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A Class of Discriminant Varieties in the Conformal 3-Sphere

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Abstract. We associate a 3-manifold to a multiple valued semi-conformal mapping on the 3-sphere and study the branching set of the corresponding covering projection. Some remarkable geometric structures occur.

1. Introduction

Let $P(z,x)$ be a polynomial in $z \in \mathbf{C}P^1$ whose coefficients are smooth complex-valued functions of $x \in M^m$, where M^m is a smooth manifold of dimension m:

$$P(z,x) \equiv a_n(x)z^n + a_{n-1}(x)z^{n-1} + \cdots a_1(x)z + a_0(x)\,. \tag{1}$$

Then we define the *discriminant variety associated to* P, to be the set K of points $x \in M^m$, for which the (algebraic) equation

$$P(z,x) = 0\,, \tag{2}$$

has a multiple root.

Recall that for two polynomials:

$$\begin{aligned} f &= a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0 \\ g &= b_l z^l + b_{l-1}z^{l-1} + \cdots + b_0\,, \end{aligned}$$

the *resultant* is the determinant

$$R_{f,g} = \begin{vmatrix} a_k & a_{k-1} & \cdots & \cdots & a_0 & & & \\ & a_k & a_{k-1} & \cdots & \cdots & a_0 & & \\ & & & \ddots & & & & \\ & & a_k & a_{k-1} & \cdots & \cdots & a_0 & \\ b_l & b_{l-1} & \cdots & \cdots & b_0 & & & \\ & b_l & b_{l-1} & \cdots & \cdots & b_0 & & \\ & & & \ddots & & & & \\ & & b_l & b_{l-1} & \cdots & \cdots & b_0 & \end{vmatrix}$$

As is well-known in algebra, f and g have a common zero if and only if $R_{f,g} = 0$, so that the discriminant variety consists of those points $x \in M^m$ for which the resultant $R_{P,\partial P/\partial z}$ (i.e. the *discriminant*) vanishes. In general it can be extremely hard to calculate the zero set of $R_{P,\partial P/\partial z}$. One of our objectives is to find other practical ways of doing this. We are able to achieve this goal for certain classes of polynomials P.

The study of discriminant varieties is motivated by the following covering construction, well-known in Riemann surface theory and important in 3-manifold theory (see the Introduction of [7] for an account of related constructions).

Set $\widetilde{M} = \{(z,x) \in \mathbf{C}P^1 \times M^m : P(z,x) = 0\}$. Then provided $\mathrm{d}P$ has maximal rank 2 along $P(z,x) = 0$, $\widetilde{M}$ is a smooth submanifold of $\mathbf{C}P^1 \times M^m$ of dimension m. There are natural projections $\pi : \widetilde{M} \to M$ and $\varphi : \widetilde{M} \to \mathbf{C}P^1$ given by $\pi(z,x) = x$, $\varphi(z,x) = z$, respectively. We then view π as a covering map, branched over the discriminant variety. Thus K is the image of the set C of critical points of π. We refer to the image of C under φ as the *set of singular values* and denote this set by Σ.

A particular case of this construction was considered by Gudmundsson and Wood [9] (see also [1]), with M^3 equal to either $\mathbf{R}^3$ or S^3 endowed with their Euclidean metrics and with any local solution $z = z(x)$ to (2) a harmonic morphism (see [5] for definitions) – in particular this is equivalent to z being semi-conformal (see below) with geodesic fibres. One consequence of our study is to show how it is possible to determine explicitly the discriminant variety in this case. A class of examples, which we refer to as (p,q)-examples, displays some fascinating properties. Here, p, q are positive co-prime integers satisfying $1 < p < q < 2p$. One component of the discriminant variety lies on a sphere whose radius is bounded above and below independently of p and q. By suitably choosing p and q, this component can be made to make an arbitrary number of rotations about an axis and forms a geometric object resembling a shell in space.

We shall take a more general viewpoint than that described above which is tied up intimately with the conformal geometry of S^3. We will suppose

1. $M^3 = S^3$ endowed with any smooth metric conformal to the standard Euclidean one;
2. any smooth local solution to 2 is *semi-conformal* (see below);
3. the regular fibres of such a solution are arcs of circles.

Let us explain these assumptions. A fundamental theorem of topology asserts that every closed connected orientable 3-manifold can be viewed as a cover of S^3, branched over some link (see [12] for an account), hence our choice of S^3. We will describe examples of discriminant varieties consisting of multi-component links with non-trivial topological structure.

