < i_q \leq s + 1$. The equation $um_l = wm_r$ implies

$$uX_{i_q} \cdots X_{i_d} = wX_{j_q} \cdots X_{j_d}$$

and thus we have $X_{j_q} | u$ which yields $u \in (M_s)$. Conversely, let $u \in M_s$. Then there exists $r \in \{1, \ldots, s\}$ such that $X_r | u$. We define $w = X_{i_1} \cdots X_{i_{d-1}} X_r$. It follows $w >_{lex} m_l$ and hence $w \in J_{l-1}$. Since

$$um_l = (\frac{u}{X_r} X_{i_d}) w,$$

it follows that $u \in J_{l-1} : (m_l)$.

We now consider the general case $R = S/I$. Let $x_i = X_i + I$ for $i = 1, \ldots, n$. Since I is monomial ideal, the set of all monomials which do not belong to I forms a K-basis of R. Thus each monomial $u = x_{j_1} x_{j_2} \ldots x_{j_r} \in R$ is either 0 or has a unique presentation $u = X_{j_1} X_{j_2} \ldots X_{j_r} + I$. Therefore we may identify each monomial with its residue class. We have the following relations:

$(*)$ For any two non-zero monomials $m, m' \in R$ we have $mm' = 0$ if and only if there are $i, j \in \{1, \ldots, n\}$ such that $X_i \mid m$, $X_j \mid m'$ and $X_i X_j \in I$.

$R^{(d)}$ is generated as a K-algebra by the set M of all non-zero monomials of degree d in R. As in the first case we order the monomials of M by $m_1 >_{lex} m_2 >_{lex} \ldots >_{lex} m_t$ and set $J_i = (m_1, \ldots, m_i)$ for $i = 0, \ldots, t$. We define

$$N(m_l) = \{m \in M \colon \text{there exists } i \leq j \text{ with } X_i | m_l \text{ , } X_j | m \text{ and } X_i X_j \in I\}$$

and assert that

$$J_{l-1} : (m_l) = (M_{\max(m_l)-1}, N(m_l))$$

for $l = 1, \ldots, t$. Let $a \in J_{l-1} : (m_l)$, $a \neq 0$. We may assume that a is a monomial. There are two cases to consider:

(a) $am_l = 0$. We have a relation as in $(*)$. If $a \notin (M_{\max(m_l)-1})$ then, for each index t with $X_t \mid a$, it holds $t \geq \max(m_l)$. Thus, if $X_i \mid m_l$ and $X_j \mid a$ with $X_i X_j \in I$, it follows $i \leq j$ which yields $a \in N(m_l)$.

(b) $am_l \neq 0$. We have $am_l = bm_i$ for some monomial $b \in R^{(d)}$ and some $i < l$. There is a K-linear, injective map $\sigma : R = S/I \to S$ with $m + I \mapsto m$ for all non-zero monomials $m \in R$. If $mm' \neq 0$ for two monomials $m, m' \in R$, we get $\sigma(m)\sigma(m') = \sigma(mm')$. Let $\pi : S \to R = S/I$ be the natural epimorphism. Then it holds $\pi \circ \sigma = \mathrm{id}_R$. Since σ and π respect the standard grading, these maps restrict to $R^{(d)}$ respectively $S^{(d)}$. We apply σ to the equation above and, since $am_l \neq 0$, obtain that $\sigma(a)\sigma(m_l) = \sigma(b)\sigma(m_i)$ in $S^{(d)}$. The case $R = S$ yields $\sigma(a) \in (M_{\max(\sigma(m_l))-1})$. Applying π we get $a \in (M_{\max(m_l)-1})$.

The converse inclusion $(M_{\max(m_l)-1}, N(m_l)) \subset J_{l-1} : (m_l)$ follows immediately from the case $R = S$ and the relations in $(*)$.

