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The Second Extension of the Thas-Walker Construction

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Abstract. We extend the extended Thas-Walker construction by introducing the concept "flocklet" of the Klein quadric. Thus we have a tool to construct spreads with asymplecticly complemented regulization.

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1. Introduction

Independently, M. Walker [15] and J. A. Thas (unpublished) discovered that to each flock of an elliptic or hyperbolic quadric or of a quadratic cone of $\mathrm{PG}(3,q)$ a spread of $\mathrm{PG}(3,q)$ and, consequently, a (finite) translation plane is constructable; compare [1, p. 8], [12, p. 441] or the surveys [3], [13, p.95-96], [14]. The Thas-Walker construction remains valid for flocks of $\mathrm{PG}(3,\mathbb{K})$ with arbitrary commutative field $\mathbb{K}$; cf. [6, p. 146–149]. In [7] the author used flocks of $\mathrm{PG}(3,\mathbb{R})$ to construct spreads representing topological translation planes. The extended Thas-Walker construction exhibited in [8], hereafter called [ETW], starts with a flockoid of a Lie quadric of $\mathrm{PG}(4,\mathbb{K})$ and yields also a spread; a *Lie quadric* L_4 is a hyperquadric of a Pappian projective 4-space such that L_4 has no vertex and contains a line; a collection $\mathcal{D}$ of conics contained in a Lie quadric L_4 of a Pappian projective 4-space is called a *flockoid* of L_4, if the following two conditions hold:

(FD1) For each generatrix g of L_4 there exists exactly one conic $k \in \mathcal{D}$ with $g \cap k \neq \emptyset$.
(FD2) There are at most two improper conics in $\mathcal{D}$.

In view of the present article, the paper [ETW] deals with the first extension of the Thas-Walker construction.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a Pappian projective 3-space with point set $\mathcal{P}$ and line set $\mathcal{L}$. As we deal with spreads composed of reguli and at most two exceptional lines, so we standardize by defining: A *proper regulus* $\mathcal{R}$ is the set of lines meeting three mutually skew lines; the directrices of $\mathcal{R}$ form the complementary (opposite) regulus $\mathcal{R}^c$; if $x \in \mathcal{L}$, then $\{x\}$ is called an *improper regulus*; $\{x\}^c := \{x\}$.

Definition 1. *Let $\mathcal{S}$ be a spread of Π and let Σ be a collection of (proper or improper) reguli contained in $\mathcal{S}$. We call Σ a regulization of $\mathcal{S}$, if the following hold:*

(RZ1) *Each line of $\mathcal{S}$ belongs either to exactly one regulus of Σ or to all reguli of Σ.*

(RZ2) *There are at most two improper reguli in Σ.*

The set $\cup(\mathcal{R}^c | \mathcal{R} \in \Sigma) =: \mathcal{S}^c_\Sigma$ is named complementary congruence of $\mathcal{S}$ with respect to Σ. If $\mathcal{S}^c_\Sigma$ is an elliptic linear congruence of lines, then Σ is called an elliptic regulization of $\mathcal{S}$. If $\mathcal{S}^c_\Sigma$ belongs to a linear complex of lines, then we say that Σ is a symplecticly complemented regulization, otherwise we speak of an asymplecticly complemented regulization. If $\mathcal{S}^c_\Sigma$ belongs to a single linear complex of lines, then Σ is called a unisymplecticly complemented regulization.

The papers [ETW], [9], and [10] are devoted to the construction and investigation of spreads with symplecticly complemented regulization. For the real projective 3-space PG$(3, \mathbb{R})$ an example of a non-regular spread with an asymplecticly complemented regulization is given in [6, (4.1.7)].

Let λ be the well-known Klein mapping of $\mathcal{L}$ onto the Klein quadric H_5 which is embedded into a projective 5-space Π_5 with point set $\mathcal{P}_5$; cf. e.g. [4] and the translation table in [2, p. 29–30]. A Latin <Greek> plane on H_5 is the λ-image of a star of lines <a ruled plane>. If $\mathcal{R}$ is a proper or improper regulus, then $\lambda(\mathcal{R})$ is an irreducible conic or a point. For obvious reasons, we speak of *proper* or *improper conics*. If $\mathcal{S}$ is a spread of Π with the unisymplecticly complemented regulization Ω, then $\{\lambda(\mathcal{R}^c) | \mathcal{R} \in \Omega\}$ is a flockoid of a uniquely determined Lie quadric L_4 with $L_4 \subset H_5$; cf. [ETW, Prop. 1].

