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Shaking Compact Sets

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Abstract. If C is a compact subset of $\mathbb{R}^d$ and H is a halfspace bounded by a hyperplane π, the set $\tilde{C}$ obtained by shaking C on π is defined as the set contained in H, such that for every line ℓ orthogonal to π, $\tilde{C} \cap \ell$ is a segment of the same length as $C \cap \ell$, and one of its endpoints is on π. It is shown that there exist $d+1$ hyperplanes such that every compact set can be reduced to a simplex, via repeated shaking processes on these hyperplanes.

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Introduction

Symmetrizations and more general rearrangements are a powerful tool used to solve many problems in mathematics. We mention for instance their employment in the proof of isoperimetric and functional inequalities (see e.g. [13], [14], [5]). The best known symmetrization is the one introduced by Steiner. Given a compact set $C \subset \mathbb{R}^d$ and a hyperplane π, the Steiner symmetral C' of C with respect to π is obtained as follows: for every line ℓ orthogonal to π, $C' \cap \ell$ is a segment of the same length as $C \cap \ell$, having its midpoint on π. An essential feature of this process is that every compact set can be reduced to a ball of the same volume via countably many Steiner symmetrizations (see e.g. [2], [7], [8]).

The aim of the present paper is to establish an analogous result for another type of rearrangement, namely the *shaking* process. The set $\tilde{C}$ obtained by shaking C on π is the

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set, contained in one of the halfspaces bounded by π, such that $\tilde{C} \cap \ell$ is again a segment of the same length as $C \cap \ell$, having one endpoint on π.

The shaking process (*Schüttelung*) was introduced by Blaschke in [3] for solving the Sylvester problem in the plane. Blaschke's argument relies on the fact that, for every line ℓ, the set obtained by shaking a triangle T on ℓ is affinely equivalent to T. This property characterizes triangles similarly as the affine invariance under Steiner symmetrizations characterizes ellipses.

In higher dimensions, while the class of ellipsoids is still closed under Steiner symmetrizations, it is not true that the shaking process maps simplices into simplices.

For convex bodies both Steiner and shaking processes can be seen as particular instances of a more general class of transformations which move continuously each chord of a set parallel to a fixed direction. These transformations were used by the authors to approach extremal problems of Sylvester's type (see [6]).

The idea underlying the shaking process can be found in several papers. For instance, Uhrin [15] used it for strengthening the Brunn-Minkowski-Lusternik inequality, and Kleitman [9] introduced a discrete version of this process to obtain discrete isoperimetric inequalities (see also Bollobás and Leader [4]).

In [1] Biehl showed that given a convex body K in $\mathbb{R}^2$ there exists a sequence of lines π_i, $i \in \mathbb{N}$, such that the process of shaking K successively on π_i transforms it into a triangle. Furthermore Biehl suggests that his argument can be extended to arbitrary dimension. In fact such an extension is performed by Schöpf in [11]. We notice that in the procedure used by Biehl and Schöpf the choice of π_i is recursive, i.e. it depends on the resulting body at the previous step.

In the present paper we improve this result in two directions: We extend it to the class of compact sets and we prove that the sequence π_i can be chosen independently of K. More precisely, we consider the simplex S in $\mathbb{R}^d$ whose vertices are at the points $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ and an arbitrary compact set C. We show that by shaking repeatedly C on the hyperplanes bounding S we obtain a sequence of sets converging (up to translations) to a simplex homothetic to S, in the Hausdorff metric. The simpler case when C is convex, which will be treated in Section 2, plays a key role in proving the result in its generality (Section 3).

We note that since the sequence of hyperplanes can be chosen independently of the compact set, we can transform two compact sets into homothetic simplices simultaneously. This fact can be used to obtain the Brunn-Minkowski-Lusternik inequality (see Remark 3.3).

1. Notations and preliminaries

In the d-dimensional Euclidean space $\mathbb{R}^d$ let O denote the origin, $e_1, e_2, \dots, e_d$ the standard orthonormal basis, and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0, i = 1, 2, \dots, d\}$. Furthermore S^{d-1} and B^d stand for the unit sphere and the unit ball of $\mathbb{R}^d$, respectively.

We denote by $\mathcal{C}^d$ the family of all compact sets in $\mathbb{R}^d$ and by $\mathcal{K}^d$ the subset of $\mathcal{C}^d$ of all convex bodies, i.e. all compact convex sets in $\mathbb{R}^d$ with non-empty interior. Both $\mathcal{C}^d$ and $\mathcal{K}^d$ are endowed with the Hausdorff distance d_H and the Minkowski or vector sum (see e.g. [10]). If $A \in \mathcal{C}^d$, then $|A|_d$ stands for its d-dimensional Lebesgue measure, $conv(A)$, $int(A)$ and ∂A for the convex hull, the interior and the boundary of A, respectively.