By analogy with Riemann surface theory, we expect the conformal geometry of S^3 to play an important role. A smooth local solution $z : U \to \mathbf{C}$, $U \subset S^3$ to (2) is said to be *semi-conformal* if and only if

$$\nabla z \cdot \nabla z = 0\,, \tag{3}$$

where the gradient is taken with respect to the x-variable and is defined in terms of any choice of metric conformal to the standard one on S^3. Note that (3) is independent of this choice. There are various equivalent definitions of semi-conformal (cf. [5]), one is as follows. Define the horizontal space $H_x \subset T_x S^3$ at $x \in S^3$, to be the subspace $H_x = (\ker \mathrm{d}\varphi_x)^\perp$. Then z is semi-conformal if and only if the restriction

$$\mathrm{d}z|_{H_x} : H_x \to T_{z(x)}\mathbf{C}$$

is surjective and conformal at each point $x \in U$ where $\mathrm{d}z(x) \neq 0$. Differentiating (2) implicitly with respect to x yields

$$\frac{\partial P}{\partial z}\nabla z + \nabla_x P = 0$$

so that, under the assumption $\partial P/\partial z \neq 0$, (3) is satisfied if and only if

$$\nabla_x P \cdot \nabla_x P = 0\,, \tag{4}$$

at every point along $P(z,x) = 0$.

Finally, the space of circles in conformal S^3 has a particularly nice description in terms of holomorphic geometry which we will exploit. This classical representation is due to Laguerre [11]. A comprehensive account is found in the treatise of Coolidge ([6], Theorem 34). We give a modern treatment in the next section.

2. The space of circles in S^3

The complex quadric hypersurface

$$\mathcal{H} = \{[w_0, \ldots, w_4] \in \mathbf{C}P^4 : w_0^2 + w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0\} \subset \mathbf{C}P^4$$

can be identified with the space of oriented circles in S^3, compactified to include points, as follows [2]. Given $[w_0, \ldots, w_4] \in \mathcal{H}$, define the oriented 2-plane P^2 in $\mathbf{R}^4$ (or 3-plane if all the w_i are real) by the equation

$$w_0 y_0 + w_1 y_1 + w_2 y_2 + w_3 y_3 + w_4 = 0 \quad (y = (y_0, \ldots, y_3) \in \mathbf{R}^4). \tag{5}$$

(The orientation is determined by that on the normal space to P^2 in $\mathbf{R}^4$, which in turn is determined by $\mathrm{Re}\, w$ and $\mathrm{Im}\, w$). Then the intersection with the sphere $S^3 = \{y \in \mathbf{R}^4 : |y| = 1\}$ gives either a point or an oriented circle. If we identify $S^3 \setminus \{(1,0,0,0)\}$ with $\mathbf{R}^3$ via stereographic projection: $(x_1, x_2, x_3) = (y_1, y_2, y_3)/(1 - y_0)$, then its equation is given explicitly by

$$(w_4 + w_0)|x|^2 + 2w_2 x_1 + 2w_2 x_2 + 2w_3 x_3 + w_4 - w_0 = 0\,, \tag{6}$$

or, after inversion $\hat{x} = x/|x|^2$, by

$$(w_4 + w_0) + 2w_2\hat{x}_1 + 2w_2\hat{x}_2 + 2w_3\hat{x}_3 + (w_4 - w_0)|\hat{x}|^2 = 0\,. \tag{7}$$

There is a natural antiholomorphic involution or *real structure* σ on $\mathcal{H}$, defined by $\sigma([w_0, \ldots, w_4]) = [\overline{w}_0, \ldots, \overline{w}_4]$. The set of fixed points (w_i real) corresponds to circles of zero radius, i.e. to points of S^3. Indeed, (5) is now a 3-plane, tangent to S^3 at the point $y = (w_0/w_4, w_1/w_4, w_2/w_4, w_3/w_4)$. Otherwise, σ has the effect of reversing the orientation of the circle. Finally, $w_0 + w_4 = 0$ if and only if the corresponding circle in S^3 passes through the point $(1, 0, 0, 0)$. Such circles can be identified with the space of lines in $\mathbf{R}^3$, compactified to include the point ∞ (given by $[-1, 0, 0, 0, 1] \in \mathcal{H}$). Indeed the hyperplane $w_0 + w_4 = 0$ is tangent to $\mathcal{H}$ at $[-1, 0, 0, 0, 1]$. Its intersection $\mathcal{L}$ with $\mathcal{H}$ is a cone on the point $[-1, 0, 0, 0, 1]$ and $\mathcal{L} \setminus \{[-1, 0, 0, 0, 1]\}$ is biholomorphic to the complex surface $T\mathbf{C}P^1$.