It remains to show that for all $l = 1, \dots, t$ the ideal $J_{l-1} : (m_l)$ is generated by an initial sequence $m_1, \dots, m_{k(l)}$. Since the elements of $M_{\max(m_l)-1}$ form already an initial sequence it suffices to prove the following: If $m_s \in N(m_l)$ for some s, then $m_{s-1} \in M_{\max(m_l)-1} \cup N(m_l)$. Let $m_l = X_{i_1} \dots X_{i_d}$ with $i_1 \leq \dots \leq i_d$. It is $i_d = \max(m_l)$. Since $m_s \in N(m_l)$ there are $i \leq j$ with $X_i | m_l$ and $X_j | m_s$ and $X_i X_j \in I$. By the chosen order we have $m_{s-1} >_{lex} m_s$. Thus there exists a k with $X_k \mid m_{s-1}$ and $k \leq j$. If $k < i_d$, we have $m_{s-1} \in M_{\max(m_l)-1}$. Otherwise we have $i \leq k \leq j$. Since R is i-Koszul we have $X_i X_k \in I$ by 2.3. This yields $m_{s-1} \in N(m_l)$. □

We now consider

Definition 3.5. (see e.g. [2]) *Let L be a finite, distributive lattice, and $K[\{X_\alpha\}_{\alpha \in L}]$ the polynomial ring over K. Consider the ideal $I_L = (X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta} \colon \alpha, \beta \in L)$ of $K[\{X_\alpha\}_{\alpha\in L}]$. The quotient algebra*

$$R_K[L] = K[\{X_\alpha\}_{\alpha\in L}]/L$$

is called the Hibi ring of L over K.

Hibi has shown that I_L has a quadratic Gröbner basis for any term order which selects, for any two incomparable elements $\alpha, \beta \in L$, the monomial $X_\alpha X_\beta$ as the initial term of $X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta}$ (see [10]). Such a term order $>$ is, for example, the reverse lexicographic term order induced by a total ordering of the variables satisfying $X_\alpha < X_\beta$, if $\operatorname{rank}(\alpha) > \operatorname{rank}(\beta)$ (see [2]). We get the following characterization:

Remark 3.6. *Let L be a finite distributive lattice and $>$ a term order on $S = K[\{X_\alpha\}_{\alpha\in L}]$ as above. Then the Hibi-Ring $R = S/I_L$ is i-Koszul if and only if R is a polynomial ring.*

Proof. If $I_L \neq (0)$, we have $X_\alpha X_\beta \in \operatorname{in}(I)$ where α and β are some incomparable elements of L, say $X_\alpha < X_\beta$. Since R is i-Koszul, it follows $X_\alpha^2 \in \operatorname{in}(I)$ by 2.1. But this would mean that α is incomparable with itself, a contradiction. □

4. u-i-Koszulness

Let R be i-Koszul. In 1.2 we have seen that K has a linear R-free resolution. If we consider R as an S-module, we can study the minimal S-free resolution of R. For the next statement we take $\operatorname{Gin}(I)$ with respect to the reverse lexicographic order induced by $X_1 > \dots > X_n$. It holds

Proposition 4.1. *Let K be an infinite field,* $\operatorname{char}(K) \neq 2$, *$I \subset S$ a graded ideal and $I \neq (0)$. The following statements are equivalent:*

(a) *I has a 2-linear S-resolution.*

(b) *$S/\operatorname{Gin}(I)$ is i-Koszul.*

(c) *$S/\operatorname{Gin}(I)$ is Koszul.*

Proof. We use some results about $\mathrm{Gin}(I)$. It is known that $\mathrm{Gin}(I)$ is a Borel-fixed ideal and $\operatorname{reg}\mathrm{Gin}(I) = \operatorname{reg}(I)$ (see [7] 20.21). Since $\mathrm{char}(K) \neq 2$ by hypothesis, we obtain that $\mathrm{Gin}(I)_2$ is stable (see [7]15.23b).

Let us assume (a). Then we have $\operatorname{reg}(I) = 2 = \operatorname{reg}\mathrm{Gin}(I)$ which implies that $\mathrm{Gin}(I)$ is generated in degree 2. Using 3.2 $S/\mathrm{Gin}(I)$ is i-Koszul. This is condition (b) which implies (c) by 1.2.

Assuming (c) we have that $\mathrm{Gin}(I)$ is generated in degree 2. Since by hypothesis K is an infinite field and $\mathrm{Gin}(I)_2$ is stable, we can use Prop.10 in [8] which yields that $\mathrm{Gin}(I)$ is 2-regular. In case $I \subset \mathfrak{m}^2$ this implies $\operatorname{reg}(\mathrm{Gin}(I)) = 2 = \operatorname{reg}(I)$. Thus I has a 2-linear resolution. □

Proposition 4.1 can be interpreted as follows:

Corollary 4.2. *I has a 2-linear resolution if and only if all generic flags are Gröbner flags.*

We may now ask for which algebras all flags are Gröbner flags. This leads us to the following