Recall the extended Thas-Walker construction [ETW, Prop. 2]: If $\mathcal{D}$ is a flockoid of a Lie quadric L_4 with $L_4 \subset H_5$, then $\cup((\lambda^{-1}(k))^c | \, k \in \mathcal{D})$ is a spread of Π with the regulization $\{(\lambda^{-1}(k))^c | k \in \mathcal{D}\}$ which is either unisymplecticly complemented or elliptic.

In Proposition 1 we start with a spread $\mathcal{S}$ of Π admitting an asymplecticly complemented regulization[1] Γ and investigate the set $\{\lambda(\mathcal{R}^c) | \mathcal{R} \in \Gamma\} =: \mathcal{C}$ of conics. In the proof of Proposition 1 we shall find that $\mathcal{C}$ is a "flocklet" of the Klein quadric H_5; we define the concept "flocklet", as follows.

Definition 2. *A collection $\mathcal{B}$ of (proper or improper) conics contained in the Klein quadric H_5 is called a flocklet <flockling> of H_5, if the following two conditions hold:*

(FT1)<(FG1)> *For each Latin <Greek> plane γ on H_5 there exists exactly one conic $k \in \mathcal{B}$ with $\gamma \cap k \neq \emptyset$.*

(FT2)<=(FG2)> *There are at most two improper conics in $\mathcal{B}$.*

Depending on the intersection M of the carrier planes of the participating proper conics we distinguish three types of flocklets:

[1] The assumption in Proposition 1 is more comprehensive; cf. Remark 1 and Definition 3.

1) M is empty (*unbundled flocklet*)
2) M consists of exactly one point (*star flocklet*)
3) M is a line (*linear flocklet*)[2].

In the same way *unbundled flocklings, star flocklings, and linear flocklings* are defined.

The second extension of the Thas-Walker construction starts with a flocklet $\mathcal{A}$ of the Klein quadric H_5. Then $\cup((\lambda^{-1}(k))^c|k \in \mathcal{A})$ is a spread of Π admitting the regulization $\{(\lambda^{-1}(k))^c|k \in \mathcal{A}\}$ which is either asymplecticly complemented or unisymplecticly complemented or elliptic; cf. Proposition 2 and Remark 1. Each flockoid of a Lie quadric L_4 can be interpreted as flocklet and also as flockling of a Klein quadric H_5 containing L_4; cf. Remark 3. Each flock of an elliptic quadric Q_e can be interpreted as flocklet and also as flockling of a Klein quadric H_5 containing Q_e; cf. Remark 4. Note, that a flock of a quadric Q covers Q, but a flocklet of a Klein quadric H_5 is no covering of H_5.

In Section 3 we introduce Thas-Walker plane sets of Latin type to get further properties of the second extension of the Thas-Walker construction.

2. The second extension of the Thas-Walker construction

Definition 3. *Let Σ be an arbitrary regulization of a spread $\mathcal{S}$ of Π. We call*

$$i(\Sigma) := \#\left(\bigcap_{\mathcal{R}\in\Sigma} \mathcal{R}\right) \tag{1}$$

the intersection number of Σ. If $i(\Sigma) = 0$, then we say that the regulization Σ is of intersection number 0.

By [6, Remark 2.4], $i(\Sigma) \in \{0, 1, 2\}$. From [6, Remark 2.5 and 2.6] we deduce:

Remark 1. *A regulization Σ is of intersection number 0 if, and only if, Σ is asymplecticly complemented or unisymplecticly complemented or elliptic.*

In the following remark we sum up further properties of a regulization of intersection number 0; for (1) and (2) see [6, Remark 2.8 resp. 2.9], the statements (3) and (4) are evident.