Let $C \in \mathcal{C}^d$ and fix a hyperplane π and a unit vector v orthogonal to π. We define the set $C_{\pi,v}$ obtained by shaking C on π with respect to v as follows. For every $x \in \pi$, let $C(x)$ be the intersection of C with the straight line through x parallel to v. We define

$$C_{\pi,v}(x) = \begin{cases} \emptyset & \text{if } C(x) = \emptyset \\ conv(\{x, x + |C(x)|_1 \, v\}) & \text{if } C(x) \neq \emptyset \end{cases}$$

and

$$C_{\pi,v} = \bigcup_{x \in \pi} C_{\pi,v}(x) \, .$$

In the following lemma we collect some basic properties of the shaking process. The proof immediately follows from the definition.

Lemma 1.1.
(i) *If $C \in \mathcal{C}^d$, then $C_{\pi,v} \in \mathcal{C}^d$.*
(ii) *If $C \in \mathcal{K}^d$, then $C_{\pi,v} \in \mathcal{K}^d$.*
(iii) *If $C, D \in \mathcal{C}^d$ and $C \subset D$, then $C_{\pi,v} \subset D_{\pi,v}$.*

Let $e_0 = -\frac{1}{\sqrt{d}} \sum_{i=1}^d e_i$ and $\pi_0 = \{x \in \mathbb{R}^d : \langle x, \, e_0 \rangle = \frac{-1}{\sqrt{d}}\}$, where $\langle \; , \; \rangle$ is the standard scalar product in $\mathbb{R}^d$. Furthermore let $\pi_i = \{x \in \mathbb{R}^d : x_i = 0\}$, $i = 1, 2, \dots, d$. The hyperplanes π_0, π_1, ..., π_d bound a simplex that will be referred to as S_0 throughout.

For every $i = 0, 1, \dots, d$, we define a mapping $T_i : \mathcal{C}^d \to \mathcal{C}^d$ by

$$T_i(C) = C_{\pi_i, e_i}$$

and a mapping $T : \mathcal{C}^d \to \mathcal{C}^d$ by

$$T = T_d \circ T_{d-1} \circ \cdots \circ T_1 \circ T_0 \, .$$

We shall denote by $\mathcal{P}^d$ the class of all $C \in \mathcal{C}^d$ satisfying the following properties:
(i) $C \subset \overline{\mathbb{R}^d_+} = \{x \in \mathbb{R}^d \, : \, x_i \geq 0, i = 1, 2, \dots, d\}$,
(ii) if $x = (x_1, \, x_2, \, \dots, \, x_d) \in C$ and $y = (y_1, \, y_2, \, \dots, \, y_d)$ are such that $0 \leq y_i \leq x_i$, $i = 1, 2, \dots, d$, then $y \in C$.

Clearly $\mathcal{P}^d$ is closed with respect to the Hausdorff metric.

Lemma 1.2. *If $C \in \mathcal{C}^d$ then $T(C) \in \mathcal{P}^d$.*

Proof. The proof can be easily obtained by taking the following facts into account. If $A \in \mathcal{C}^d$ and $x \in T_i(A)$, $i = 1, 2, \dots, d$, then the segment joining x and its projection onto π_i is contained in $T_i(A)$. Moreover, this property is preserved by $T_j \circ T_i(A)$, $i < j \leq d$. $\square$

It is easy to exhibit examples showing that the shaking process, and a fortiori T, is not continuous in $\mathcal{C}^d$. Continuity can be recovered if T is restricted to some special subsets of $\mathcal{C}^d$. For example, T is continuous in $\mathcal{K}^d$. Indeed, if K_n, $n \in \mathbb{N}$, is a sequence of convex bodies converging to $K \in \mathcal{K}^d$, then the measure of the symmetric difference $|K_n \Delta K|_d$ tends to 0 as $n \to \infty$. Clearly $|T(K_n) \Delta T(K)|_d \leq |K_n \Delta K|_d$. This implies that $T(K_n)$ converges to $T(K)$ in the Hausdorff metric (see [12]).

A further example, which is useful for our purposes, is given by the following lemma.

Lemma 1.3. *The restriction of the mapping T to $\mathcal{P}^d$ is continuous.*

Proof. Up to a rescaling, we can restrict ourselves to the elements of $\mathcal{P}^d$ contained in S_0.

We notice that if C is an element from $\mathcal{P}^d$, then C can be seen as the region enclosed by the graphs of two Lipschitz functions, f and g, defined on π_0, with Lipschitz constant less than or equal to one. Thus $T_0(C)$ is the region bounded by π_0 and the graph of $f - g$. From this fact the continuity of T_0 in $\mathcal{P}^d$ easily follows.