Definition 4.3. *A K-algebra $R = S/I$ is called universally initially Koszul (for short u-i-Koszul), if R is i-Koszul with respect to every K-basis $x_1, \dots, x_n \in R_1$.*

In the case that R is u-i-Koszul the property of i-Koszulness is preserved under any change of coordinates in S_1. Since this is a strong condition, we can classify all u-i-Koszul algebras in the following case:

Theorem 4.4. *Let K be algebraically closed,* $\mathrm{char}(K) \neq 2$ *and $I \subset \mathfrak{m}^2$. Then $R = S/I$ is u-i-Koszul if and only if $I = (g^2)$ for some linear form $g \in S_1$ or $I = \mathfrak{m}^2$.*

We need some preparation.

Lemma 4.5. *Let R be u-i-Koszul and $x \in R_1$. Then R/xR is also u-i-Koszul.*

Proof. Let $\bar{R} = R/xR$ and $x_2, \dots, x_n \in \bar{R}_1$ an arbitrary K-basis of $\bar{R}_1$. We have to show that $\bar{R}$ is i-Koszul with respect to this sequence. Since R is u-i-Koszul, R is i-Koszul with respect to $x, x_2, \dots, x_n$. This yields the assertion. □

Lemma 4.6. *Let R be u-i-Koszul,* $\mathrm{char}(K) \neq 2$ *and let $N \subset R_1$ denote the set of all zerodivisors in R_1. Then N is a linear subspace of R_1 and $N^2 = 0$.*

Proof. Since R is u-i-Koszul we have $(x) \subset 0 : (x)$ for all $x \in N$. This implies $x^2 = 0$ for all $x \in N$. Thus, for $x, y \in N$ we have $(x+y)(x-y) = x^2 - y^2 = 0$ and therefore $x + y \in N$. Since $\mathrm{char}(K) \neq 2$, it follows $N^2 = 0$. □

Lemma 4.7. *Let $I = (L^2)$ for some linear subspace L of S_1. Then $R = S/I$ is u-i-Koszul if and only if* $\dim_K L \in \{0, 1, n\}$.

Proof. Let R be u-i-Koszul. After a change of coordinates we may assume that $L = (X_1, \dots, X_i)$ with $i = \dim_K L$ and $I = (X_1, \dots, X_i)^2$. If $i \notin \{0, 1, n\}$, we interchange X_i and X_{i+1}. We obtain a new defining ideal J with $X_1 X_{i+1} \in J$, but $X_1 X_i \notin J$ which is a contradiction to i-Koszulness of S/J by 2.3. Conversely, let $i = \dim_K L \in \{0, 1, n\}$. If $i = 0$, there is nothing to prove. If $i = 1$, then $I = (g^2)$ for some $g \in S_1$. For any transformation we obtain a new defining ideal $J = (h^2)$ with $h \in S_1$. We observe that $\mathrm{in}(h^2)$ is a square in the term order of 2.1. The assertion follows from 2.1. If $i = n$, we have $I = (X_1, \ldots, X_n)^2$. In this case the defining ideal does not change of any transformation and we get the claim by 2.1. $\square$

Lemma 4.8. *Let K be algebraically closed,* $\mathrm{char}(K) \neq 2$ *and $R = S/I$. If $I \subset \mathfrak{m}^2$ is a principal ideal, then R is u-i-Koszul if and only if $I = (g^2)$ for some $g \in S_1$.*

Proof. If $I = (g^2)$ for some $g \in S_1$, then R is u-i-Koszul by 4.7. Let R be u-i-Koszul. Since K is algebraically closed and since $\mathrm{char}(K) \neq 2$, there exists a K-basis $Y_1, \ldots, Y_n$ of S_1 such that the generator of I is of the form $Y_1^2 + \ldots + Y_i^2$ for some $i \leq n$ (see [13]). We claim that $i = 1$ and argue by contradiction. If $i > 1$, we apply $Y_{i-1} \mapsto Y_{i-1} + \sqrt{-1}Y_i$ and $Y_j \mapsto Y_j$ for $j \neq i-1$. Then the generator f in the new coordinates has $\mathrm{in}(f) = -2\sqrt{-1}Z_{r-1}Z_r$ and thus R is not i-u-Koszul by 2.1. Therefore we have $i = 1$, and $f = Y_1^2$. $\square$