Explicitly, from (6), the equation of the line is given by

$$2w_1x_1 + 2w_2x_2 + 2w_3x_3 + w_4 - w_0 = 0\,,$$

where $w_1^2 + w_2^2 + w_3^2 = 0$. The case $(w_1, w_2, w_3) = 0$ corresponds to the point ∞, otherwise, if we set $g = -w_3/(w_1 - iw_2)$, $h = (w_0 - w_4)/2(w_1 - iw_2)$, its equation becomes

$$q - g^2\overline{q} - 2gx_3 - 2h = 0\,, \tag{8}$$

where $q = x_1 + ix_2$. Then $g \in \mathbf{C} \cup \{\infty\}$ represents the direction u of the line in the chart given by stereographic projection $\rho : S^2 \to \mathbf{C} \cup \{\infty\}$, while γ represents the unique vector c with endpoint on the line orthogonal to the line under the derivative of $\rho : \gamma = \mathrm{d}\rho_u(c)$ [4].

It will be convenient to introduce the following chart on $\mathcal{H}$. Provided $w_0 + w_4 \neq 0$, set

$$\xi = (\xi_1, \xi_2, \xi_3) = \frac{1}{w_0 + w_4}(w_1, w_2, w_3) \tag{9}$$

Then $\xi^2 = (w_4 - w_0)/(w_4 + w_0)$ and we can parametrize those points of $\mathcal{H}$ where $w_0 + w_4 \neq 0$, by setting

$$[w] = \left[\frac{1 - \xi^2}{2}, \xi_1, \xi_2, \xi_3, \frac{1 + \xi^2}{2}\right].$$

Equation (6) now takes the form

$$|x|^2 + 2\xi \cdot x + \xi^2 = 0\,, \tag{10}$$

or, more economically

$$(x + \xi)^2 = 0\,,$$

which, by definition is equivalent to saying that the vector $x + \xi \in \mathbf{C}^3$ is *isotropic.* The chart given by ξ parametrizes all circles and points in S^3 except those passing through ∞. Under inversion $\hat{x} = x/|x|^2$, (10) becomes

$$\xi^2|\hat{x}|^2 + 2\xi \cdot \hat{x} + 1 = 0\,. \tag{11}$$

In terms of the coordinate $\hat{x}$ for S^3, a line in $\mathbf{R}^3$ which does not pass through the origin now corresponds to a point in $\mathcal{H}$ with $\xi^2 = 0$. It will be convenient later on to take this viewpoint. Thus we will identify lines in $\mathbf{R}^3$ which do not pass through the origin (after inversion) with circles in $\mathbf{R}^3$ passing through the origin, i.e. with circles given by (10) such that $\xi^2 = 0$.

3. Semi-conformality and the discriminant variety

Locally, a semi-conformal map $z : U \subset S^3 \to \mathbf{C}P^1$ with arcs of circles as fibres is given by defining the point $[w] \in \mathcal{H}$ as a holomorphic function of $z \in \mathbf{C}P^1$ [2]. Thus each w_j is holomorphic in $z \in \mathbf{C}P^1$, i.e. meromorphic with respect to a local coordinate $z \mapsto [1, z]$ on CP^1. Then (6) takes the form

$$P(z,x) \equiv (w_4(z) + w_0(z))|x|^2 + 2w_1(z)x_1 + 2w_2(z)x_2 + 2w_3(z)x_3 + w_4(z) - w_0(z) = 0. \quad (12)$$

If (z_0, x_0) satisfies $P(z_0, x_0) = 0$ and $\frac{\partial P}{\partial z}(z_0, x_0) \neq 0$, then by the Implicit Function Theorem, there is a local smooth solution $z = z(x)$ satisfying $z(x_0) = z_0$. To see that z is semi-conformal, it suffices to note that

$$\nabla_x P = 2\left\{(w_4 + w_0)x + (w_1, w_2, w_3)\right\} ,$$

so that

$$\begin{aligned} \nabla_x P \cdot \nabla_x P &= 4\left\{(w_4 + w_0)^2|x|^2 + 2(w_4 + w_0)x \cdot (w_1, w_2, w_3) + w_1^2 + w_2^2 + w_3^2\right\} \\ &= 4(w_4 + w_0)P(z,x)\,, \end{aligned}$$

which vanishes along $P = 0$, so that z is semi-conformal by (4).

