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On the Classification of 16-dimensional Planes

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The concluding section *Principles of classification* of the book [18] on compact projective planes contains an outline of how a classification of 16-dimensional planes admitting a group Δ of dimension $\dim\Delta \geq h$ (where h is at least 35) might be accomplished. In the second part, proofs of several claims (of Proposition 87.4 in particular) have been indicated only very sketchily; some details will be supplied in the following, compare Theorem A below. Implications of Theorem A will be discussed elsewhere.

The automorphism group Σ of a compact plane $\mathcal{P}$ will always be taken with the compact-open topology. Only closed subgroups Δ of Σ will be considered, Δ is then a locally compact transformation group of the point space P.

Theorem L. *If* $\dim\Delta \geq 29$, *or if* Δ *is connected and* $\dim\Delta \geq 27$, *then* Δ *is a Lie group.*

This suffices for all classification purposes. A weaker result is given in [18] (87.1). For proofs see Salzmann [17] and Priwitzer-Salzmann [11].

From now on, assume that P is a compact 16-dimensional space, and that Δ is a connected Lie group. By the structure theory of Lie groups, there are 3 possibilities: Δ is semi-simple, or Δ contains a central torus subgroup, or Δ has a minimal normal vector subgroup $\Theta \cong \mathbb{R}^t$, compare [18] (94.26). In the first two cases, the results mentioned in [18] (87.2 and 3) have been improved in the meantime:

Theorem S. *Let* Δ *be a semi-simple group of automorphisms of the* 16*-dimensional plane* $\mathcal{P}$. *If* $\dim\Delta > 28$, *then* $\mathcal{P}$ *is the classical Moufang plane, or* $\Delta \cong \mathrm{Spin}_9(\mathbb{R}, r)$ *and* $r \leq 1$, *or* $\Delta \cong \mathrm{SL}_3\mathbb{H}$ *and* $\mathcal{P}$ *is a Hughes plane as described in* [18], §86.

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The proof can be found in Priwitzer [9], [10].

Theorem T. *Assume that* Δ *has a normal torus subgroup* $\Theta \cong \mathbb{T}$*. If* $\dim \Delta > 30$*, then* Θ *fixes a Baer subplane,* $\Delta' \cong \mathrm{SL}_3\mathbb{H}$*, and* $\mathcal{P}$ *is a Hughes plane.*

This is proved in Salzmann [15].

Hence only the case $\mathbb{R}^t \cong \Theta \lhd \Delta$ has to be considered. For convenience, the classical Moufang plane over the octonions will be excluded from the discussion. So-called *stiffness* theorems on the size of the stabilizer of a quadrangle play a decisive rôle:

(‡) *Suppose that the fixed elements of the connected closed subgroup* Λ *of* Δ *form a non-degenerate subplane* $\mathcal{E}$*.*

(a) *If* $\dim \Lambda > 11$*, or if* $\mathcal{E}$ *is a Baer subplane, then* Λ *is compact.*

(b) *If* Λ *is compact, or if* Λ *is a Lie group and* $\mathcal{E}$ *is connected, then* $\Lambda \cong \mathrm{G}_2$*, or* $\Lambda \cong \mathrm{SU}_3\mathbb{C}$*, or* $\dim \Lambda \leq 7$*.*

(c) *If* Λ *is a compact Lie group and* $\dim \Lambda < 8$*, then* $\Lambda \cong \mathrm{SO}_4\mathbb{R}$ *or* $\dim \Lambda \leq 4$*.*

(d) *If* Λ *is a Lie group and* $\mathcal{E}$ *is a Baer subplane, then* Λ *is isomorphic to* $\mathrm{SU}_2\mathbb{C}$ *or* $\dim \Lambda \leq 1$*.*

The first results are essentially due to Bödi [1], [2]. For (c) and (d) see Salzmann [13] and [18] (83.22).

Lemma 0. *If* $\mathbb{R}^t \cong \Theta \lhd \Delta$ *and if* $\dim \Delta \geq 24$*, then* Δ *fixes a point or a line, say a line* W*.*

Proof. Grundhöfer-Salzmann [8], Proposition XI.10.19.

Theorem A. *Assume that* Δ *is not semi-simple and that* $\mathcal{P}$ *is not a Hughes plane. If* $\dim \Delta \geq 33$*, then, up to duality,* Δ *has a minimal normal subgroup* $\Theta \cong \mathbb{R}^t$ *consisting of axial collineations with common axis* W*. Either* $\Theta \leq \Delta_{[a,W]}$ *is a group of homologies and* $t = 1$*, or* Θ *is contained in the group* $\mathsf{T} = \Delta_{[W,W]}$ *of elations with axis* W*.*

Remarks. This has been stated in [18], p. 587 under the stronger hypothesis $\dim \Delta \geq 36$. The theorem does not assert that every given minimal normal subgroup is axial. The proof is fairly easy for $t < 8$ and rather involved for even $t \geq 8$. The different cases will be treated in separate propositions. The result may well be true for even smaller dimensions of Δ, but a proof would become unreasonably complicated.

A group Ξ of collineations is called *straight* if each point orbit x^Ξ is contained in some line. The following result (Stroppel [19] Lemma 3 or Priwitzer-Salzmann [11], Th. B) is a clue to the proof of the existence of axial collineations:

Baer's theorem. *If* Ξ *is a straight subgroup of* Δ*, then* Ξ *is contained in a group* $\Delta_{[z]}$ *of central collineations with common center* z*, or the fixed elements of* Ξ *form a Baer subplane* $\mathcal{F}_\Xi$ *of* $\mathcal{P}$ *and* Ξ *is compact by* (‡).

Corollary 1. *If* $\Pi \cong \mathbb{R}$ *and* Π *is straight, then* $\Pi \leq \Delta_{[z,A]}$ *for some center* z *and axis* A.