Moreover, it follows that there exists in $\mathbb{R}^d$ a closed convex cone K with non-empty interior and vertex at O satisfying the following condition: if $C \in \mathcal{P}^d$ and $x \in T_0(C)$ then $(x + K) \cap \Sigma \subseteq T_0(C)$, where Σ is the closed half-space bounded by π_0 containing O.

Notice that, by the above considerations, $|\partial T_0(C)|_{d-1} \leq M$ for every $C \in \mathcal{P}^d$, $C \subset S_0$, where $M > 0$ is independent of C.

Now let us fix $x \in T_1(T_0(C))$, and let ℓ be the line through x parallel to e_1. Clearly $|\ell \cap T_0(C)|_1 = |\ell \cap T_1(T_0(C))|_1 \geq \langle x, e_1 \rangle$. The set

$$V = \bigcup_{y \in \ell \cap T_0(C)} ((y + K) \cap \Sigma)$$

is contained in $T_0(C)$. By (iii) of Lemma 1.1, $T_1(V) \subseteq T_1(T_0(C))$. In particular there exists a closed convex cone K' with non-empty interior and vertex at O such that

$$(x + K') \cap \mathbb{R}^d_+ \subseteq T_1(T_0(C)) .$$

Furthermore K' can be chosen independently of C and x.

The next step is to show that the above property gives the continuity of T_1 in $T_0(\mathcal{P}^d)$. Owing to the monotonicity of T_1 it is sufficient to prove that for every $\varepsilon > 0$ and $D \in T_0(\mathcal{P}^d)$, $D \subset S_0$, there exists $\delta > 0$ such that

$$d_H\left(T_1(D), T_1(D_\delta)\right) < \varepsilon , \tag{1.1}$$

where D_δ is the parallel set $D + \delta B^d$, and δ depends only on ε. Let $K_0 = K \cap \Sigma$, and fix $\mu > 0$ so that μK_0 contains a translate of the unit ball. We have that

$$d_H\left(T_1(D), T_1(D_\delta)\right) \leq d_H\left(T_1(D), T_1(D + \delta\mu K_0)\right) . \tag{1.2}$$

Recall that $|\partial D|_{d-1} \leq M$, where M is independent of D. Therefore

$$\lim_{\delta \to 0^+} |D \Delta (D + \delta\mu K_0)|_d = 0 , \tag{1.3}$$

uniformly with respect to D.

Let $K'_0 = K' \cap \{x_1 \geq -1\}$, and λ be such that

$$|\lambda K'_0|_d = 2\,|D\Delta(D + \delta\mu K_0)|_d \,. \tag{1.4}$$

Then, by (1.3), λ tends uniformly to 0 as $\delta \to 0$.

We prove that

$$T_1(D + \delta\mu K_0) \subseteq T_1(D) + \rho B^d \,, \tag{1.5}$$

where $\rho = 2\max\{\delta\, diam(\mu K_0), \lambda\, diam(K'_0)\}$.

Fix $x \notin T_1(D)+\rho B^d$, we have that $(x+\lambda K'_0)\cap T_1(D) = \emptyset$. If we assume that $x \in T_1(D+\delta\mu K_0)$, then $(x+\lambda K'_0)\cap\{x_1 \geq 0\} \subseteq T_1(D+\delta\mu K_0)$. Thus, by (1.4), $\|x-x'\| \leq \lambda\, diam(K'_0)$, where x' is the orthogonal projection of x onto π_1. Since $d_H\,(x', T_1(D)) \leq \delta\, diam(\mu K_0)$, we get

$$d_H\,(x, T_1(D)) < \rho\,,$$

i.e. a contradiction.

Finally by (1.2) and (1.5), inequality (1.1) follows.

The same argument used for showing that T_1 is continuous in $T_0(\mathcal{P}^d)$ can be repeated to prove the continuity of T_i in $T_{i-1}(T_{i-2}(\ldots T_0(\mathcal{P}^d)))$, $i = 2, 3, \ldots, d$, and the proof of the lemma is complete. □

2. The convex case

In this section we prove the following:

Theorem 2.1. *Let $K \in \mathcal{K}^d$, then the sequence*

$$K_0 = K\ ,\quad K_{i+1} = T(K_i)\,,\ i \in \mathbb{N}\,,$$

converges to $S = \rho S_0$, where $\rho > 0$ is such that $|S|_d = |K|_d$.

Proof. Let us define $\varphi : \mathcal{P}^d \cap \mathcal{K}^d \to \mathbb{R}$ by

$$\varphi(C) = \max\{\lambda \geq 0 : \lambda S_0 \subseteq C\}\,.$$

We show that φ is continuous.