Remark 4.9. *Let $I \subset S$ have a quadratic Gröbner basis and let $f_1, \ldots, f_k$ be a minimal system of generators of I. Then there exists a minimal Gröbner basis of I which consists of K-linear combinations of $f_1, \ldots, f_k$.*

Proof of 4.4. In 4.7 and 4.8 we have already observed that R is u-i-Koszul, if $I = (g^2)$ or $I = \mathfrak{m}^2$. Let R be u-i-Koszul. By 4.6, the set N of all zerodivisors in R_1 is a linear subspace of R_1 and $N^2 = 0$. Thus, in the case that $\dim(R) = 0$ we have $N = R_1$ and so $I = (X_1, \ldots, X_n)^2$. Let now $\dim(R) > 0$. We have to show that $I = (g^2)$ for some $g \in R_1$. We use induction on $d = \dim(R)$. Let $d = 1$. We have two cases:

(a) $N = 0$. In this case R is a 1-dimensional Cohen-Macaulay ring with minimal multiplicity and every $l \in R_1$ is a non-zerodivisor. Suppose $I \neq (0)$. We show that R must be a domain and deduce a contradiction. Since $\dim(R) = 1$ and $I \neq (0)$ we have $\mathrm{emb\,dim}(R) > 1$. Let $x_i = X_i + I$ for $i = 1, \ldots, n$. x_1 is a non-zerodivisor of R. Since $\dim(R/x_1R) = 0$ and since R/x_1R is u-i-Koszul by 4.5, we get $R/x_1R = K[X_2, \ldots, X_n]/(X_2, \ldots, X_n)^2$ as we have already observed above. Since x_1 is a non-zerodivisor of R, we have $X_1^2 \notin I$. By 2.1, $S/\mathrm{in}(I)$ is i-Koszul. The term order of 2.1 implies that $X_1^2 \notin \mathrm{in}(I)$. By 2.3, we get $\mathrm{in}(I) = (X_2, \ldots, X_n)^2$. It is a general fact that the set of monomials which do not belong to $\mathrm{in}(I)$ forms a K-basis of R. In our case $x_1^i x_1, \ldots, x_1^i x_n$ form a K-basis of R_{i+1} for all $i \geq 0$. If $a \in R_i$, $i \geq 2$, is a homogeneous element we have $a \in (x_1)^{i-1}$. Suppose $ar = 0$ for some $r \in R$. We can write $a = x_1^{i-1}l$ with some linear form $l \in R_1$. It is $ar = x_1^{i-1}lr = 0$. Since x_1 and l are non-zerodivisors by assumption it follows that $r = 0$. Thus, every homogeneous element of R is a non-zerodivisor which implies that R is a domain. Therefore R is a polynomial ring in one variable because K is algebraically closed and I is homogeneous. This is a contradiction to $\mathrm{emb\,dim}(R) > 1$.

(b) $N \neq 0$. It is $I \neq (0)$. We start induction on $n = \mathrm{emb\,dim}(R)$. Let $n = 2$. By 2.1 and 4.9 $I \subset K[X_1, X_2]$ has a minimal system of generators $f_1, \ldots, f_k$ which forms a minimal Gröbner basis. Since we are in the case that $d = \dim(R) = 1$ we have $I \neq \mathfrak{m}^2$ and thus $k \leq 2$. If $k = 0$, then R is a polynomial ring. For $k = 1$ we get the assertion by 4.8. If $k = 2$, we deduce a contradiction. Since R is i-Koszul, we obtain by 2.1 that $\mathrm{in}(I) = (X_1^2, X_1X_2)$

with respect to the term order of 2.1. It follows that $I = (X_1^2, X_1X_2)$ because X_1^2, X_1X_2 are the smallest two monomials of degree two. Thus, by interchanging X_1 and X_2 we get the defining ideal $J = (X_1X_2, X_2^2)$. By 2.1 S/J is not i-Koszul which is a contradiction to R being u-i-Koszul. Let $n > 2$. We choose $x \in N$, $x \neq 0$. We may assume $x = x_1 = X_1 + I$. Since $x_1^2 = 0$ by 4.6, we have that $\dim(R/x_1R) = 1$ and $\operatorname{emb}\dim(R/x_1R) = n-1$. R/x_1R is u-i-Koszul by 4.5. Let $\bar{N}$ be the set of all zerodivisors of R/x_1R. If $\bar{N} \neq 0$, by induction hypothesis on n, if $\bar{N} = 0$, by case (a), it follows that R/x_1R is a hypersurface ring of the form $R/x_1R = K[X_2, \ldots, X_n]/(g^2)$ for some $g \in K[X_2, \ldots, X_n]_1$. Let $L \subset S_1$ be the linear subspace with $(I : X_1)_1 = L$. Then we have $I = (X_1L, g^2 + X_1l)$ for some linear form $l \in S_1$. By 4.6, we get $X_1 \in L$ and thus $X_1 \in \operatorname{Rad}(I)$. It follows that $g \in \operatorname{Rad}(I)$ which implies $g + I \in N$. Again by 4.6, we get $g^2 \in I$ and $X_1g \in I$. This implies $g, l \in L$ and therefore $I = (L^2)$. Since $d = 1$ we have $L^2 = I \neq \mathfrak{m}^2$. By 4.7 we get the assertion.