Proof. Note that Π is not compact. By the dual of [18] (61.8), all elements in Π have the same axis. □

Corollary 2. *If* $\Theta \cong \mathbb{R}^t$, *and if each one-parameter subgroup* Π *of* Θ *is straight, then* Θ *satisfies the assertions of Theorem A.*

Proof. By [18] (61.7), the center map $\Theta \setminus \{\mathbb{1}\} \to P$ is continuous, and the centers of all one-parameter subgroups of Θ form a compact and connected set Z. Commutativity of Θ implies that either Z is a single point, or Z is contained in the common axis W of all elements of Θ. The dual is also true. If there exist homologies in Θ, then $t = 1$ by [18] (61.2). □

Corollary 3. *If* $\Theta \cong \mathbb{R}^t$ *is a minimal normal subgroup of* Δ *and if some one-parameter subgroup* Π *of* Θ *is straight, then* Θ *satisfies the assertions of Theorem A.*

Proof. Let $\Pi \leq \Delta_{[z,A]}$ as in Corollary 1, and assume that Δ fixes the line W. Commutativity of Θ implies that $z^\Delta = z$ or $z^\Delta \subseteq A$. If $A = W$, then $\Theta = \Theta_{[A]}$ by minimality of Θ. If $A \neq W$, then $z^\Delta \subseteq W$ and hence $z^\Delta = z$. This case is dual to the first one. □

Lemma 1. *Assume that the one-parameter subgroup* Π *of* Θ *is not straight. Then there is an orbit* b^Π *which generates a connected subplane; its closure will be denoted by* $\mathcal{E} = \langle b^\Pi \rangle$. *For* $\varrho \in \Pi \setminus \{\mathbb{1}\}$, *the stabilizer* Δ_ϱ *in the action of* Δ *on* Θ *is the centralizer of* Π, *and the connected component* Λ *of* $\Delta_{b,\varrho}$ *induces the identity on* $\mathcal{E}$. *Hence* (‡) *applies, and the dimension formula* [18] (96.10) *gives*

$$(*) \qquad \dim \Delta = \dim b^\Delta + \dim \varrho^{\Delta_b} + \dim \Lambda \leq 16 + t + \dim \Lambda. \qquad \square$$

Proposition 1. *If* $t < 8$ *and if some one-parameter subgroup* Π *of* Θ *is not straight, then* $\dim \Delta \leq 32$.

Proof. Use Lemma 1. If $\Lambda \cong \mathrm{G}_2$, then $\dim \mathcal{E} = 2$ by [18] (83.24), and $\mathcal{E}$ consists of all fixed elements of Λ. Moreover, Λ acts trivially on Θ because Λ fixes ϱ and each non-trivial representation of G_2 is at least 7-dimensional [18] (95.10). The connected component ${\Theta_b}^1$ of the stabilizer Θ_b is contained in the compact group Λ, hence ${\Theta_b}^1 = \mathbb{1}$, $\dim \Theta_b = 0$, and $\dim b^\Theta = t$. Since Θ centralizes the group Λ, the fixed plane $\mathcal{E}$ of Λ is Θ-invariant. Consequently $b^\Theta \subseteq \mathcal{E}$, and it follows that $t \leq 2$ and $\dim \Delta \leq 32$. In all other cases $\dim \Lambda \leq 8$ by (‡), and $\dim \Delta < 32$. □

Proposition 1 and Corollary 2 imply that Theorem A is true in the case $t < 8$ and $\dim \Delta > 32$.

Lemma 2. *Let* $\mathbb{R}^t \cong \Theta \lhd \Delta$ *and* $\mathbb{R} \cong \Pi \leq \Theta$. *Assume that* $\dim \Delta > 32$, *and that the orbit* b^Π *is not contained in any line. Then* $\Theta_b = \mathbb{1}$ *and* $8 \leq t \leq 16$. *Moreover, the orbit* b^Θ *is not contained in any proper closed subplane of* $\mathcal{P}$, *and* Δ_b *acts effectively on* Θ.

Proof. Proposition 1 shows that $t \geq 8$. Let $\mathcal{F} = \langle b^\Theta \rangle$ be the smallest closed subplane containing b^Θ. Note that $\mathcal{F}$ is connected and Δ_b-invariant, and that Θ_b fixes $\mathcal{F}$ pointwise. From $(*)$ it follows that $17 - t \leq \dim \Lambda$. Obviously, $b^\Theta \subseteq \mathcal{F}$ and ${\Theta_b}^1 \leq \Lambda$. If $\dim \mathcal{F} = 2$, then $t - 2 \leq \dim \Lambda$ and $15 \leq 2 \dim \Lambda$. Therefore, $\dim \Lambda \geq 8$, and Λ is compact by (‡). Now ${\Theta_b}^1 = \mathbb{1}$ and $\dim b^\Theta = t \leq 2$, a contradiction. If $\dim \mathcal{F} = 4$, then Δ_b induces on $\mathcal{F}$ a group Δ_b/Φ of dimension at most 8 (see [18] (72.8) and note that $\Delta_b \leq \Delta_W$ and $b \notin W$). The kernel Φ of this action satisfies $\dim \Phi > 8$. The connected component of Φ is isomorphic to G_2 by (‡), but this would imply $\dim \mathcal{F} = 2$, see [18] (83.24). Hence $\dim \mathcal{F} \geq 8$ and $\mathcal{F}$ is a Baer subplane or $\mathcal{F} = \mathcal{P}$, for short, $\mathcal{F} \leq^\bullet \mathcal{P}$. According to [18] (83.6), the group Θ_b is compact, and then $\Theta_b = \mathbb{1}$. Suppose, finally, that $\mathcal{F} < \mathcal{P}$. Then $t = \dim \mathcal{F} = 8$, and Δ_b induces on $\mathcal{F}$ a group $\Gamma \cong \Delta_b/\Phi$ of dimension $\dim \Gamma \geq 14$ (note that $\dim \Phi \leq 3$ by (‡),(d). Hence $\dim \Gamma\Theta \geq 22$, $\mathcal{F}$ is isomorphic to the quaternion plane, and Θ acts on $\mathcal{F}$ as a group of translations, see [18] (84.13) or Salzmann [14]. This contradicts the assumption that b^Π is not contained in a line. □

Note. For the proof of Theorem A in the case $t \geq 8$ it is essential that Θ is chosen as a minimal normal subgroup. Write $\Xi = \mathrm{Cs}\,\Theta$ for the centralizer of Θ. Then $\Gamma = \Delta/\Xi$ is an irreducible subgroup of $\mathrm{GL}_t\mathbb{R}$. The structure of such groups is well known, compare [18] (95.6). In particular, the commutator subgroup Γ' is semi-simple, and Γ is the product of Γ' and the center Z of Γ. The irreducible representations of almost simple groups in dimension at most 16 are listed in [18] (95.10). Extensive use will be made of this list, compare also Bödi-Joswig [3]. Whenever X is an almost simple factor of Γ, the dimension of a minimal X-invariant subgroup of Θ divides t by Clifford's Lemma [18] (95.5). If t is odd, then Γ' is also irreducible and $\dim \mathsf{Z} \leq 1$. Hence these cases are less difficult, they will be discussed next. A special argument is needed for large values of t.