We will henceforth suppose that the $w_j = w_j(z)$ are given as rational functions of $z \in \mathbf{C}$, equivalently, as holomorphic functions of $[z_1, z_2] \in \mathbf{C}P^1$. We give two approaches for determining the discriminant variety in S^3, both useful in studying examples.

Consider first equation (5):

$$Q(z,y) \equiv w_0(z)y_0 + w_1(z)y_1 + w_2(z)y_2 + w_3(z)y_3 + w_4 = 0\,. \quad (13)$$

We clearly have

Lemma 1. *The discriminant variety K is given by those points $y \in \mathbf{R}^4$, $|y| = 1$ such that there exists a point $z \in \mathbf{C}P^1$ with*

$$\left\{ \begin{array}{rcl} Q(z,y) &=& 0 \\ \frac{\partial Q}{\partial z}(z,y) &=& 0 \end{array} \right. \quad (14)$$

In general, (14) can be solved explicitly, indeed, taking the two equations of (14) together with their complex conjugates yields the linear system

$$\left\{ \begin{array}{rcl} w_0y_0 + w_1y_1 + w_2y_2 + w_3y_3 &=& -w_4 \\ \overline{w}_0y_0 + \overline{w}_1y_1 + \overline{w}_2y_2 + \overline{w}_3y_3 &=& -\overline{w}_4 \\ w_0'y_0 + w_1'y_1 + w_2'y_2 + w_3'y_3 &=& -w_4' \\ \overline{w}_0'y_0 + \overline{w}_1'y_1 + \overline{w}_2'y_2 + \overline{w}_3'y_3 &=& -\overline{w}_4' \end{array} \right. \quad (15)$$

Set

$$W = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ \overline{w}_0 & \overline{w}_1 & \overline{w}_2 & \overline{w}_3 \\ w_0' & w_1' & w_2' & w_3' \\ \overline{w}_0' & \overline{w}_1' & \overline{w}_2' & \overline{w}_3' \end{pmatrix}$$

to be the matrix of coefficients and let W_j be the matrix obtained from W by replacing column j with the vector $\begin{pmatrix} -w_4 \\ -\overline{w}_4 \\ -w_4' \\ -\overline{w}_4' \end{pmatrix}$ (indexing the first column as 0). Then, provided $|W| \neq 0$, by Cramer's rule, the solution to (15) is given by

$$y = \frac{1}{|W|}\left(|W_0|, |W_1|, |W_2|, |W_3|\right) . \tag{16}$$

Proposition 2. *Let $P(z,x) = 0$ be the polynomial equation* (12)*, then provided $|W(z)|$ is non-zero, z belongs to Σ, the set of singular values, if and only if*

$$|W_0(z)|^2 + |W_1(z)|^2 + |W_2(z)|^2 + |W_3(z)|^2 = |W(z)|^2 , \tag{17}$$

i.e. $[|W_0(z)|, |W_1(z)|, |W_2(z)|, |W_3(z)|, |W(z)|]$ *is a real point of* $\mathcal{H}$ *.*

Proof. Equation (17) is simply the condition that the solution y given by (16) lies on the sphere S^3. □

In general the set of singular values is a real-algebraic subset of $\mathbf{C}P^1$ of dimension 1. Having found Σ, the discriminant variety is then given by (16). If, on the other hand, $|W(z)|$ is identically zero, other methods must be used, one of which we give below. First we give an example.

Example 3. Let $\psi : \mathbf{C}P^1 \to \mathcal{H}$ be defined by

$$\psi([z_1, z_2]) = \left[z_1^2, z_1^2 + z_2^2, i(z_1^2 - z_2^2), iz_1^2, -2z_1z_2\right] .$$

In terms of a local chart $z \mapsto [1, z]$ for $\mathbf{C}P^1$, (14) has the form

$$\begin{cases} y_0 + (1+z^2)y_1 + i(1-z^2)y_2 + iy_3 + 2z &= 0 \\ zy_1 - izy_2 + 1 &= 0 \end{cases} \tag{18}$$

It is easily checked that (17) has the form

$$(1 - |z|^2)(2 - |z|^2) = 0 ,$$

i.e. a pair of concentric circles in the complex plane. Setting $z = e^{i\theta}$, (18) gives

$$\begin{cases} y_1 + iy_2 &= -e^{i\theta} \\ y_0 + iy_3 &= 0 \end{cases}$$

i.e. an equatorial circle which we denote by $\mathcal{C}_1$.