We finish now the induction on d. Let $d > 1$. Then we have $N \neq R_1$. Thus there exists $x \in R_1 \setminus N$, $x \neq 0$. We may assume $x = x_1 = X_1 + I$. By 4.5 R/x_1R is u-i-Koszul. We have $\dim(R/x_1R) = \dim(R) - 1 \geq 1$ and thus by induction hypothesis $R/x_1R = K[X_2, \ldots, X_n]/(g^2)$. It follows $I = (g^2 + X_1l)$ for some $l \in R_1$. If $I \neq (0)$, we obtain the assertion by 4.8. □

In the monomial case we have a more precise statement.

Proposition 4.10. *Let $I \subset S$ a proper monomial ideal. $R = S/I$ is u-i-Koszul if and only if $I = \mathfrak{m}^2$ or I is of the form* $\begin{cases} (X_i^2) & \text{if } \operatorname{char}(K) \neq 2 \\ (X_{i_1}^2, \ldots, X_{i_r}^2) & \text{if } \operatorname{char}(K) = 2 \end{cases}$.

Proof. In the case that $I = \mathfrak{m}^2$ or I is of the form (X_i^2) for some i the algebra R is u-i-Koszul by 4.7. Now let $\operatorname{char}(K) = 2$ and $I = (X_{i_1}^2, \ldots, X_{i_r}^2)$ for some indices $i_1 < \ldots < i_r$. For any transformation $X_i \mapsto \sum_{j=1}^{n} a_{ji}X_j$ $i = 1, \ldots, n$, we obtain a new defining ideal $J = (g_1, \ldots, g_r)$ with $g_k = \sum_{j=1}^{n} a_{ji_k}^2 X_j^2$ for $k = 1, \ldots, r$. Then J has a minimal system of generators which forms a Gröbner basis of J. In the term order of 2.1 $\operatorname{in}(J)$ is of the form $(X_{j_1}^2, \ldots, X_{j_s}^2)$ for some indices $j_1 < \ldots < j_s$. By 2.1, S/J is i-Koszul and thus $R = S/I$ is u-i-Koszul. Conversely, let us assume that R is u-i-Koszul. There are two cases:

(a) $\operatorname{char}(K) \neq 2$. By 4.6 and 4.7, we get $I = (X_i^2)$ for some i or $I = \mathfrak{m}^2$.

(b) $\operatorname{char}(K) = 2$. Let $G(I)$ be the set of the minimal generators of I. We need some facts which follow immediately from 2.3. If

(1) $X_iX_j \in G(I)$ with $i < j$ and $X_iX_k \notin G(I)$ for some $k > j$ or if

(2) $X_iX_j \in G(I)$ with $i < j$ and $X_k^2 \notin G(I)$ for some $k > i$ or if

(3) $X_i^2, X_iX_{i+1}, X_iX_n \in G(I)$, $X_1^2, \ldots, X_n^2 \in G(I)$ and $X_{i+1}X_{i+2} \notin G(I)$ for some $i < n-1$ or if

(4) $X_i^2, X_iX_{i+1}, X_{i+1}^2 \in G(I)$ and $X_{i-1}X_i \notin G(I)$ for some $1 < i < n$,

then R is not u-i-Koszul.