Proposition 2. *If $t < 15$ and t is odd, and if $\dim \Delta > 32$, then Θ satisfies the assertions of Theorem A.*

Proof. As in Lemma 2, assume that the orbit b^Π is not contained in a line. Since Δ_b acts effectively on Θ, the Note shows that $\dim \Gamma' \geq 16$, and Λ is mapped injectively into Γ. Now let $t = 9$. From $(*)$ it follows that $\dim \Lambda \geq 8$, and (‡) implies that $\Lambda \cong \mathrm{SU}_3\mathbb{C}$ or $\Lambda \cong G_2$. Therefore, $\dim \Delta \leq 39$ and $\dim \Gamma \leq 30$. Moreover, Λ acts on a 6- or 7-dimensional subspace of Θ and fixes a complement, see the List [18] (95.10). By Clifford's Lemma, Λ is *properly* contained in an almost simple factor Υ of Γ, and Υ acts effectively and irreducibly on Θ. Noting that $8 < \dim \Upsilon \leq 30$, the List shows that $\Upsilon \cong \mathrm{PSL}_3\mathbb{C}$, but this group does not contain Λ. Hence $t \in \{11, 13\}$, and t is a prime number. Clifford's Lemma implies that Γ' is almost simple. Since $\dim \Gamma' > 3$, it follows from the List that $\dim \Gamma' \geq 55$, an obvious contradiction. □

Lemma 3. *A group Δ of dimension ≥ 31 fixes at most one point $a \notin W$. Assume that $t \geq 12$ and that $\Theta|_W \neq \mathbb{1}$. Let L be a line such that $L \cap W = u \neq u^\Theta$. If L does not contain the exceptional point a, then $\Theta_L = \mathbb{1}$ and $\dim L^\Theta \geq 12$. Hence L^Θ is not contained in any proper closed subplane of $\mathcal{P}$, and Δ_L acts effectively on Θ.*

Proof. The first assertion follows immediately from (‡) and the dimension formula. Either $\langle L^{\Theta}\rangle = \mathcal{Q}$ is a closed subplane, or L^{Θ} is contained in a pencil $\mathfrak{L}_x$, and $x^{\Theta} = x$. Suppose that $x^{\Delta} \neq x$. Then Θ_u induces the identity on the connected subplane $\mathcal{D} = \langle x^{\Delta}, u^{\Theta}\rangle$. By the dimension formula, $12 \le t = \dim u^{\Theta} + \dim \Theta_u$, and $\dim \Theta_u \ge 4$. Since Θ_u is not compact, (‡) implies $\dim \Theta_u \le 7$, and $\dim u^{\Theta} \ge 5$. Consequently, $\mathcal{D} = \mathcal{P}$ and $\Theta_u = \mathbb{1}$. This contradiction shows that $x^{\Delta} = x$ is the unique point a. Hence $L \neq au$ leads to the first alternative, and Θ_L acts trivially on $\mathcal{Q}$. Either $\dim \Theta_L \ge 8$ or $\dim L^{\Theta} > 4$ and $\mathcal{Q} \le\!\bullet\, \mathcal{P}$. In both cases, Θ_L is compact by (‡), and then $\Theta_L = \mathbb{1}$. Consequently, $\mathcal{Q} = \mathcal{P}$ and $\Delta_L \cap \mathrm{Cs}\,\Theta = \mathbb{1}$. □

The lemma leads to a useful modification of condition $(*)$:

Proposition 3. *In the situation of Lemma 3, each one-parameter subgroup* Π *of* Θ *has some orbit* b^{Π} *which is not contained in any line. If* u *and* L *are chosen as above, let* $\varrho \in \Pi \le \Theta_u$, $\varrho \neq \mathbb{1}$. *Then* $\Theta_u \lhd \mathsf{H} = \Delta_L\Theta$, $\dim \varrho^{\mathsf{H}} \le \dim \Theta_u \le 8$, *and*

$$(**) \qquad \mathsf{H} < \Delta,\ \ \dim \Delta/\mathsf{H} \le 16 - t,\ \ \textit{and}\ \ \dim \mathsf{H} \le 16 + \dim \varrho^{\mathsf{H}_b} + \dim \mathsf{K} \le 24 + \dim \Lambda,$$

where K *denotes the connected component of* $\mathsf{H}_{b,\varrho} = \Delta_b \cap \mathrm{Cs}_{\mathsf{H}}\Pi$ *and* Λ *has the same meaning as in Lemma* 1.

Proof. Because Δ_L fixes the point u and Θ is commutative, Θ_u is invariant in $\mathsf{H} = \Delta_L\Theta$. By assumption, $\Theta_u < \Theta$ and Θ is a minimal normal subgroup of Δ. Moreover, $\dim \Theta_u \ge t - 8 \ge 4$. Consequently, $\mathsf{H} \neq \Delta$. From $\Delta_L \cap \Theta = \mathbb{1}$ it follows that $\dim \mathsf{H} = \dim \Delta_L + t$, and the dimension formula shows $\dim \Delta = \dim \Delta_L + \dim L^{\Delta} \le \dim \Delta_L + 16$. Therefore, $\dim \Delta - \dim \mathsf{H} \le 16 - t$. Since $\dim b^{\mathsf{H}} \le 16$, the last inequality in $(**)$ is immediate from the dimension formula. □

Proposition 4. *Suppose that* $\dim \Delta \ge 31$ *and that* $u^{\Theta} \neq u \in W = W^{\Delta}$. *If* $\dim \Lambda > 8$, *then* $\Lambda \cong \mathrm{G}_2$ *by* (‡), *and the centralizer* $\mathsf{X} = \mathrm{Cs}_{\Theta}\Lambda$ *has dimension at most* 2. *Moreover,* $t \in \{1, 2, 8, 9\}$.

Proof. Use the same notation as in Proposition 3. By assumption, $\Pi \le \mathsf{X}$ and $\dim \mathsf{X} \ge 1$. The orbit b^{X} is contained in the 2-dimensional subplane $\mathcal{E}$ of the fixed elements of Λ. Obviously, $\mathsf{X}_b{}^1 \le \Theta \cap \Lambda = \mathbb{1}$. Therefore, $\dim \mathsf{X} \le 2$. From [18] (95.3 and 10) it follows that the complement of X in Θ has a dimension divisible by 7. Hence the proposition is true unless $t \ge 15$. Then Proposition 3 applies, and the first statements of $(**)$ exclude the possibility $t = 16$. In the case $t = 15$, condition $(**)$ implies that $\dim \Delta \le 39$. Hence Δ induces on Θ an irreducible group Γ of dimension at most 24. According to the note, Γ' is irreducible on Θ, and Λ is properly contained in Γ'. By Clifford's Lemma, Γ' is almost simple. Inspection of the List [18] (95.10) shows that there is no group with these properties. □