Remark 2. *Let $\mathcal{S}$ be a spread of Π admitting the regulization Σ of intersection number 0. Then following statements hold true:*

(1) *Each element of $\mathcal{S}$ belongs to exactly one regulus of Σ.*

(2) *The complementary congruence $\mathcal{S}^c_\Sigma = \cup(\mathcal{R}^c|\mathcal{R} \in \Sigma)$ of $\mathcal{S}$ with respect to Σ is also a spread of Π.*

(3) $\{\mathcal{R}^c|\mathcal{R} \in \Sigma\} =: \Sigma^c$ *is a regulization of $\mathcal{S}^c_\Sigma$ with $i(\Sigma^c) = 0$.*

(4) *The complementary congruence $(\mathcal{S}^c_\Sigma)^c_{\Sigma^c}$ of $\mathcal{S}^c_\Sigma$ with respect to Σ^c coincides with $\mathcal{S}$; in symbols:*

$$(\mathcal{S}^c_\Sigma)^c_{\Sigma^c} = \mathcal{S}. \tag{2}$$

[2]In contrast to a linear flock, a linear flocklet is not uniquely determined by the common line of the carrier planes of the participating proper conics.

The rest of this section generalizes [ETW, Section 3].

Proposition 1. *Let $\mathcal{S}$ be a spread of* Π *and let* Γ *be a regulization of* $\mathcal{S}$ *with intersection number* 0. *Then* $\{\lambda(\mathcal{R}^c)|\mathcal{R} \in \Gamma\} =: \mathcal{C}$ *is a flocklet of the Klein quadric* H_5.

Proof. Clearly, (RZ2) implies (FT2). By Remark 2 (3), $\mathcal{S}^c_\Gamma$ is a spread admitting the regulization Γ^c with $i(\Gamma^c) = 0$; we can say that $\mathcal{C}$ is the λ-image of the regulization Γ^c. Let η be an arbitrary Latin plane on H_5, then $\lambda^{-1}(\eta)$ is a star of lines with a vertex, say Y. A conic k of $\mathcal{C}$ has a non-empty intersection with η if, and only if, the regulus $\lambda^{-1}(k)$ of Γ^c contains a line incident with Y. In the spread $\mathcal{S}^c_\Gamma$ there exists exactly one line, say s_Y, incident with Y. Because of $i(\Gamma^c) = 0$ and Remark 2 (1), s_Y belongs to exactly one regulus, say $\mathcal{R}^c_Y$, of Γ^c. Now $Y \in s_Y \in \mathcal{R}^c_Y \in \Gamma^c$ implies $\lambda(s_Y) \in \eta \cap \lambda(\mathcal{R}^c_Y)$ and $\lambda(\mathcal{R}^c_Y) \in \mathcal{C}$, i.e., the conic $\lambda(\mathcal{R}^c_Y)$ is the only element of $\mathcal{C}$ having at least one common point with η. □

Remark 3. *Let $\mathcal{D}$ be a flockoid of the Lie quadric L_4 with $L_4 \subset H_5$. Then $\mathcal{D}$ is a flocklet and also a flockling of H_5.*

Proof. Let ξ be an arbitrary (Latin or Greek) plane on H_5. Then $\xi \cap \operatorname{span} L_4$ is always a line, say x. Because of (FD1)[3], there exists exactly one conic $k_x \in \mathcal{D}$ with $k_x \cap x \neq \emptyset$ and thus $k_x \cap \xi \neq \emptyset$. □

By [ETW, Remark 4], each Lie quadric of $\mathrm{PG}(4, \mathbb{K})$ is embeddable into the Klein quadric H_5 of $\mathrm{PG}(5, \mathbb{K})$, hence each flockoid of a Lie quadric can be interpreted as flocklet and also as flockling of a suitable Klein quadric.

Remark 4. *Let $\mathcal{F}$ be a flock of the elliptic quadric Q_e with $Q_e \subset H_5$. Then $\mathcal{F}$ is a flocklet and also as flockling of H_5.*

Proof. Let ξ be an arbitrary (Latin or Greek) plane on H_5. Then $\xi \cap \operatorname{span} Q_e$ is always a point, say X. In the flock $\mathcal{F}$ there exists exactly one conic k_X with $X \in k_X$. □

By [ETW, Remark 9], each elliptic quadric of $\mathrm{PG}(3, \mathbb{K})$ is embeddable into a Lie quadric of $\mathrm{PG}(4, \mathbb{K})$ which in turn is embeddable into the Klein quadric of $\mathrm{PG}(5, \mathbb{K})$, by [ETW, Remark 4]. Hence each elliptic flock can be interpreted as flocklet and also as flockling of a suitable Klein quadric.