We have to show the following: If $I \neq \mathfrak{m}^2$ and I is not of the form $(X_{i_1}^2, \ldots, X_{i_r}^2)$, then R is not u-i-Koszul. Under this assumption we have $X_iX_j \in G(I)$ for some $i < j$. By

2.3 and (1), we have $X_i^2, \dots, X_iX_n \in G(I)$. By (2), we get $X_{i+1}^2, \dots, X_n^2 \in G(I)$. Then (4) implies $X_{i-1}X_i \in G(I)$. By iteration and using (3), we obtain that $I = \mathfrak{m}^2$ which is a contradiction. □

As a direct consequence from 4.4 and 4.10 we have

Corollary 4.11. *Let* $\operatorname{char}(K) \neq 2$ *and* K *algebraically closed. If* $R = S/I$ *is u-i-Koszul, then* $R' = S/\operatorname{in}(I)$ *is also u-i-Koszul.*

The converse of 4.11 is not true. For example, take $n = 3$ and $I = (X_1X_3 - X_2^2)$. Since $\operatorname{in}(I) = (X_2^2)$ the algebra R' is u-i-Koszul by 4.10, but R is not, as follows from 4.4. We get immediately from 4.4:

Corollary 4.12. *Let* K *be algebraically closed,* $\operatorname{char}(K) \neq 2$ *and* $R = S/I$ *a u-i-Koszul domain. Then* $I = (0)$.

The statements in 4.4 and 4.12 are not true for more general base fields. Take, for example,

$$R = \mathbb{Q}[X_1, X_2]/(X_1^2 - \frac{1}{2}X_2^2).$$

Then $X_1^2 - \frac{1}{2}X_2^2$ is not a square in $\mathbb{Q}[X_1, X_2]$ and R is a u-i-Koszul domain. Moreover,

$$R = \mathbb{Z}/2\mathbb{Z}[X_1, \dots, X_4]/(X_1^2 + X_2^2, X_3^2 + X_4^2)$$

is u-i-Koszul. Therefore we need $\operatorname{char}(K) \neq 2$ in 4.4.

In [4] we find the following concept: A homogeneous K-algebra R is called universally Koszul, if the set of all ideals of R which are generated in degree 1 defines a Koszul filtration of R. There is no direct relation to i-Koszulness. Since on the one hand the algebra

$$K[X_1, X_2]/(X_1X_2)$$

is u-Koszul by [4] 1.5., but not i-Koszul by 2.3. On the other hand

$$K[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$$

is i-Koszul due to 2.3, but not u-Koszul, if $n > 3$ and $\operatorname{char}(K) \neq 2$ (see [4]).

5. i-Koszulness of semigroup rings

In this chapter we want to study semigroup rings. In this case we only consider flags spanned by semigroup generators. We identify a monomial X^α with the corresponding exponent $\alpha \in \mathbb{N}^n$. Thus if $R = K[\alpha_1, \dots, \alpha_k]$ is a homogeneous semigroup ring with minimal set of semigroup generators $G = \{\alpha_1, \dots, \alpha_k\}$, then R is called *i-Koszul* if R is i-Koszul with respect to $\alpha_1, \dots, \alpha_k$. R is said to be *u-i-Koszul* , if R is i-Koszul with respect to $\alpha_{\pi(1)}, \dots, \alpha_{\pi(k)}$ for any permutation $\pi \in S_k$. We will see that i-Koszulness implies a certain shellability of the finite intervals in the divisor poset of R. The set Σ of all monomials in R is partially ordered by divisibility. If there is an injective map $\lambda : G \to \Lambda$, Λ totally ordered, then all unrefineable finite chains of divisors

$$C : \beta_0 \overset{\cdot \alpha_{i_1}}{\to} \beta_1 \to \cdots \overset{\cdot \alpha_{i_r}}{\to} \beta_r$$

are labeled by $\lambda(C) = (\lambda(\alpha_{i_1}), \dots, \lambda(\alpha_{i_r})) \in \Lambda^r$. We have a total order $>$ on Λ^r induced by the lexicographic order and the order on Λ.

Definition 5.1. (see [1],[3]) *R is called naturally shellable, if for every monomial $\alpha \in \Sigma$ the order complex $\Delta([1, \alpha])$ is shellable with order $\lambda(C_1) < \cdots < \lambda(C_r)$ where $\{C_1, \dots, C_r\}$ is the set of all unrefineable chains in the interval $[1, \alpha]$.*

Let $R = K[Y_1, \dots, Y_k]/I$. Natural shellability can be translated into a condition on $\mathrm{in}(I)$ with respect to the reverse lexicographic order induced by $Y_1 < \cdots < Y_k$.