Propositions 3 and 4 imply:

Corollary 4. *If* $t \geq 12$, *then* $\dim \Delta \leq 48 - t$ *or* Θ *satisfies the assertions of Theorem A.*

Corollary 5. *If* $t = 16$ *and* $\dim \Delta > 32$, *then* $\Theta = \mathsf{T}$ *is a transitive elation group.*

Proposition 5. *If* $t = 15$ *and* $\dim \Delta > 32$, *then* Θ *is a group of elations.*

Proof. Assume that $u^\Theta \neq u$ for some point u on the fixed line W of Δ. Then $\dim \Delta = 33$ by Corollary 4, and Propositions 3 and 4 give $\dim \mathsf{H} = 32$ and $\Lambda \cong \mathrm{SU}_3\mathbb{C}$. Moreover, $(**)$ implies $\dim \varrho^{\mathsf{H}} = 8$ for each admissible ϱ. Hence Δ_L is transitive on $\Theta_u \cong \mathbb{R}^8$. With [18] (96.22) or the List (95.10) it follows that the commutator subgroup Υ of Δ_L is isomorphic to $\mathrm{SU}_4\mathbb{C}$. According to the Note and to Clifford's Lemma in particular, Υ is properly contained in an almost simple factor of $\Gamma \cong \Delta/\mathrm{Cs}\,\Theta$. This implies $\dim \Gamma \geq 21$ and $\dim \Delta \geq 36$, a contradiction. □

Lemma 4. *Let* p *and* q *be prime numbers. A semi-simple irreducible subgroup* G *of* $\mathrm{SL}_{pq}\mathbb{R}$ *has not more than two almost simple factors.*

Proof. G is either almost simple or a product of two proper semi-simple factors A and B such that $B \leq \mathrm{Cs}\,A$. Let U be a minimal A-invariant subspace of $V = \mathbb{R}^{pq}$. If $U = V$, then $B \leq \mathbb{H}^\times$ by Schur's Lemma [18] (95.4). Hence $4 \mid pq = 4$ and $G \cong \mathrm{SO}_4\mathbb{R}$. If $U < V$, however, then A acts effectively on U (because the fixed elements of the kernel of the action of A on U form a G-invariant subspace of V). Clifford's Lemma shows that $\dim U \in \{p, q\}$ and that A is almost simple.

Alternative proof (suggested by the referee). All semi-simple irreducible subgroups G of $\mathrm{SL}_t\mathbb{R}$ have been determined by Dynkin [5/6] Th. 1.5: either G is maximal in $\mathrm{SL}_t\mathbb{R}$ or in a symplectic or orthogonal group, and the claim follows from Theorems 1.3 and 1.4, or G belongs to a long list of exceptions, but these are even almost simple. □

The following well-known theorem will be needed several times:

Complete reducibility. *If a semi-simple group* G *acts on a real vector space* V, *then each* G*-invariant subspace of* V *has a* G*-invariant complement in* V.

This follows directly from an analogous result about representations of semi-simple Lie algebras, see e.g. Bourbaki [4] I.6.2 Theorem 2, p. 52 or Freudenthal-de Vries [7] § 35 or § 50.

Proposition 6. *If* $t = 14$ *and* $\dim \Delta > 32$, *then* Θ *is a group of elations.*

Proof. Assume that Θ does not act trivially on W. Put again $\Xi = \mathrm{Cs}\,\Theta$.

(a) Corollary 4 shows that $\dim \Delta \leq 34$. According to Lemma 2 and the Note, the stabilizer Δ_b acts effectively on Θ and may be considered as a subgroup of $\Gamma = \Delta/\Xi$, and the semi-simple commutator group Γ' satisfies $15 \leq \dim \Gamma' \leq 20$. No almost simple group with dimension between 15 and 20 has an irreducible representation in dimension 7 or 14.

Therefore, Lemma 4 implies that $\mathsf{\Gamma}'$ is a product of exactly two almost simple subgroups A and B. Let $\dim \mathsf{A} \leq \dim \mathsf{B}$. Then $\dim \mathsf{B} \geq 8$, and B acts effectively and irreducibly on $\mathbb{R}^7$. By the List, $\dim \mathsf{B} = 14$, and B is one of the two simple groups of type G_2. (Later it will be seen that B is in fact the compact form.) As in the proof of Lemma 4, it follows that A acts effectively on $\mathbb{R}^2$. Consequently, $\mathsf{A} \cong \mathrm{SL}_2\mathbb{R}$ and $\dim \mathsf{\Gamma}' = 17$, moreover, the center Z of $\mathsf{\Gamma}$ consists of real dilatations of Θ, see [18] (95.6).

(b) The last statement gives $\dim \mathsf{\Gamma} \leq 18$ and $\dim \Xi \geq 15$. On the other hand, $\dim \Xi \leq 16$ since $\dim \Delta/\Xi \geq \dim \Delta_b \geq \dim \Delta - 16$. Therefore, Ξ is contained in the radical $\sqrt{\Delta}$, and a maximal semi-simple subgroup Ψ of Δ is locally isomorphic to $\mathsf{\Gamma}'$.

(c) The group Δ has a subgroup $\widehat{\Delta}$ of codimension 3 which induces on Θ the group BZ and acts on a 7-dimensional subgroup N of Θ. Note that $\dim \widehat{\Delta}_b \geq 14$. If $\widehat{\Delta}_b$ is transitive on N, then it is also transitive on the 6-sphere consisting of the rays in N, and $\widehat{\Delta}_b$ contains the compact group G_2, compare [18] (96.19 and 22). If $\widehat{\Delta}_b$ is not transitive on N, then for some $\varrho \in \mathsf{N}$ the connected component $\widehat{\Lambda}$ of $\widehat{\Delta}_{b,\varrho}$ is at least 8-dimensional, and (‡) shows that $\widehat{\Lambda} \cong \mathrm{SU}_3\mathbb{C}$. This group is not contained in the non-compact group $\mathrm{G}_2(2)$. Therefore B is compact, and steps (a) and (b) imply $\mathsf{\Gamma}' = \mathsf{A} \times \mathsf{B} \cong \mathrm{SL}_2\mathbb{R} \times \mathrm{G}_2 \cong \Psi$.