Let $C \in \mathcal{P}^d\cap\mathcal{K}^d$, thus $(C+\varepsilon B^d)\cap\overline{\mathbb{R}^d_+} = C+\varepsilon(B^d\cap\overline{\mathbb{R}^d_+})$. Since $B^d\cap\overline{\mathbb{R}^d_+} \subset \sqrt{d}S_0$, we have that $(C+\varepsilon B^d)\cap\overline{\mathbb{R}^d_+} \subset C+\varepsilon\sqrt{d}S_0$. Hence, $\varphi((C+\varepsilon B^d)\cap\overline{\mathbb{R}^d_+}) \leq \varphi(C+\varepsilon\sqrt{d}S_0) = \varphi(C)+\varepsilon\sqrt{d}$, which implies the continuity of φ.

As $T(\lambda S_0) = \lambda S_0$, for every $\lambda \geq 0$, it follows from the monotonicity of T (see Lemma 1, (iii)), that

$$\varphi(C) \leq \varphi(T(C))\,. \tag{2.1}$$

Let us show that in (2.1) equality holds if and only if

$$C = \bar{\lambda} S_0\,, \text{ for some } \bar{\lambda} > 0\,. \tag{2.2}$$

Assume that C does not satisfy (2.2). Let $S' = \lambda' S_0$, with $\lambda' = \varphi(C)$. The orthogonal projection of the origin onto the facet of S' parallel to π_0 is contained in $int\, C$. This implies that for a suitable $t \in \mathbb{R}$, $S' + te_0 \subseteq T_0(C)$ and $te_0 \in int\, T_0(C)$. Hence, for some $\varepsilon > 0$

$$te_0 - \varepsilon e_i \in T_0(C)\,,\ i = 1, 2, \ldots, d\,.$$

It is easy to check that this implies that

$$(\lambda' + \varepsilon) S_0 = S' + \varepsilon S_0 \subseteq T(C)\,.$$

Thus $\varphi(T(C)) > \varphi(C)$.

Let K_{λ_i} be a subsequence of K_i converging to some $\tilde{K}$ and assume that $\tilde{K}$ is not of the form λS_0. Clearly $\tilde{K} \in \mathcal{P}^d$, and therefore

$$\varphi(T(\tilde{K})) > \varphi(\tilde{K})\,. \tag{2.3}$$

On the other hand, by Lemma 1 and the continuity of φ,

$$\varphi(T(\tilde{K})) = \lim_{i\to\infty} \varphi(T(K_{\lambda_i})) = \lim_{i\to\infty} \varphi(K_{\lambda_i+1})\,. \tag{2.4}$$

Moreover, by (2.1)

$$\varphi(K_{\lambda_i+1}) \le \varphi(K_{\lambda_{i+1}})\,, \text{ for all } i \in \mathbb{N}\,. \tag{2.5}$$

From (2.4) and (2.5) we obtain a contradiction to (2.3). Thus we have proved that every converging subsequence of K_i tends to S. This fact, together with the Blaschke selection theorem (see e.g. [10], p.50) concludes the proof. □

Remark 2.2. The proof of Theorem 2.1 can be adapted to show the following more general result. Let S be a simplex in $\mathbb{R}^d$ and let $\nu_1, \nu_2, \ldots, \nu_{d+1}$ be the outer unit normals to its facets. Assume that $\langle \nu_i, \nu_j \rangle \le 0$, $i, j = 1, 2, \ldots, d+1$, $i \neq j$, i.e. the angles between the facets of S are acute. Denote by $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_d$ the hyperplanes bounding S and define the mapping T as above with π_i replaced by $\tilde{\pi}_i$, $i = 0, 1, \ldots, d$. Then, for every $K \in \mathcal{K}^d$, the sequence $K_0 = K$, $K_i = T(K_{i-1})$, $i = 1, 2, \ldots$, converges to a simplex homothetic to S, in the Hausdorff metric.

3. The general case

Theorem 3.1. *Let $C \in \mathcal{C}^d$, then the sequence*

$$C_0 = C\ ,\quad C_{i+1} = T(C_i)\,,\ i \in \mathbb{N}\,,$$

converges to $S = \rho S_0$, where $\rho \ge 0$ is such that $|S|_d = |C|_d$.

The proof is based on Lemma 3.2 stated below. Let us define $\psi : \mathcal{P}^d \to \mathbb{R}$ by $\psi(C) = \min\{\lambda \ge 0 : C \subset \lambda S_0\}$. It is easy to verify that the map ψ is continuous.