If now $z = \sqrt{2}e^{i\theta}$, then

$$\begin{cases} y_1 + iy_2 &= -\frac{1}{\sqrt{2}}e^{i\theta} \\ y_0 + iy_3 &= -\frac{1}{\sqrt{2}}e^{i\theta} \end{cases}$$

i.e. a circle $\mathcal{C}_2$ on a torus. In fact $\mathcal{C}_1$ and $\mathcal{C}_2$ represent two leaves of the Hopf fibration of S^3 and are linked with linking number 1. The discriminant variety can also be calculated directly in this case from the discriminant of the quadratic equation given by the first equation in (18). It is easily checked, in respect of the covering construction discussed in the introduction, that the covering manifold $\widetilde{M}$ is smooth.

Let us now project $S^3 \setminus \{\infty\}$ to $\mathbf{R}^3$ via stereographic projection and take equation (10) as our starting point, with ξ given by (9) meromorphic in z. Then the discriminant variety K is given by those $x \in \mathbf{R}^3$ satisfying

$$\left\{ \begin{array}{rcl} |x|^2 + 2\xi \cdot x + \xi^2 & = & 0 \\ \xi' \cdot (x + \xi) & = & 0 \end{array} \right. \tag{19}$$

for some $z \in \mathbf{C}P^1$. Whether or not $\infty \in S^3$ belongs to K can be investigated after inversion $\hat{x} = x/|x|^2$. Equations (19) imply the following system

$$\left\{ \begin{array}{lcl} (\xi - \overline{\xi}) \cdot x & = & \frac{-\xi^2 + \overline{\xi}^2}{2} \\ \xi' \cdot x & = & -\xi \cdot \xi' \\ \overline{\xi}' \cdot x & = & -\overline{\xi} \cdot \overline{\xi}' \end{array} \right. \tag{20}$$

Set A to be the matrix $A = \begin{pmatrix} \xi - \overline{\xi} \\ \xi' \\ \overline{\xi}' \end{pmatrix}$ (i.e. the row vectors are given by $\xi - \overline{\xi}, \xi', \overline{\xi}'$) and B to be the column vector $B = \begin{pmatrix} \frac{-\xi^2 + \overline{\xi}^2}{2} \\ -\xi \cdot \xi' \\ -\overline{\xi} \cdot \overline{\xi}' \end{pmatrix}$. Then we have

Proposition 4. *Provided $A(z) \neq 0$, then z is a singular value if and only if*

$$|A(z)^{-1}B(z)|^2 + 2\xi(z) \cdot \left(A(z)^{-1}B(z)\right) + \xi(z)^2 = 0\,. \tag{21}$$

If on the other hand $|A(z)| = 0$, then there is a non-trivial relation between the coefficients of the form

$$a(z)\left(\xi(z) - \overline{\xi}(z)\right) + b(z)\xi'(z) + c(z)\overline{\xi}'(z) = 0\,, \tag{22}$$

for some $a(z)$, $b(z)$, $c(z) \in \mathbf{C}$. In this case z is a singular value if and only if

$$a(z)\frac{\left(-\xi^2(z) + \overline{\xi}^2(z)\right)}{2} - b(z)\,\xi(z) \cdot \xi'(z) - c(z)\,\overline{\xi}(z) \cdot \overline{\xi}'(z) = 0\,. \tag{23}$$

Proof. The first part of the proposition is obtained by solving (20) and substituting into (10). If on the other hand $|A(z)| = 0$, taking the inner product with x on the left-hand-side of (22) and applying (20) yields (23). □

Example 5. Set $\xi = (iz, 0, 0))$, then, in the notation introduced above $|A(z)| = 0$ for all z. The rows of A are related by the (non-unique) relation

$$\xi'(z) - \overline{\xi}'(z) = 0\,.$$

Then (23) is equivalent to $z - \overline{z} = 0$, giving the singular values as the imaginary axis $z = iu$, $u \in \mathbf{R}$. The discriminant variety can be obtained by substituting into (10), which yields

$$x_1^2 + x_2^2 + x_3^2 - 2ux_1 + u^2 = 0$$

having as solution $x_2 = x_3 = 0$, $x_1 = u$, i.e. the x_1-axis in $\mathbf{R}^3$. In respect of the covering construction, $dP(z,x) = 0$ along K, where

$$P(z,x) = |x|^2 + 2\xi \cdot x + \xi^2 .$$

The set $\tilde{M}$ is not a smooth manifold, but consists of two 3-spheres glued along a great circle.