Before formulating and proving the converse of Proposition 1 in Proposition 2 we expose two lemmas about flocklets. The statements of the following Lemma 1 are immediate consequences of (FT1) and the properties of a plane section of a quadric.

Lemma 1. *Let $\mathcal{A}$ be a flocklet of the Klein quadric H_5.*
(i) *Then different conics of $\mathcal{A}$ are disjoint.*
(ii) *If $\{P_1\}$ and $\{P_2\}$ are different improper conics of $\mathcal{A}$, then $P_1 \vee P_2 \not\subset H_5$.*
(iii) *If η is a Latin plane on H_5 and $k \in \mathcal{A}$ satisfies $k \cap \eta \neq \emptyset$, then $\eta \neq \operatorname{span} k$, $\eta \cap \operatorname{span} k$ is no line, and $\#(k \cap \eta) = 1$.*

Lemma 2. *Let $\mathcal{A}$ be a flocklet of the Klein quadric H_5 and let k_1 be a proper conic of $\mathcal{A}$. If $k_2 \in \mathcal{A} \setminus \{k_1\}$, then there exists no tangent cone C_4 of H_5 with $k_1 \cup k_2 \subset C_4$.*

[3] Compare [ETW, Definition 3].

Proof. Let C_4 be a tangent cone of H_5 with a vertex, say V, and with $k_1 \subset C_4$. In span C_4 there exists a 3-dimensional subspace, say S_3, with $V \notin S_3$. Now $C_4 \cap S_3 =: Q_h$ is a hyperbolic quadric and the Latin planes on C_4 give rise to a regulus $\mathcal{R}_{Q_h}$ on Q_h. By k_1^π we denote the image of k_1 under the projection from centre V onto S_3. The proper conic $k_1^\pi \subset Q_h$ meets each line of the regulus $\mathcal{R}_{Q_h}$ in exactly one point. Consequently, k_1 and each Latin plane on C_4 have exactly one common point and, because of (FT1), C_4 cannot contain further (proper or improper) conics of $\mathcal{A}$. □

Proposition 2. *If $\mathcal{A}$ is a flocklet of the Klein quadric H_5, then*

$$\cup((\lambda^{-1}(k))^c | k \in \mathcal{A}) =: T_{E_2}(\mathcal{A}) \tag{3}$$

is a spread of Π *admitting the regulization*

$$\{(\lambda^{-1}(k))^c | k \in \mathcal{A}\} =: T_{R_2}(\mathcal{A}) \tag{4}$$

and $T_{R_2}(\mathcal{A})$ is of intersection number 0.

Proof. Let X be an arbitrary point of Π and denote the star of lines with vertex X by $\mathcal{L}[X]$. In $T_{E_2}(\mathcal{A})$ there exists a line incident with X if, and only if, there is a conic $k_X \in \mathcal{A}$ such that X is on a line h of the regulus $\lambda^{-1}(k_X)$, i.e., $h \in \mathcal{L}[X] \cap \lambda^{-1}(k_X)$ and thus $\lambda(h) \in \lambda(\mathcal{L}[X]) \cap k_X$. As $\lambda(\mathcal{L}[X])$ is a Latin plane on H_5, so there is a unique $k_X \in \mathcal{A}$ with $k_X \cap \lambda(\mathcal{L}[X]) \neq \emptyset$, by (FT1) . Hence there is a unique regulus in $T_{E_2}(\mathcal{A})$, namely $(\lambda^{-1}(k_X))^c$, which contains a line incident with X. Consequently, $T_{E_2}(\mathcal{A})$ is a spread.

Next we prove the validity of (RZ1) and (RZ2) for $T_{R_2}(\mathcal{A})$. Clearly, (FT2) implies (RZ2). Instead of (RZ1) we show even more:

(RZ1*) *Each line of $T_{E_2}(\mathcal{A})$ belongs to exactly one regulus of $T_{R_2}(\mathcal{A})$.*

Let $b \in T_{E_2}(\mathcal{A})$ be arbitrary. We assume, to the contrary,

$$b \in (\lambda^{-1}(k_1))^c \cap (\lambda^{-1}(k_2))^c, \quad \{k_1, k_2\} \subseteq \mathcal{A}, \quad k_1 \neq k_2. \tag{5}$$