(d) Because $\Delta_b \hookrightarrow \mathsf{\Gamma}$ and $\dim \mathsf{\Gamma}/\Delta_b \leq 1$, it follows that $\mathrm{G}_2 \cong \mathsf{B} \hookrightarrow \Delta_b$. One can now conclude that Δ_b acts irreducibly on Θ. In fact, the action of B and complete reducibility force a proper Δ_b-invariant subgroup N of Θ to be 7-dimensional. Lemma 1 with $\varrho \in \mathsf{N}$ gives $\dim \Delta_{b,\varrho} \geq 10$ and then $\Lambda \cong \mathrm{G}_2$, but this contradicts the fact $\varrho^{\Lambda} = \varrho$. As a consequence of [18] (95.6(b)), even the action of the semi-simple group $\Delta_b{}'$ on Θ is irreducible, hence $\dim \Delta_b{}' = 17$ and $\Delta_b{}' \cong \Psi$.

(e) In particular, the involution in A corresponds to an involution α in the center of Δ_b. By [18] (84.9), the group $\mathsf{B} \cong \mathrm{G}_2$ cannot act on a Baer subplane. Consequently, α is a reflection, see [18] (55.29). Because of (‡), the group A acts effectively on the 2-dimensional plane $\mathcal{E} = \mathcal{F}_{\mathsf{B}}$ of the fixed elements of B. If A would fix a flag in $\mathcal{E}$, then A would be solvable by [18] (33.8). Therefore, b and W are the only fixed elements of A in $\mathcal{E}$, and $\alpha \in \Delta_{[b,W]}$. Since $\Theta_b = \mathbb{1}$, it follows from [18] (61.19b) that $\alpha^{\Theta}\alpha$ is a 14-dimensional subset of $\mathsf{T} = \Delta_{[W,W]}$. Note that $\Theta\mathsf{T} \leq \sqrt{\Delta}$ and that $\dim \sqrt{\Delta} \leq 17$. Minimality of Θ implies $\Theta \leq \mathsf{T}$. □

Proposition 7. *If* $t = 10$ *and* $\dim \Delta > 32$, *then* Θ *is a group of elations.*

Proof. From $(*)$ and Proposition 4 one obtains $17 \leq \dim \Delta_b \leq \dim \varrho^{\Delta_b} + \dim \Lambda \leq 10 + 8$. Either Δ_b is transitive on Θ, or $\Lambda \cong \mathrm{SU}_3\mathbb{C}$ and $\dim \varrho^{\Delta_b} = 9$ for some $\varrho \in \Theta$. In the first case, Δ_b would contain the group $\mathrm{SU}_5\mathbb{C}$, and $\dim \Delta_b$ would be to large, compare [18] (96.16–22). Similarly, Δ_b cannot be transitive on a 9-dimensional subspace of Θ. Hence Δ_b acts effectively and irreducibly on Θ. Since each non-trivial representation of $\mathrm{SU}_3\mathbb{C}$ on Θ is either 6- or 8-dimensional, it follows from Clifford's Lemma that Λ is not normal in Δ_b. Therefore, Λ is properly contained in an almost simple and irreducible factor X of $\Delta_b{}'$, and $\mathsf{X} = \Delta_b{}'$ by Schur's Lemma [18] (95.4). The List shows that $\Delta_b{}'$ must be locally isomorphic to $\mathrm{SL}_4\mathbb{R}$ or to $\mathrm{SL}_2\mathbb{H}$, but these groups do not contain $\mathrm{SU}_3\mathbb{C}$. □

The only remaining cases $t = 8$ and $t = 12$ are more difficult. If the given group Θ does not consist of elations, it will be shown that some other normal vector group $\widetilde{\Theta}$ of Δ satisfies the conditions of Theorem A.

Proposition 8. *If $t = 8$ and $\dim \Delta > 32$, then either Θ or some minimal normal subgroup $\widetilde{\Theta} \cong \mathbb{R}^7$ consists of elations.*

Proof. (a) According to Corollary 3, we may assume that for each one-parameter subgroup $\mathsf{\Pi} < \Theta$ there is some point b such that $b^{\mathsf{\Pi}}$ generates a subplane. Lemma 1 then shows that the connected component $\mathsf{\Lambda}$ of $\Delta_b \cap \mathrm{Cs}\,\mathsf{\Pi}$ has dimension at least 9, and $\mathsf{\Lambda} \cong \mathrm{G}_2$ by (‡). Consider the action of the connected component B of Δ_b on the 7-sphere S consisting of the rays in Θ, and let r, r' denote the two opposite rays contained in $\mathsf{\Pi}$. Since $\dim \mathsf{B} \geq 17$ and $\dim \mathsf{B}_r/\mathsf{\Lambda} \leq 1$, it follows that r^{B} is a connected set of positive dimension. For each $s \in S \setminus \{r, r'\}$, the orbit $s^{\mathsf{\Lambda}}$ is a 6-sphere, and r^{B} is a connected union of $\mathsf{\Lambda}$-orbits. Consequently, r^{B} contains an open neighbourhood of r in S, and r^{B} is open in S, see also [18] (96.25). The dimension formula implies $\dim \mathsf{B}/\mathsf{\Lambda} \geq 7$ and $21 \leq \dim \Delta_b \leq 22$.

(b) In particular, r^{Δ} is open in S, and this is true for each ray $r \in S$ because step (a) is valid for an arbitrary choice of r. Therefore, Δ acts transitively on S. Put $\Xi = \mathrm{Cs}\,\Theta$ as in the Note. Then $\mathsf{\Gamma} = \Delta/\Xi$ is the effective group induced by Δ on Θ. Remember from Lemma 2 that Δ_b is embedded into $\mathsf{\Gamma}$. Step (a) and Lemma 1 imply $21 \leq \dim \mathsf{\Gamma} \leq 30$. According to [18] (96.19), a maximal compact subgroup $\mathsf{\Phi}$ of $\mathsf{\Gamma}$ acts transitively on S, and from [18] (96.20) it follows that $\mathsf{\Phi}$ is isomorphic to a subgroup of $\mathrm{SO}_8\mathbb{R}$. Moreover, $\mathsf{\Phi}$ has a subgroup $\mathsf{\Lambda} \cong \mathrm{G}_2$. There are only two groups $\mathsf{\Phi}$ which satisfy these conditions, viz. $\mathsf{\Phi} \cong \mathrm{Spin}_7\mathbb{R}$ and $\mathsf{\Phi} \cong \mathrm{SO}_8\mathbb{R}$, see [18] (96.21 and 22). The centralizer of $\mathsf{\Phi}$ in $\mathrm{GL}_8\mathbb{R}$ is isomorphic to $\mathbb{R}^{\times}$, and the remarks in the Note show that $\mathsf{\Gamma}$ is the product of $\mathsf{\Phi} = \mathsf{\Gamma}'$ and the center Z of $\mathsf{\Gamma}$. By [18] (94.27), the group Δ contains a subgroup $\mathsf{\Psi}$ which is a covering group of $\mathsf{\Gamma}'$. In fact, $\mathsf{\Psi}$ is simply connected, since Δ cannot contain a group $\mathrm{SO}_7\mathbb{R}$, see [18] (55.34 or 40).