Lemma 3.2. *If $C \in \mathcal{P}^d$, then*

$$\psi(T(C)) \leq \psi(C)\,. \tag{3.1}$$

Moreover if C is not homothetic to S_0, then there exists $j \in \mathbb{N}$ such that

$$\psi(T^j(C)) < \psi(C)\,. \tag{3.2}$$

Proof. The proof of (3.1) follows easily from the monotonicity of the mapping T with respect to the inclusion.

In order to establish (3.2) we argue by contradiction. Assume that C is not homothetic to S_0. Up to a rescaling we may take $\psi(C) = 1$. If (3.2) were not true, then

$$\psi(T^j(C)) = 1\,, \text{ for all } j \in \mathbb{N}\,. \tag{3.3}$$

Step 1. Equality (3.3) implies

$$conv(T^j(C)) = conv(C) = S_0\,, \text{ for all } j \in \mathbb{N}\,. \tag{3.4}$$

Indeed

$$conv(T^j(C)) \subseteq \psi(T^j(C))S_0 = S_0\,.$$

If, for some $j \in \mathbb{N}$, $conv(T^j(C))$ were strictly contained in S_0, then by Theorem 2.1 we would have

$$\lim_{i\to\infty} T^i(conv(T^j(C))) = \lambda S_0$$

for a certain $\lambda < 1$. Therefore for sufficiently large i and a suitable $\lambda' < 1$, we have

$$T^{i+j}(C) \subseteq T^i(conv(T^j(C))) \subset \lambda' S_0$$

which contradicts (3.3). Hence (3.3) implies (3.4).

Step 2. Let F be the closed facet of S_0, orthogonal to e_0 and $E_j = T^j(C) \cap F$. We prove that $|E_j|_{d-1}$, $j \in \mathbb{N}$, is a non-increasing sequence.

For $i = 1, 2, \ldots, d$, we denote by P_i the vertex of F lying on the x_i-axis. Let Q be the centroid of F and

$$F_i = conv\{Q, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_d\}\,,\ i = 1, 2, \ldots, d\,.$$

We have that

$$\left|T_0(T^j(C)) \cap \pi_i\right|_{d-1} = \sqrt{d}\,|F_i \cap E_j|_{d-1}\,.$$

Hence

$$\begin{aligned} |F_i \cap E_j|_{d-1} &= \frac{1}{\sqrt{d}} \left|T_{i-1}(T_{i-2}(\ldots T_0(T^j(C)))) \cap \pi_i\right|_{d-1} \\ &\geq \frac{1}{d} \left|T_i(T_{i-1}(\ldots T_0(T^j(C)))) \cap F\right|_{d-1} \\ &\geq \frac{1}{d} \left|E_{j+1}\right|_{d-1}\,. \end{aligned}$$

If we sum up these inequalities then we find

$$|E_{j+1}|_{d-1} \leq |E_j|_{d-1} \tag{3.5}$$

and equality holds only if

$$|F_1 \cap E_j|_{d-1} = |F_2 \cap E_j|_{d-1} = \cdots = |F_d \cap E_j|_{d-1} .$$

Step 3. Let C_{λ_j} be any converging subsequence of $C_j = T^j(C)$, $j \in \mathbb{N}$, and call $\tilde{C}$ its limit. By using Lemma 1.3, the continuity of ψ and (3.3) one proves that

$$\psi(T^j(\tilde{C})) = \psi(\tilde{C}) = 1\,, \text{ for all } j \in \mathbb{N}\,. \tag{3.6}$$

Furthermore, by (3.4)

$$conv(T^j(\tilde{C})) = conv(C) = S_0\,, \tag{3.7}$$

and we define

$$\tilde{E} = F \cap \tilde{C}\ , \quad \tilde{E}_j = F \cap T^j(\tilde{C})\,.$$

We now prove that

$$\left|\tilde{E}_{j+1}\right|_{d-1} = \left|\tilde{E}_j\right|_{d-1}\ , \text{ for all } j \in \mathbb{N}\,. \tag{3.8}$$

To do this, a crucial ingredient is the following equality

$$\left|\tilde{E}\right|_{d-1} = \lim_{i\to\infty} |E_i|_{d-1}\ . \tag{3.9}$$

Since, by (3.5), $|E_i|_{d-1}$ is monotone, it suffices to prove that

$$\left|\tilde{E}\right|_{d-1} = \lim_{j\to\infty} |E_{\lambda_j}|_{d-1}\ .$$

For $H \in \mathcal{P}^d$, and $t > 0$, we set

$$H_t = H \cap \Sigma_t\,,$$

where $\Sigma_t = \{x \in \mathbb{R}^d_+ : \langle x, e_0\rangle = -t\}$. Since H is a compact set, $|H_t|_{d-1}$ is a lower semicontinuous function of t. If we denote by $c(d)$ the volume of the $(d-1)$-dimensional regular simplex with edge length $\sqrt{2d}$, then $|H_t|_{d-1} - c(d)t^{d-1}$ is a non-increasing function of t. To see this let Σ' be the orthogonal projection of Σ_{t_1} onto Σ_{t_2}, for $0 \leq t_1 < t_2$. Since $H \in \mathcal{P}^d$,