In the case that both $(\lambda^{-1}(k_1))^c$ and $(\lambda^{-1}(k_2))^c$ are improper reguli with $(\lambda^{-1}(k_i))^c = \{g_i\}$ and $g_i \in \mathcal{L}$, $(i = 1, 2)$, the lines g_1 and g_2 are skew and (5) yields the absurdity $b \in \{g_1\} \cap \{g_2\} = \emptyset$. Hence we may assume, without loss of generality, that $(\lambda^{-1}(k_1))^c$ is a proper regulus. Each line of $(\lambda^{-1}(k_1)) \cup (\lambda^{-1}(k_2))$ meets b. Thus $k_1 \cup k_2$ is contained in the tangent cone of H_5 at the point $\lambda(b)$, a contradiction to Lemma 2. Thus $T_{R_2}(\mathcal{A})$ is a regulization and the validity of (RZ1*) implies $i(T_{R_2}(\mathcal{A})) = 0$. □

The process of gaining a spread from a flocklet via formula (3) is called *second extension of the Thas-Walker construction.*

We combine Proposition 1 and 2 and get

Corollary 1. *To each spread of* $\mathrm{PG}(3, \mathbb{K})$ *admitting a regulization of intersection number* 0 *there corresponds a flocklet of the Klein quadric of* $\mathrm{PG}(5, \mathbb{K})$*, and vice versa.*

3. Thas-Walker plane sets of latin type

Let Q be an elliptic or hyperbolic quadric of $\mathrm{PG}(3,\mathbb{K})$ with bijective polarity π_Q. A point set T of $\mathrm{PG}(3,\mathbb{K})$ is called a *Thas-Walker point set with respect to* Q, if $\{\pi_Q(X)\cap Q \mid X \in T \wedge \pi_Q(X)\cap Q \neq \emptyset\}$ is a flock of Q. Let L_4 be a Lie quadric of $\mathrm{PG}(4,\mathbb{K})$ with polarity π_4. A line set T_ℓ of $\mathrm{PG}(4,\mathbb{K})$ is called a *Thas-Walker line set with respect to* L_4, if $\{\pi_4(X)\cap L_4 \mid X \in T_\ell \wedge \pi_4(X)\cap L_4 \neq \emptyset\}$ is a flockoid of L_4. In [7, Section 2.2], we considered a Thas-Walker point set T with respect to an elliptic quadric $E \subset H_5$ and got the λ-image of a spread $\mathcal{T}$ by projecting T from the line $e = \pi_5(\text{span } E)$, with π_5 the polarity defined by the Klein quadric H_5, into H_5, in symbols $\lambda(\mathcal{T}) = \cup((X \vee e)\cap H_5 | X \in T)$. In [ETW, Section 4], we took a Thas-Walker line set T_ℓ with respect to a Lie quadric $L_4 \subset H_5$ and got the λ-image of a spread $\mathcal{T}_\ell$ by projecting T_ℓ from the point $Z = \pi_5(\text{span } L_4)$ into the Klein quadric H_5, in symbols $\lambda(\mathcal{T}_\ell) = \cup((x \vee Z)\cap H_5 | x \in T_\ell)$. In the present section we continue this process: we shall take a Thas-Walker plane set T_{La} of Latin type[4] with respect to the Klein quadric H_5 and shall get the λ-image of a spread $\mathcal{T}_{La}$ by projecting T_{La} from $\pi_5(\text{span } H_5) = \emptyset$ into the Klein quadric H_5. As this projection is the identity of $\mathcal{P}_5$, so the following statements and formulas concerning Thas-Walker plane sets of Latin type are of simplier appearance as their analogues in the case of Thas-Walker point resp. line sets.

A set T_{La} of planes of Π_5 is called *Thas-Walker plane set of Latin type with respect to* H_5, if

$$D_2(T_{La}) := \{\pi_5(\xi)\cap H_5 \mid \xi \in T'_{La}\} \quad \text{with} \quad T'_{La} := \{\xi \in T_{La} \mid \pi_5(\xi)\cap H_5 \neq \emptyset\} \qquad (6)$$

is a flocklet of H_5. We put

$$T^p_{La} := \{\xi \in T_{La} \mid \#(\pi_5(\xi)\cap H_5) > 1\} \qquad (7)$$

for the set of those planes of T_{La} which yield proper conics.