Although equations (17) and (21) both give means to determine the set of singular values implicitly, unless we can find an explicit parametrization, it may be difficult to describe the discriminant variety. However, there is one case where this is always possible.

Suppose that all the regular circles defined by (10) pass through a single point, for instance this is the case if all the circles pass through ∞ and so correspond to lines in $\mathbf{R}^3$. As described at the end of Section 2, after inversion we may identify this with the space of circles passing through $0 \in \mathbf{R}^3$. From (10), this is the case if and only if the meromorphic function ξ satisfies

$$\xi(z)^2 \equiv 0 \qquad \text{(which implies } \xi(z) \cdot \xi'(z) \equiv 0). \tag{24}$$

In this case the system of equations (20) becomes

$$\begin{cases} \left(\xi - \overline{\xi}\right) \cdot x & = & 0 \\ \xi' \cdot x & = & 0 \\ \overline{\xi}' \cdot x & = & 0 \end{cases}$$

which has a non-trivial solution z if and only if

$$|A(z)| = 0 . \tag{25}$$

As is well-known from the Weierstrass representation for minimal surfaces, we may parametrize meromorphic ξ satisfying $\xi(z)^2 \equiv 0$ in the form

$$\xi = \frac{1}{2h} \left(1 - g^2, i(1+g^2), -2g\right) , \tag{26}$$

where g and h are meromorphic in z (see [10], also [4]).

Proposition 6. *A point $z \in \mathbf{C}P^1$ is a singular value if and only if*

$$2|g'(z)|^2 \mathrm{Im}\,(\overline{g(z)}\, h(z)) = (1+|g|^2)\mathrm{Im}\,(\overline{g'(z)}\, h'(z)) . \tag{27}$$

If z satisfies (27), then, setting $q = x_2 + ix_3$ and $\zeta(z) = h'(z)/g'(z)$, the corresponding point in the discriminant variety is given by

$$\begin{cases} q & = & \frac{2}{(1-|g(z)|^4)} \left\{ h(z) - g(z)\zeta(z) - g^2 \left(\overline{h(z)} - \overline{g(z)\zeta(z)} \right) \right\} \\ x_3 & = & -g(z)\overline{q} + \zeta(z) \end{cases} \tag{28}$$

Proof. Equation (27) is obtained by substituting (26) into (25), after a straightforward calculation. Then (28) is obtained by solving (8) for (q, x_3). In fact condition (27) is precisely the condition that x_3 be real. $\square$

Remark. The discriminant variety depends only on the holomorphic curve in $\mathcal{L}$, rather than on its parametrization, as is easily seen from the form of (28) above. At a singular point of such a curve $\zeta(z) = h'(z)/g'(z)$ is well-defined (see [8]), so that the corresponding point of the discriminant variety is also well-defined by (28).

4. Links on tori

For integers k, l, not both zero, let $[w] \in \mathcal{H}$ be defined as a holomorphic function of $z \in \mathbf{C} \cup \{\infty\}$, by

$$[w(z)] = \left[1 + z^{2l}, i(1 - z^{2l}), z^k, iz^k, 2z^l\right] .$$

If we set $p = y_0 + iy_1,\ q = y_2 + iy_3$, then (14) has the form

$$\left\{ \begin{array}{rcl} p + z^{2l}\overline{p} + z^k q + 2z^l & = & 0 \\ 2lz^{2l-1}\overline{p} + kz^{k-1}q + 2lz^{l-1} & = & 0 \end{array} \right. \tag{29}$$

This implies the following equation together with its complex conjugate:

$$kp + (k - 2l)z^{2l}\overline{p} + 2(k - l)z^l = 0 . \tag{30}$$

There are the special cases $k = 0, l, 2l$.

Case 1. $k = 0$: After a reparametrization of the z-variable (replacing z^l by z), we may suppose $l = 1$ and the example is equivalent after isometry to Example 3, having two circles with linking number 1 as discriminant variety.