$$|H \cap \Sigma'|_{d-1} \leq |H_{t_1}|_{d-1}\ .$$

Consequently

$$|H_{t_2}|_{d-1} \leq |H \cap \Sigma'|_{d-1} + |\Sigma_{t_2} \setminus \Sigma'|_{d-1} \leq |H_{t_1}|_{d-1} + c(d)(t_2^{d-1} - t_1^{d-1})\,.$$

From the lower semicontinuity of $|H_t|_{d-1}$ and the monotonicity of $|H_t|_{d-1} - c(d)t^{d-1}$, it follows that

$$\lim_{t\to t_0^-} |H_t|_{d-1} = |H_{t_0}|_{d-1} \ , \ \text{for all } t_0 > 0 \, . \tag{3.10}$$

For every $t > 0$, the function

$$f_H(t) = \int_0^t |H_s|_{d-1} \, ds$$

admits left-hand side derivative $f_H'^-(t)$, since

$$f_H(t) - \frac{c(d)}{d} t^d$$

is concave for $t \geq 0$. Moreover, by (3.10),

$$f_H'^-(t) = |H_t|_{d-1} \ , \ \text{for all } t > 0 \, . \tag{3.11}$$

Let us reconsider the sequence C_j. We define

$$f_j(t) = f_{C_j}(t) \, , \ j \in \mathbb{N} \ , \quad \tilde{f}(t) = f_{\tilde{C}}(t) \, , \ t > 0 \, .$$

Since $f_j(t) = \left| C_j \cap t\sqrt{d} S_0 \right|_d$, for every $t \geq 0$, the monotonicity of T implies

$$f_j(t) \leq f_{j+1}(t) \, , \ t \geq 0 \, . \tag{3.12}$$

Moreover, since $\frac{1}{\sqrt{d}}$ is the distance of F from the origin, we have that

$$f_j(t) = |C|_d \ , \ \text{for all } j \in \mathbb{N}, \, t \geq \frac{1}{\sqrt{d}} \, . \tag{3.13}$$

By (3.12) and (3.13), the sequence f_j converges pointwise to a bounded function in $[0, +\infty)$. For every $t > 0$,

$$\lim_{j\to\infty} f_j(t) = \lim_{j\to\infty} f_{\lambda_j}(t) = \lim_{j\to\infty} \left| C_{\lambda_j} \cap t\sqrt{d} S_0 \right|_d \, .$$

Since C_{λ_j} tends to $\tilde{C}$ and $\left| \partial(C_{\lambda_j} \cap t\sqrt{d} S_0) \right|_d = \left| \partial(\tilde{C} \cap t\sqrt{d} S_0) \right|_d = 0$, it follows that

$$\lim_{j\to\infty} f_j(t) = \left| \tilde{C} \cap t\sqrt{d} S_0 \right|_d = \tilde{f}(t) \, , \ t \geq 0 \, .$$

By (3.11), in order to establish (3.9), it is sufficient to prove that

$$\lim_{j\to\infty} f_j'^-\left(\frac{1}{\sqrt{d}}\right) = \tilde{f}'^-\left(\frac{1}{\sqrt{d}}\right) .$$

For every $j \in \mathbb{N}$, by (3.12) and (3.13), we have that

$$\tilde{f}'^{-}(\frac{1}{\sqrt{d}}) = \lim_{h \to 0^+} \frac{f_j(\frac{1}{\sqrt{d}}) - \tilde{f}(\frac{1}{\sqrt{d}} - h)}{h} \leq \lim_{h \to 0^+} \frac{f_j(\frac{1}{\sqrt{d}}) - f_j(\frac{1}{\sqrt{d}} - h)}{h} = f_j'^{-}(\frac{1}{\sqrt{d}}) .$$

On the other hand, if we fix $\varepsilon > 0$, and take $h > 0$ so that

$$\tilde{f}'^{-}(\frac{1}{\sqrt{d}}) \geq \frac{\tilde{f}(\frac{1}{\sqrt{d}}) - \tilde{f}(\frac{1}{\sqrt{d}} - h)}{h} - \varepsilon$$

and

$$-\frac{c(d)}{d} \cdot \frac{\frac{1}{d^{d/2}} - (\frac{1}{\sqrt{d}} - h)^d}{h} + c(d)\frac{1}{d^{(d-1)/2}} < \varepsilon \, ,$$

then

$$\tilde{f}'^{-}(\frac{1}{\sqrt{d}}) \geq \lim_{j \to \infty} \frac{f_j(\frac{1}{\sqrt{d}}) - f_j(\frac{1}{\sqrt{d}} - h)}{h} - \varepsilon \geq \limsup_{j \to \infty} f_j'^{-}(\frac{1}{\sqrt{d}}) - 2\varepsilon \, ,$$

where in the last inequality we used the concavity of $f_j(t) - \frac{c(d)}{d} t^d$.