Remark 5. Let $\{P\} \subset H_5$ be an improper conic. In the case $\mathbb{K} = \mathbb{R}$ there are infinitely many planes α with $\pi_5(\alpha)\cap H_5 = \{P\}$, since there are infinitely many planes incident with P and contained in $\pi_5(P)$ and intersecting the tangent cone $\pi_5(P)\cap H_5$ only in P (compare also the description of a tangent cone of H_5 in the proof of Lemma 2). In other words, if T_{La_1} and T_{La_2} are Thas-Walker plane sets of Latin type with respect to H_5, then $D_2(T_{La_1}) = D_2(T_{La_2})$ implies $T^p_{La_1} = T^p_{La_2}$, but not $T'_{La_1} = T'_{La_2}$.

For the discussion of Thas-Walker line sets in [ETW] we had to assume $\mathrm{Char}\,\mathbb{K} \neq 2$ througout Section 4. Here we need this additional assumption only in the following

Lemma 3. *Assume* $\mathrm{Char}\,\mathbb{K} \neq 2$. *Denote by* $\mathcal{P}_L[H_5]$ *the set of all Latin planes on the Klein quadric* H_5*, by the way,* $\pi_5(\mathcal{P}_L[H_5]) = \mathcal{P}_L[H_5]$. *A set* A *of planes of* Π_5 *is a Thas-Walker plane set of Latin type with respect to* H_5 *if, and only if, the following three conditions hold true*[5]*:*

[4] This concept has to be defined still.

[5] In order to have full conformity with the corresponding Lemma 5 from [ETW] we start the numbering of the conditions with 2.

(TWLa2) $\#(A_e) \le 2$ *with* $A_e := \{\alpha \in A \mid \alpha \cap \pi_5(\alpha) \neq \emptyset\}$.
(TWLa3) *If* $\alpha_e \in A_e$, *then* $\#(\pi_5(\alpha_e) \cap H_5) = 1$.
(TWLa4) *For each plane* $\xi \in \mathcal{P}_L[H_5]$ *there exists exactly one plane* $\alpha \in A$ *with* $\xi \cap \alpha \neq \emptyset$.

Proof. If the intersection of the plane $\alpha \in A$ and the plane $\pi_5(\alpha)$ is empty, then $\pi_5(\alpha) \cap H_5$ is either a proper conic or empty, and conversely[6]. We define $D_2(A)$ according to (6). Now (TWLa2) and (TWLa3) imply that all elements of $D_2(A)$ are proper or improper conics and that $D_2(A)$ satisfies (FT2), and vice versa. Finally, (TWLa4)⇔(FT1). □

If $k \subset H_5$ is a proper conic, then $(\lambda^{-1}(k))^c = \lambda^{-1}(\pi_5(\text{span } k))$. If α is a plane of Π_5 such that $\alpha \cap H_5$ is an improper conic, say $\{A\}$, then, as the reader proves easily, $\pi_5(\alpha) \cap H_5 = \{A\}$ and hence $(\lambda^{-1}(\{A\}))^c = \lambda^{-1}(\pi_5(\alpha))$. Thus we have the subsequent modification of the second extension of the Thas-Walker construction:

Lemma 4. *If* T_{La} *is a Thas-Walker plane set of Latin type with respect to the Klein quadric* H_5, *then*

$$\mathcal{T}_{La} := \cup(\lambda^{-1}(\xi) \mid \xi \in T_{La}) \tag{8}$$

is a spread of Π *admitting the regulization*

$$\Theta_{La} := \{\lambda^{-1}(\xi) \mid \xi \in T'_{La}\} \tag{9}$$

wherein T'_{La} *is defined by* (6); Θ_{La} *is of intersection number* 0.

We say that $\Phi(T'_{La}) := \cup(\tau \mid \tau \in T'_{La})$ is the 3-*surface determined by* T'_{La} and that each plane $\tau \in T'_{La}$ is a T'_{La}-*generatrix of* $\Phi(T'_{La})$. The following lemma is evident.