Case 2. $k = l$: Now we have the simultaneous equations

$$\left\{ \begin{array}{rcl} p - z^{2l}\overline{p} & = & 0 \\ \overline{p} - \overline{z}^{2l}p & = & 0 \end{array} \right.$$

This has a non-trivial solution if and only if $|z| = 1$. Setting $z = e^{i\theta}$ yields $p = \lambda e^{il\theta}$ for an arbitrary real constant λ. Substituting back into (29) gives $q = -2(1 + \lambda)$. Then $|p|^2 + |q|^2 = 1$ if and only if $(5\lambda + 3)(\lambda + 1) = 0$, giving two unlinked circles in S^3 as the discriminant variety.

Case 3. $k = 2l$: Then $p + z^l = 0$. Substituting back into (29) gives $q = (|z|^{2l} - 1)/z^l$ (the solution $z = 0$ is excluded. Then $|p|^2 + |q|^2 = 1$ if and only if $\left(2|z|^{2l} - 1\right)\left(|z|^{2l} - 1\right) = 0$. We once more obtain two linked circles, isometric to Case 1.

Other cases. From (30) and its conjugate, we obtain

$$p\left\{k^2 - (k - 2l)^2|z|^{4l}\right\} + 2(k - l)z^l\left\{k - (k - 2l)|z|^{2l}\right\} = 0 .$$

If $k-(k-2l)|z|^{2l} = 0$, then it is easily checked that there is no solution satisfying $|p|^2+|q|^2 = 1$. Suppose then that $k - (k - 2l)|z|^{2l} \neq 0$, then

$$\begin{array}{rcl} p & = & \frac{-2(k-l)z^l}{k+(k-2l)|z|^{2l}} \\ q & = & \frac{-2lz^{l-k}(1-|z|^2)}{k+(k-2l)|z|^{2l}} . \end{array}$$

Then $|p|^2 + |q|^2 = 1$ if and only if

$$|z|^{2k}\left\{|z|^{2l} - 1\right\}\left\{(k - 2l)^2|z|^{2l} - k^2\right\} - 4l^2|z|^{2l}\left(1 - |z|^2\right)^2 = 0 \quad (z \neq 0).$$

If $|z| = 1$, setting $z = e^{i\theta}$ gives $(p, q) = (-e^{i\theta}, 0)$. Otherwise, if we let $z = re^{i\theta}$, we obtain the following polynomial in r:

$$r^{2k}\left\{r^{2(l-1)} + r^{2(l-2)} + \cdots + r^2 + 1\right\}\left\{(k-2l)^2 r^{2l} - k^2\right\} + 4l^2 r^{2l}(1 - r^2) = 0\,. \qquad (31)$$

Then for each real positive root $r = a$, we have a link on a torus of the form:

$$(p, q) = \left(Ae^{il\theta}, Be^{i(l-k)\theta}\right)\,.$$

These curves are all distinct and any two have linking number $l(l-k)$. For example, if $l = 1$ and $k = 3$, (31) has two positive roots and one negative. Note also that the polynomial on the left-hand-side of (31) always has a real root lying strictly between 0 and 1. To summarize:

Theorem 7. *Let k, l be integers which are not both zero. The holomorphic curve $\mathbf{C}P^1 \to \mathcal{H}$ given by*

$$[z_1, z_2] \mapsto \left[z_1^{2l} + z_2^{2l}, i\left(z_1^{2l} - z_2^{2l}\right), z_1^{2l-k}z_2^k, iz_1^{2l-k}z_2^k, 2z_1^l z_2^l\right]$$

has associated discriminant variety K given by

(i) *in the case $k = 0$: two linked circles with linking number 1;*

(ii) *in the case $k = l$: two unlinked circles;*

(iii) *in the case $k = 2l$: two linked circles with linking number 1;*

(iv) *in all other cases: one great circle and circles $C_1, C_2, \ldots, C_k$ ($k \geq 1$) determined by the distinct positive roots of (31). These are all disjoint and any two C_i, C_j ($i \neq j$) have linking number $l(l-k)$.*

5. (p, q)-examples and their deformation

Let (p, q) be integers with $0 < p < q < 2p$. Consider the polynomial equation (8), where $g = z^p$, $h = z^q$. The discriminant variety can be calculated from (27) and (28). The singular values are given by

$$|z|^{2p-2}\left\{q - (2p-q)|z|^{2p}\right\}\operatorname{Im} z^{q-p} = 0\,,$$

with solution set:

(i) $z = 0$;

(ii) $\operatorname{Im} z^{q-p} = 0$;

(iii) $|z|^{2p} = q/(2p-q)$.