Now we can apply the above argument to the sequence $C_{\lambda_j + i}$ converging to $T^i(\tilde{C})$ for every $j \in \mathbb{N}$. In this way we prove that

$$\lim_{j \to \infty} |E_{j+i}|_{d-1} = \left| \tilde{E}_i \right|_{d-1} \, , \text{ for all } i \in \mathbb{N} \, ,$$

which implies (3.8).

Step 4. We claim that (3.8) implies that either

$$\left| \tilde{E}_j \right|_{d-1} = 0 \, , \text{ for all } j \in \mathbb{N} \tag{3.14}$$

or

$$\left| \tilde{E}_j \right|_{d-1} = |F|_{d-1} \, , \text{ for all } j \in \mathbb{N} \, . \tag{3.15}$$

By the same argument used in Step 2 we obtain

$$\left| \tilde{E}_j \cap F_i \right|_{d-1} = \frac{1}{d} \left| \tilde{E}_j \right|_{d-1} = \frac{1}{d} \left| \tilde{E} \right|_{d-1} \, , \; j \in \mathbb{N}, \; i = 1, 2, \ldots, d \, . \tag{3.16}$$

Let $V = S_0 \cap \pi_d$, and Q' its centroid. Set

$$V_1 = conv\{Q', P_1, P_2, \ldots, P_{d-1}\}\,,\ V_2 = V \setminus V_1\,.$$

Since V_1 is the orthogonal projection of F_d onto π_d, we have that

$$\begin{aligned}\left|\tilde{E}_{j+1} \cap F_d\right|_{d-1} &= \left|F_d \cap T_d(T_{d-1}(\ldots(T_0(\tilde{C}_j))))\right|_{d-1} \\ &\le \sqrt{d}\left|V_1 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1}\,. \end{aligned} \tag{3.17}$$

Analogously one obtains

$$\left|\tilde{E}_{j+1} \setminus F_d\right|_{d-1} \le \sqrt{d}\left|V_2 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1}\,. \tag{3.18}$$

On the other hand

$$\begin{aligned}&\left|V_1 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1} + \left|V_2 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1} \\ &= \left|V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1} = \left|V \cap T_0(\tilde{C}_j)\right|_{d-1} \\ &\le \sqrt{d}\left|\tilde{E}_j \cap F_d\right|_{d-1} = \frac{\sqrt{d}}{d}\left|\tilde{E}_{j+1} \cap F\right|_{d-1}\,.\end{aligned}$$

Hence, equality holds in (3.17), (3.18) and consequently

$$\left|V_1 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1} = \frac{1}{d-1}\left|V_2 \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))\right|_{d-1}\,. \tag{3.19}$$

Notice that $T_{d-1}(\ldots(T_0(\tilde{C}_j)))$ is the resulting set after $d-1$ repeated shakings of $T_0(\tilde{C}_j)$ along e_1, e_2, $\ldots$, e_{d-1}. Thus $V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))$ satisfies the following property (see Lemma 1.2): if $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{d-1}, 0) \in V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))$ then

$$V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j))) \supseteq P\,, \tag{3.20}$$

where $P = \{(x_1, x_2, \ldots, x_{d-1}, 0) \in V : 0 \le x_i \le \bar{x}_i,\ i = 1, 2, \ldots, d-1\}$.

Let r be a line contained in π_d and parallel to $(1, 1, \ldots, 1, 0)$ and let $\sigma_r = r \cap V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j)))$. By (3.20), if $\sigma_r \neq \emptyset$, so that it is a segment having an endpoint on π_i for some $i \in \{1, 2, \ldots, d-1\}$. On the other hand

$$|r \cap V_1|_1 = \frac{1}{d-1}\,|r \cap V_2|_1\,,$$

thus

$$|\sigma_r \cap V_1|_1 \le \frac{1}{d-1}\,|r \cap V_2|_1\,.$$

Now, if $|\sigma_r \cap V_1|_1 \neq 0$ then $|r \cap V_2|_1 = |\sigma_r \cap V_2|_1$. Therefore