Lemma 5. *Suppose that the conditions (and notations) of Lemma 4 hold. If each proper conic* k *with* $k \subseteq \Phi(T'_{La}) \cap H_5$ *is contained in a* T'_{La}*-generatrix of* $\Phi(T'_{La})$, *then*
(1) *each proper regulus contained in the spread* $\mathcal{T}_{La}$ *belongs to* Θ_{La};
(2) $\mathcal{T}_{La}$ *admits exactly one regulization, namely* Θ_{La}.

Put[7]

$$v' := \dim(\bigvee_{\xi \in T'_{La}} \xi) \quad \text{and} \quad v^p := \dim(\bigvee_{\xi \in T^p_{La}} \xi). \tag{10}$$

(A) *If* $v' = v^p = 5$, *then* $\mathcal{T}_{La}$ *is an asymplectic*[8] *spread and* $D_2(T_{La})$ *is an unbundled flocklet.*

In order to give a survey of all imaginable cases of $v' = v^p$ we exhibit the subsequent table. The first row of the table represents an abbreviation of statement (A). The first column of the table gives the assumption, middle and left column are the conclusions.

$v' = v^p$	$\mathcal{T}_{La}$	$D_2(T_{La})$
5	asymplectic	unbundled
4	unisymplectic	star
3	regular	linear

Table 1. Deductions from $v' = v^p$

[6]This assertion is wrong for Char $\mathbb{K} = 2$.
[7]Compare also the definition of a 4-spatial or solid or planar Thas-Walker line set in [9, Definition 1]; furthermore, see [9, Remark 10].
[8]A spread $\mathcal{S}$ of Π is called *symplectic*, if $\mathcal{S}$ is part of a linear complex of lines; otherwise we say that $\mathcal{S}$ is an *asymplectic* spread. By a *unisymplectic* spread we mean a non-regular symplectic spread.

Definition 4. *If a flocklet $\mathcal{B}$ of the Klein quadric H_5 is neither a flockoid of a Lie quadric $L_4 \subset H_5$ nor a flock of an elliptic quadric $Q_e \subset H_5$, then we say that $\mathcal{B}$ is a genuine flocklet*[9].

Put

$$d' := \dim(\bigcap_{\xi \in T'_{La}} \xi) \quad \text{and} \quad d^p := \dim(\bigcap_{\xi \in T^p_{La}} \xi). \tag{11}$$

(B) *If $d' = d^p = -1$, then $D_2(T_{La})$ is a genuine flocklet and Θ_{La} is an asymplecticly complemented regulization.*

The following table gives a survey.

$d' = d^p$	Θ_{La}	$D_2(T_{La})$
-1	asymplecticly complemented	genuine flocklet
0	unisymplecticly complemented	genuine flockoid
1	elliptic	elliptic flock

Table 2. Deductions from $d' = d^p$

4. Spreads with regulizations of intersection number 0: types

Given a spread $\mathcal{S}$ of Π with regulization Σ we can distinguish 9 combinations:

Combination 1: $\mathcal{S}$ regular and Σ elliptic: Examples are well-known.

Combination 2: $\mathcal{S}$ regular and Σ unisymplecticly complemented: See [9, Section 5, Type 7].

Combination 3: $\mathcal{S}$ regular and Σ asymplecticly complemented: See the subsequent Remark 6.

Combination 4: $\mathcal{S}$ unisymplectic and Σ elliptic: See [7, Theorem 3.3.1].

Combination 5: $\mathcal{S}$ unisymplectic and Σ unisymplecticly complemented: See [9, Section 5, Type 3, Type 4, Type 5, and Type 6]. The case from [9, Section 5, Type 4] is investigated in [9, Section 6] shortly.

Combination 6: $\mathcal{S}$ unisymplectic and Σ asymplecticly complemented: See [6, Remark 4.1.2 and (4.1.7)].

Combination 7: $\mathcal{S}$ asymplectic and Σ elliptic: See [7, Theorem 3.2.1].

Combination 8: $\mathcal{S}$ asymplectic and Σ unisymplecticly complemented: See [9, Section 5, Type 1 and Type 2] which are investigated in [10] thoroughly.

Combination 9: $\mathcal{S}$ asymplectic and Σ asymplecticly complemented: For sake of completeness, we construct an example in the subsequent Remark 7.