(c) Assume that $\dim \mathsf{\Gamma}' > 21$. Then $\mathsf{\Psi} \cong \mathrm{Spin}_8\mathbb{R}$, and the center of $\mathsf{\Psi}$ contains 3 reflections α, σ, and $\alpha\sigma$ with centers a, u, and v. By (‡), the stabilizer ∇ of the triangle $\{a, u, v\}$ satisfies $\dim \nabla \leq 30$, and $\mathsf{\Psi} \leq \nabla < \Delta$. It suffices to consider the case $\Delta = \mathsf{\Psi}\Theta$. Note that Δ is not transitive on W (otherwise Δ would induce on W the group $\mathrm{SO}_9\mathbb{R}$). On $W \setminus v$ the action of $\mathsf{\Psi}$ is equivalent to a linear action, and for each $z \neq u, v$ the orbit $z^{\mathsf{\Psi}}$ is a 7-sphere, compare [18] (96.36). Hence z^{Δ} is open in W whenever $z^{\Delta} \neq z^{\mathsf{\Psi}}$, see also [18] (96.25). If $v^{\Delta} = v$, then $u^{\Delta} = u^{\Theta}$ is open in W, and $\sigma^{\Theta}\sigma = \{\vartheta^{-1}\vartheta^{\sigma} \mid \vartheta \in \Theta\}$ would generate a transitive group of elations with axis av in Θ. Therefore $u^{\Theta} \neq u$ and $v^{\Theta} \neq v$, and W contains some orbit $z^{\Delta} = Z \approx \mathbb{S}_7$. This leads to a contradiction as follows: Since z^{Θ} is $\mathsf{\Psi}_z$-invariant, the argument of [18] (96.25) shows that either $z^{\Theta} = z$ or z^{Θ} is open in Z. Because $\mathsf{\Psi}$ is transitive on Z and Θ is normal, all Θ-orbits in Z are equivalent. The commutative group Θ cannot be transitive on Z. Therefore the orbits are points, $\Theta|_Z = \mathbb{1}$, and Θ would act freely and transitively on $az \setminus \{a, z\} \approx \mathbb{R}^8 \setminus \{0\}$ which is impossible.

(d) The last steps imply $\mathsf{\Psi} \cong \mathsf{\Gamma}' \cong \mathrm{Spin}_7\mathbb{R}$. Since $21 \leq \dim \Delta_b \leq \dim \mathsf{\Gamma} \leq 22$, and since $\mathrm{Spin}_7\mathbb{R}$ has no subgroup of codimennsion 1, the covering group $\mathsf{\Psi}$ of $\mathsf{\Gamma}'$ can be chosen in Δ_b. Hence $\Delta_b{}' \cong \mathrm{Spin}_7\mathbb{R}$. Now it is not difficult to determine the structure of Δ. Put again $\Xi = \mathrm{Cs}\,\Theta$. Lemma 2 implies $\Xi_b = 1\!\!1$ and $\dim \Xi \leq 16$. On the other hand, the bound on $\dim \Gamma$ gives $\dim \Xi \geq 11$. Consider the group $\Upsilon = \Xi \cap \mathrm{Cs}\,\Psi$ and note that $\Lambda \leq \Psi$. The orbit b^Υ is contained in the 2-dimensional fixed plane $\mathcal{F}_\Lambda$. Consequently, $\dim \Upsilon \leq 2$. Let Ψ act on the Lie algebra $\mathfrak{l}\Xi$. By complete reducibility, $\mathfrak{l}\Theta$ has an invariant complement $\mathfrak{n}$ in $\mathfrak{l}\Xi$, and $\dim \mathfrak{n} > 2$. If $\dim \mathfrak{n} < 7$, the representation of $\mathrm{Spin}_7\mathbb{R}$ on $\mathfrak{n}$ is trivial and $\mathfrak{n} = \mathfrak{l}\Upsilon$, a contradiction. Hence $\dim \mathfrak{n} \geq 7$ and $\dim \Xi \in \{15, 16\}$. Because $\Psi \cap \Xi \leq \Xi_b = 1\!\!1$, we may assume that $\Delta = \Psi\Xi$.

(e) The central involution σ of Ψ is a reflection, its axis K is different from W (or else $\sigma^\Theta\sigma = \Theta$ would consist of elations). If $\dim \Xi = 15$, then $\mathfrak{n} \cong \mathbb{R}^7$ and Ψ induces on $\mathfrak{n}$ the group $\mathrm{SO}_7\mathbb{R}$. Therefore $\mathfrak{n}$ is the Lie algebra of $\mathsf{N} = \Xi \cap \mathrm{Cs}\,\sigma$, moreover, Ξ is a vector group and N is Δ-invariant. Proposition 1 shows that $\widetilde{\Theta} = \mathsf{N}$ is an elation group as required.

(f) Finally, let $\dim \Xi = 16$. Because $\Xi_b = 1\!\!1$, the orbit b^Ξ is open in P by [18] (96.11(a)). Note that $b^\sigma = b$. If $\xi \in \Xi$ and $b^\xi \in K$, then $\xi\sigma\xi^{-1}\sigma \in \Xi_b = 1\!\!1$ and $\sigma\xi = \xi\sigma$. This gives $\Xi_K = \Xi \cap \mathrm{Cs}\,\sigma$ and $\dim \Xi_K = 8$, moreover, $\Xi = \Xi_K \times \Theta$. Under the action of Ψ, the group Ξ_K splits into a one-parameter group and a 7-dimensional vector group $\widetilde{\Theta}$ on which Ψ induces a group $\mathrm{SO}_7\mathbb{R}$. Obviously, $\widetilde{\Theta}$ is $\Psi\Xi$- and hence Δ-invariant, and $\widetilde{\Theta}$ is an elation group, again by Proposition 1 and Corollary 2. □

Proposition 9. *If $t = 12$ and $\dim \Delta > 32$, then Δ has a normal vector subgroup $\widetilde{\Theta}$ consisting of elations. ($\widetilde{\Theta}$ may be different from Θ).*

Proof. Assume that Θ is not contained in the elation group $\mathsf{T} = \Delta_{[W,W]}$, and use the notation introduced in Proposition 3.