$$|\sigma_r \cap V_1|_1 \leq \frac{1}{d-1} |\sigma_r \cap V_2|_1 \ , \tag{3.21}$$

where equality holds if and only if

$$|\sigma_r|_1 = 0 \quad \text{or} \quad \sigma_r = r \cap V \, .$$

From Fubini's theorem and (3.19) it follows that equality holds in (3.21) for almost every line r. By using again (3.20) we conclude that for all r

$$|\sigma_r|_1 = 0 \, , \tag{3.22}$$

or

$$\sigma_r = r \cap V \, . \tag{3.23}$$

If (3.22) holds true, then $\left| V \cap T_{d-1}(\ldots(T_0(\tilde{C}_j))) \right|_{d-1} = 0$, which implies immediately

$$\left| T(\tilde{C}_j) \cap F \right|_{d-1} = \left| \tilde{C}_{j+1} \cap F \right|_{d-1} = \left| \tilde{E}_{j+1} \right|_{d-1} = 0 \, .$$

If (3.23) holds, then

$$T_{d-1}(\ldots(T_0(\tilde{C}_j))) \cap V = V \, ,$$

and

$$T_0(\tilde{C}_j) \cap V = V \, .$$

This in turn implies that

$$\left| \tilde{E}_j \cap F_d \right|_{d-1} = |F_d|_{d-1}$$

and, by (3.16),

$$\left| \tilde{C}_j \cap F \right|_{d-1} = \left| \tilde{E}_j \right|_{d-1} = |F|_{d-1} \ .$$

This concludes Step 4.

Step 5. In this step we show that both (3.14) and (3.15) lead to a contradiction. If $\left| \tilde{E}_j \right|_{d-1} = 0$, for all $j \in \mathbb{N}$, then

$$\left| T_0(\tilde{C}_j) \cap V \right|_{d-1} = 0 \, .$$

Consequently

$$T_{d-1}(\ldots(T_0(\tilde{C}_j))) \cap V \subseteq V \cap \left(\bigcup_{i=1}^{d-1} \pi_i \right) \ ,$$

and therefore

$$\tilde{C}_{j+1} \cap F \subseteq F \cap \left(\bigcup_{i=1}^{d-1} \pi_i \right) .$$

Thus $Q \notin \tilde{E}_{j+1}$ and $(0, 0, \ldots, 0) \notin T_0(\tilde{C}_{j+1})$. This yields

$$\left| conv(T_0(\tilde{C}_{j+1})) \right|_d < |S_0|_d .$$

But

$$\left| conv(\tilde{C}_{j+2}) \right|_d \leq \left| conv(T_0(\tilde{C}_{j+1})) \right|_d ,$$

so we have a contradiction to (3.7).

Finally, since $\tilde{C}_j \in \mathcal{P}^d$ for every $j \in \mathbb{N}$, if $\left| \tilde{E}_j \right|_{d-1} = |F|_{d-1}$, then $\tilde{C} = S_0$. But $\left| \tilde{C} \right|_d = |C|_d$, so we get a contradiction to the assumption that C is not a simplex. □

Proof of Theorem 3.1. Let $\tilde{C}$ be the limit of a converging subsequence C_{λ_i} of C_i and assume that $\tilde{C}$ is not homothetic to S_0. Then, by Lemma 3.2, there exists $j \in \mathbb{N}$ such that

$$\psi(T^j(\tilde{C})) < \psi(\tilde{C}) . \tag{3.24}$$

The sequence $C_{\lambda_i + j}$, $i \in \mathbb{N}$, converges to $T^j(\tilde{C})$, by Lemma 1.3. Moreover, by Lemma 3.2,

$$\psi(\tilde{C}_{\lambda_i + j}) \geq \psi(\tilde{C}_{\lambda_{i+j}}) , \text{ for all } i \in \mathbb{N} .$$

As i tends to $+\infty$ in the above inequality, we get

$$\psi(T^j(\tilde{C})) \geq \psi(\tilde{C}) ,$$

which contradicts (3.24). □

Remark 3.3. We notice that by applying Theorem 3.1 one obtains the Brunn-Minkowski-Lusternik inequality. Indeed if A and B are two compact sets in $\mathbb{R}^d$ then

$$(A + B)_{\pi,v} \supset A_{\pi,v} + B_{\pi,v} ,$$

for arbitrary π and v (see [15]). Therefore the volume of $A + B$ does not increase when A and B are shaken on the same hyperplane. On the other hand, by Theorem 3.1, A and B can be reduced to homothetic simplices through the same sequence of shaking processes. Thus

$$|A + B|_d^{1/d} \geq |A|_d^{1/d} + |B|_d^{1/d} .$$

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