Proposition 5.2. (T. Hibi) *The following statements are equivalent:*

(a) *R is naturally shellable.*

(b) *$\mathrm{in}(I)$ is quasi-poset, i.e. if $i < k < j$ and $Y_iY_j \in \mathrm{in}(I)$, then it follows $Y_iY_k \in \mathrm{in}(I)$ or $Y_kY_j \in \mathrm{in}(I)$.*

Consequently, by 2.1 the following is evident:

Corollary 5.3. *Let R be an i-Koszul semigroup ring. Then R is naturally shellable.*

It is known (see [12]) that shellability of divisor posets implies Koszulness. Thus the corollary above gives us an alternative proof for the statement that an i-Koszul semigroup ring is Koszul. We have seen in 3.3 that the property of i-Koszulness is preserved under tensor products. This is not true for Segre products of semigroup rings. For example,

$$R = K[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2] \cong K[Z_1, Z_2, Z_3, Z_4]/(Z_1Z_4 - Z_2Z_3) \tag{1}$$

is not i-Koszul with respect to any permutation of the semigroup generators by 2.1. But it can be shown that R is naturally shellable ([3]). Therefore the converse of 5.3 is not true in general.

We now compare i-Koszulness with other Koszul properties. In [11] strongly Koszul algebras are introduced. In the semigroup case this property is preserved under Segre products (see [11]). Thus the ring $R = K[X_1, X_2] \star K[Y_1, Y_2]$ in example (1) is strongly Koszul. In [11] it is shown that strongly Koszul algebra is sequentially Koszul. It is obvious from the definition that:

Remark 5.4. *Any i-Koszul algebra R is sequentially Koszul.*

As (1) shows the converse is not true in general. Furthermore i-Koszulness does not imply the strongly Koszul property. Take, for example,

$$T = K[X_1^3, X_1^2X_2, X_1X_2^2, X_1X_2X_3, X_2^2X_3, X_2X_3^2].$$

If we order the generators lexicographically descending, we get by computation that T is i-Koszul. However, T is not strongly Koszul by [11] Prop.1.4. because

$$(X_1X_2X_3) :_T (X_2^3) = (X_2^3, X_1^3X_2X_3^2).$$

Concerning u-i-Koszulness we have the following

Proposition 5.5. *Let $R = K[\alpha_1, \dots, \alpha_k] \subset S$ be a semigroup ring. If R is u-i-Koszul, then R is a polynomial ring.*

Proof. We may assume that $\alpha_1 >_{lex} \dots >_{lex} \alpha_k$ where $>_{lex}$ is the total lexicographic order on $\mathbb{N}^n$. Let $R = K[Y_1, \dots, Y_k]/I$ with $\alpha_i = Y_i + I$ for $i = 1, \dots, n$. By hypothesis, R is i-Koszul with respect to this sequence. We argue by contradiction. If $I \neq (0)$, we get by 2.1 that I has quadratic Gröbner basis with respect to the reverse lexicographic term order induced by $Y_1 < \dots < Y_k$. The toric ideal I is minimally generated by binomials of degree 2. By 4.9, I has a quadratic Gröbner basis G which consists of binomials. The chosen order of the semigroup generators implies that every $f \in G$ is of the form $f = Y_iY_j - Y_kY_l$ with $k < i \leq j < l$ where $\mathrm{in}(f) = Y_iY_j$. We choose the smallest index i such that $Y_iY_j \in \mathrm{in}(I)$ for some $i \leq j$. Since R is i-Koszul, we have $Y_i^2 \in \mathrm{in}(I)$ by 2.1. Thus there exists $f \in G$ such that $f = Y_i^2 - Y_{i-r}Y_{i+s}$ for some $r, s > 0$. Interchanging Y_i and Y_{i-r} we get a new defining ideal J and an element $g = Y_{i-r}^2 - Y_iY_{i+s} \in J$. Taking the same term order on S/J we observe that $\mathrm{in}(g) = Y_iY_{i+s}$. Since S/J is i-Koszul, it follows that $Y_i^2 \in \mathrm{in}(J)$ by 2.1. Thus there exists a binomial $h \in J$ such that $h = Y_i^2 - Y_aY_b$ for some $a, b \in \{1, \dots, k\}$. But then, there is a relation $u = Y_{i-r}^2 - Y_cY_d \in G$ for some $c, d \in \{1, \dots, k\}$ and the order of the semigroup generators implies $c < i - r < d$. Thus we have $\mathrm{in}(u) = Y_{i-r}^2 \in \mathrm{in}(I)$ which is a contradiction to the choice of i. □

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