(a) Propositions 3 and 4 imply that $29 \leq \dim \mathsf{H} \leq 32$. Consider a minimal H-invariant subgroup M of Θ_u and let $1\!\!1 \neq \varrho \in \Pi \leq \mathsf{M}$. Then H and Δ_L act irreducibly on $\mathsf{M} \supset \varrho^\mathsf{H}$, and $\dim \varrho^\mathsf{H} \geq 5$ by $(**)$. Remember from Lemma 2 that $\Theta_b = 1\!\!1$. Consequently, $\dim b^\mathsf{M} > 4$ and $\langle b^\mathsf{M} \rangle = \mathcal{B}$ is at least 8-dimensional ($\mathcal{B} \leq\!\bullet\, \mathcal{P}$). Statement (d) of (‡) shows that $\dim(\mathsf{H}_b \cap \mathrm{Cs}\,\mathsf{M}) \leq 3$, and H induces on M a group of dimension at least 10. If K denotes again the connected component of $\mathsf{H}_b \cap \mathrm{Cs}\,\Pi$, then $\dim \mathsf{K} \leq 8$ by Proposition 4. The following will be shown in the next steps: $\mathsf{M} = \Theta_u \cong \mathbb{R}^8$ and Δ_L does not act effectively on M.

(b) If $\mathsf{M} \cong \mathbb{R}^5$, then $(**)$ and (‡) imply $\mathsf{K} \cong \mathrm{SU}_3\mathbb{C}$. This group does not admit a non-trivial representation in dimension < 6. Hence $\mathsf{K} \leq \mathrm{Cs}\,\mathsf{M}$ contrary to what has been stated in (a).

(c) In the case $\mathsf{M} \cong \mathbb{R}^6$, the same argument shows that $\dim \mathsf{K} < 8$ (note that the action of K on M is not irreducible, since Π is K-invariant). Consequently, $\dim \mathsf{H}_b = 13$ by $(**)$, moreover, $\dim \varrho^{\mathsf{H}_b} = 6$ for each choice of ϱ, and H_b acts transitively on $\mathsf{M} \setminus \{1\!\!1\}$, see [18] (96.11(a)). Let $\Psi = \mathsf{H}_b|_\mathsf{M}$ denote the effective group induced on M. Since Ψ is irreducible and $\dim \Psi \leq 13$, the List of representations shows $\Psi' \cong \mathrm{SU}_3\mathbb{C}$, and $\dim \Psi = 10$. Hence H_b has a normal subgroup $\Phi \cong \mathrm{SU}_2\mathbb{C}$ acting trivially on M, cf. step (a) and (‡)(d). The involution $\omega \in \Phi$ fixes a Baer subplane $\mathcal{F}_\omega$ and the orbit b^M is contained in the pointset

F of $\mathcal{F}_\omega = \mathcal{B}$. Since $\mathsf{H}_b \le \mathrm{Cs}\,\omega$, the group Ψ' acts non-trivially on $\mathcal{B}$ and, therefore, on $S = F \cap W \approx \mathbb{S}_4$, but this contradits Richardson's theorem [18] (96.34).

(d) Finally, let $\mathsf{M} \cong \mathbb{R}^7$. Steps (b) and (c) imply that H_b acts irreducibly on M, and $(**)$ shows that $\dim \mathsf{H}_b \le 15$. Again, $\Psi = \mathsf{H}_b|_\mathsf{M}$ is at least 10-dimensional. According to the Note, Ψ' is an almost simple group of type G_2. By complete reducibility, M has a Ψ'-invariant complement $\mathsf{N} \cong \mathbb{R}^5$ in Θ. If $\langle b^\mathsf{M}\rangle = \mathcal{B}$ is a Baer subplane, then $\Psi'\mathsf{M}$ is a 21-dimensional automorphism group of $\mathcal{B}$, and $\mathcal{B}$ is isomorphic to the quaternion plane $\mathcal{H}$, see Salzmann [12], cp. also [18] (84.21(b)) or Salzmann [14], but $\mathcal{H}$ does not admit a group of type G_2. Hence $\langle b^\mathsf{M}\rangle = \mathcal{P}$ and Ψ' is a subgroup of H_b. The List shows that Ψ' centralizes N. Therefore, Ψ' induces the identity on b^N. Because $\langle b^\mathsf{M}\rangle = \mathcal{P}$, there is some point $c \in b^\mathsf{M}$ such that $\langle b^\mathsf{N}, c\rangle = \mathcal{P}$. Consequently, $\Psi'_c = \mathbb{1}$, but $\dim \Psi'_c \ge 7$, a contradiction proving the first statement at the end of step (a).

(e) Suppose now that Δ_L acts effectively on $\mathsf{M} \cong \mathbb{R}^8$. By assumption, this action is irreducible. The structure theorem [18] (95.6(b)) shows that the commutator subgroup $\Upsilon = \Delta_L{}'$ is semi-simple and that the center Z of Δ_L consists of real or complex dilatations of M. There are two possibilities: (*i*) the action of Υ on M is irreducible, and (*ii*) M is a direct sum of two 4-dimensional subspaces M_ν and Υ acts equivalently and effectively on the spaces M_ν. Note that $\dim \Delta_L \ge 17$ and $\dim \Upsilon \ge 15$. The action of the semi-simple group Υ on Θ is completely reducible. Hence there is a Υ-invariant complement $\mathsf{N} \cong \mathbb{R}^4$ of M in Θ. In case (*i*), the group N is even Δ_L-invariant: in fact, for each $\zeta \in \mathsf{Z}$ the subspace $\mathsf{N}^\zeta\mathsf{N}$ is invariant under Υ and at most 8-dimensional, hence it has trivial intersection with M. Choose a one-parameter group E in N and a point p such that $\langle p^\mathsf{E}\rangle = \mathcal{E}$ is a subplane. Remember that $\mathsf{H} = \Delta_L\Theta$ has dimension at least 29. Consider the connected component $\widetilde{\mathsf{K}}$ of $\mathsf{H}_p \cap \mathrm{Cs}\,\mathsf{E}$. The dimension formula gives $\dim \widetilde{\mathsf{K}} \ge 9$, and (‡) shows that $\widetilde{\mathsf{K}} \cong \mathrm{G}_2$, but then Proposition 4 would imply $t \le 9$. Case (*ii*) also leads to a contradiction: Υ is semi-simple and effective on $\mathbb{R}^4$, therefore, $\Upsilon \cong \mathrm{SL}_4\mathbb{R}$ and $\dim \Upsilon = 15$, moreover, $\mathsf{Z} \cong \mathbb{C}^\times$ and Z has a subgroup $\mathsf{P} \cong \mathbb{R}$ consisting of real dilatations. The group $\widehat{\mathsf{H}} = \Upsilon\mathsf{P}\Theta$ acts on $\mathsf{M}_\nu \cong \mathbb{R}^4$. Choose $\varrho \in \mathsf{M}_\nu$ and write $\widehat{\mathsf{K}}$ for the connected component of $\widehat{\mathsf{H}}_{b,\varrho}$. Then $\dim \widehat{\mathsf{K}} \ge 8$, and $\widehat{\mathsf{K}} \cong \mathrm{SU}_3\mathbb{C}$ by (‡), but the latter group is not contained in the maximal semi-simple subgroup Υ of $\widehat{\mathsf{H}}$. This proves the last claim of step (a).