Remark 6. We use the concepts and notations of [6, Section 4]. Let $\mathcal{E}$ be a regular spread of $\mathrm{PG}(3, \mathbb{R})$. We decompose the elliptic quadric $\lambda(\mathcal{E})$ by two (proper) disjoint conics c_1, c_2 of $\lambda(\mathcal{E})$ into two elliptic caps U_1, U_2, and an elliptic zone V. Put span $c_1 \cap$span $c_2 =: v$. Choose lines g_j $(j = 1, 2)$ with $g_j \subset$ span c_j, $g_j \cap c_j = \emptyset$, and $g_j \neq v$ such that g_1 and g_2 are skew. Then

$$\Phi := \tilde{\lambda}^{-1}(\mathbf{C}(U_1, g_1) \cup \mathbf{C}(V, v) \cup \mathbf{C}(U_2, g_2)) \tag{12}$$

is an asymplecticly complemented regulization of $\mathcal{E}$.

[9]Compare also the definition of a genuine flockoid of a Lie quadric given in [9, Definition 1].

Remark 7. Based on [5, Satz 5] in [6, 4.3] two spreads $\mathcal{S}_0$ and $\mathcal{S}_1$ are defined; with the notations from [5, Satz 5] we have:

$$\mathcal{S}_0 = \mathcal{E}_1^1 \cup \mathcal{R}_{1,1}^r \cup \mathcal{E}_2^2 \cup \mathcal{R}_{2,1}^r \cup \mathcal{E}_3^3 \tag{13}$$

wherein $\mathcal{R}_{1,1}^r$ and $\mathcal{R}_{2,1}^r$ are proper reguli described in [5, 3.2]; $\lambda(\mathcal{S}_0)$ is composed of the elliptic caps $\lambda(\mathcal{E}_1^1 \cup \mathcal{R}_{1,1}^r) =: W_1$, $\lambda(\mathcal{R}_{2,1}^r \cup \mathcal{E}_3^3) =: W_2$ and the elliptic zone $\lambda(\mathcal{R}_{1,1}^r \cup \mathcal{E}_2^2 \cup \mathcal{R}_{2,1}^r) =: Z$ with disjoint limiting conics. Consider the three solids

$$\begin{aligned} S_1^1 &:= \operatorname{span} \lambda(W_1) &=& \{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid p_0 + p_3 = p_1 - \varepsilon\, p_2 + p_4 + \varepsilon\, p_5 = 0\} \\ S_2^2 &:= \operatorname{span} \lambda(Z) &=& \{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid p_0 + p_3 = p_1 + p_4 = 0\} \\ S_3^3 &:= \operatorname{span} \lambda(W_2) &=& \{\mathbf{p}\mathbb{R} \in \mathcal{P}_5 \mid p_1 + p_4 = p_0 + 4\varepsilon'' p_2 + p_3 - \varepsilon'' p_5 = 0\} \end{aligned} \tag{14}$$

$$(\ \varepsilon, \varepsilon'' \in \mathbb{R} \setminus \{0\},\ |\varepsilon| < 1,\ |\varepsilon''| < \frac{1}{4}\ );$$

as $\dim(S_1^1 \vee S_2^2 \vee S_3^3) = 5$, so the spread $\mathcal{S}_0$ is asymplectic. Choose lines h_j $(j = 1, 2)$ with $h_j \subset \operatorname{span} \lambda(\mathcal{R}_{j,1}^r)$, $h_j \cap \lambda(\mathcal{R}_{j,1}^r) = \emptyset$, and $h_j \neq z$ such that h_1 and h_2 are skew. Then

$$\Psi := \tilde{\lambda}^{-1}(\mathbf{C}(W_1, h_1) \cup \mathbf{C}(Z, z) \cup \mathbf{C}(W_2, h_2)) \tag{15}$$

is an asymplecticly complemented regulization of $\mathcal{S}_0$.

The examples given for the Combinations 7, 8, and 9 show

Corollary 2. *There exist genuine linear flocklets, genuine star flocklets, and genuine unbundled flocklets.*

The spread $\mathcal{S}_0$ of Remark 7 is like "patchwork", therefore we ask for an explicit example of an algebraic asymplectic spread[10] with asymplecticly complemented regulization. Applying the second extension of the Thas-Walker construction we give in [11] such an example.

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[10] A spread $\mathcal{S}$ is called *algebraic*, if $\mathcal{S}$ is an algebraic congruence of lines.

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