(f) The group $\mathsf{M} \cong \mathbb{R}^8$ acts transitively on the affine line pencil $\mathcal{L}_u \setminus \{W\} = \mathcal{L}_u^* \approx \mathbb{R}^8$: in Lemma 3 it has been shown that $\Theta_L = \mathbb{1}$ for each line $L \in \mathcal{L}_u^*$ with at most one exception au. By [18] (96.11), each non-trivial orbit L^M is open in $\mathcal{L}_u^*$ and homeomorphic to $\mathbb{R}^8$. Hence M is sharply transitive on $\mathcal{L}_u^*$ or on $\mathcal{L}_u^* \setminus \{au\}$, but the latter space is not contractible.

(g) According to (e) there is an element $\alpha \ne \mathbb{1}$ in $\Delta_L \cap \mathrm{Cs}\,\mathsf{M}$, and (f) implies that α fixes each line in $\mathcal{L}_u$. Consequently, α is an axial collineation with center u and some axis A. If α is an elation, i.e. if $u \in A$, then $A = W$ (since $\alpha^\mathsf{M} = \alpha$), and α^Θ is a 4-dimensional subset of T. Because the elements in α^Θ have different centers, T is commutative. Any minimal invariant subgroup of T may be chosen as the group $\widetilde{\Theta}$.

(h) Now let $\alpha \in \Delta_{[u,A]}$ be a homology. Since $u^\Delta \ne u$, Hähl's results on the generation of elations by homologies can be applied. In its simplest form, Hähl's theorem says the following:

($\bullet$) *If* $\mathsf{\Gamma}$ *is a Lie subgroup of* Σ_A*, if* $\mathsf{\Gamma}_{[c,A]} \neq \mathbb{1}$ *for some center* $c \notin A$*, and if* E *is the group of elations in* $\mathsf{\Gamma}$ *with axis* A*, then* $\dim \mathsf{E} = \dim c^{\mathsf{\Gamma}}$, see [18] (61.20) for a proof. Note that commutativity of E is not known if $c^{\mathsf{\Gamma}}$ is contained in a line.

(i) Suppose that $\Theta \leq \mathsf{\Gamma} = \Delta_A$, and put $A \cap W = v$. Then ($\bullet$) shows that $\dim \Delta_{[v,A]} \geq 4$. Moreover, $\dim u^{\Theta} = 4$ implies that the commutator set $[\alpha, \Theta] = \{\alpha^{-1}\alpha^{\vartheta} \mid \vartheta \in \Theta\} \subseteq \Theta_{[A]}$ is at least 4-dimensional. Since Θ is commutative, all elements in $\Theta_{[A]}$ have the same center $z \in W$, and $\dim \Theta_{[A]} \leq 8$. By [18] (61.2), a group of homologies has a compact subgroup of codimension at most 1, but Θ is a vector group. Hence $\Theta_{[A]} = \Theta_{[v,A]}$ consists of elations. Because Θ is a minimal normal subgroup of Δ, the group $\Theta_{[A]}$ is not normal, and, therefore, $A^{\Delta} \neq A$. Since $A^{\Theta} = A$ and Θ is normal, Θ fixes each line in the orbit A^{Δ}. On the other hand, all fixed lines of the elation group $\Theta_{[A]}$ pass through the center v. Consequently, $A^{\Delta} \subseteq \mathfrak{L}_v$ and $\Theta = \Theta_{[v,v]}$ is an invariant elation group.

(j) Only the possibility $A^{\Theta} \neq A$ remains. The strategy in this case is to apply the dual ($\widetilde{\bullet}$) of ($\bullet$) to the group Ω generated by α and the connected component of Δ_u. Assume first that $A^{\Omega} = A$. Because $\dim(\Theta_u \cup \Theta_A) < 12$, there is an element $\vartheta \in \Theta$ such that $u^{\vartheta} = z \neq u$ and $A^{\vartheta} \neq A$. Remember that $\Theta \lhd \Delta$ and that $\dim u^{\Theta} = 4$. Therefore, $z^{\Omega} \subseteq u^{\Theta}$ and $\dim \Omega_z \geq 21$. Let $a \in A \setminus W$, $a \notin A^{\vartheta}$. Then the connected component Λ of $\Omega_{a,z}$ satisfies $\dim \Lambda \geq 13$, moreover, $\Lambda \leq \Omega^{\vartheta} \leq \Delta_{A^{\vartheta}}$, and Λ fixes a non-degenerate quadrangle. Now (‡) implies $\Lambda \cong \mathrm{G}_2$. By assumption, $\mathsf{\Gamma} = \Delta/\mathrm{Cs}\,\Theta$ is an irreducible subgroup of $\mathrm{GL}_{12}\mathbb{R}$, and $\dim \mathsf{\Gamma} \leq 24$ by Corollary 4. Since $\Lambda \hookrightarrow \mathsf{\Gamma}'$, it follows from [18] (95.6) that $\mathsf{\Gamma}'$ is almost simple and irreducible on Θ. Hence $\mathsf{\Gamma}' \cong \mathrm{Spin}_7\mathbb{R}$, but, according to the List, this group does not have an irreducible representation in dimension 12.

(k) Consequently, $A^{\Omega} \neq A$, and then $\dim \Delta_{[u,u]} \geq \dim A^{\Omega} > 0$ by ($\widetilde{\bullet}$). If some one-parameter group in $\Delta_{[u,u]}$ has an affine axis, then, because of (f), all groups $\Delta_{[u,L]}$ with $L \in \mathfrak{L}_u \setminus \{W\}$ have the same positive dimension, and the dual of [18] (61.12) implies $\Delta_{[u,W]} \cong \mathbb{R}^8$. Since $u^{\Delta} \neq u$, it follows that T is transitive. If each one-parameter subgroup of $\Delta_{[u,u]}$ has axis W however, then $\dim \Delta_{[z,W]} > 0$ for each center $z \in u^{\Delta}$. Hence $\dim \mathsf{T} > 4$, and T contains a normal subgroup $\widetilde{\Theta}$ as claimed.

Theorem A is now an immediate consequence of the propositions.

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