This consists of the circle (iii) together with $(q-p)$ lines passing through the origin. An example with $q - p = 3$ is sketched in Figure 1. To investigate the component of the discriminant variety corresponding to the singular values given by (iii), we set $z = a\,e^{i\theta}$, $(0 \leq \theta \leq 2\pi)$, where $a = \left(\frac{q}{2p-q}\right)^{1/2p}$. Substituting into (28) yields

$$\begin{aligned} q &= \frac{a^q(2p-q)^2}{2p^2}\left(e^{qi\theta} - \frac{q}{(2p-q)}e^{(2p-q)i\theta}\right) \\ x_3 &= -\frac{q(2p-q)}{p^2}a^{q-p}\cos(q-p)\theta\,. \end{aligned}$$

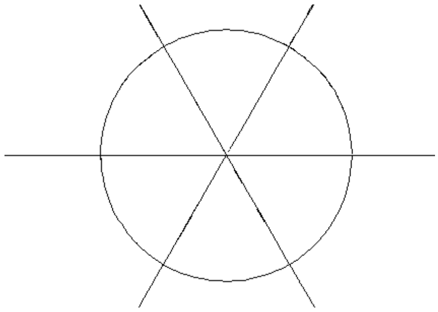


Figure 1.

It is a remarkable fact that this curve, which we denote by $\mathcal{C}_{p,q}$, lies on a 2-sphere in $\mathbf{R}^3$. Its radius can be calculated and is given by $R_{p,q}$, where

$$R^2_{p,q} = \frac{(2p-q)^2(2p^2-2pq+q^2)a^{2q}}{2p^4} \tag{32}$$

The curve is continuous with $2(p-q)$ cusps symmetrically placed at intervals $\theta = (2p-q)k\pi/(q-p)$, $k = 0, 1, 2, \ldots, 2(q-p)-1$, alternating from the top of the sphere ($x_3 > 0$) to the bottom ($x_3 < 0$). Each cusp is joined by a smooth arc which rotates through $(2p-q)\pi/(q-p)$ before joining the next cusp. A calculation (best done with a computer) shows that each of these arcs has constant curvature in the sphere (Figure 2 (a), (b), (c) illustrate the cases $(p, q) = (3, 5), (4, 7), (5, 6)$ respectively).

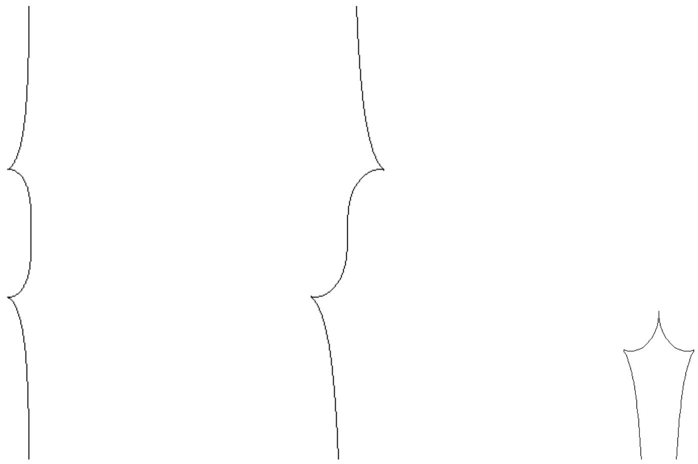
(a) (b) (c)

Figure 2.

On the other hand, the $(q-p)$ lines of (ii) determine $(q-p)$ planar curves in $\mathbf{R}^3$, each one obtained from the other by a rotation through $2\pi/(q-p)$, followed possibly by a reflection. It suffices to set $z = u$ real to determine the behaviour of each curve. Then

$$\begin{aligned} q &= -\frac{2(q-p)u^q}{p(1+u^{2p})} \\ x_3 &= -\frac{(2p-q)u^{p+q}+qu^{q-p}}{p(1+u^{2p})} \end{aligned}$$

Each planar curve has two cusps which touch the curve $\mathcal{C}_{p,q}$ at its cusps and is one of three types (or a reflection of these) depending on the positioning of the cusps of $\mathcal{C}_{p,q}$. These are sketched in Figure 3 (a), (b), (c).



(a) (b) (c)

Figure 3.

The cusps are the image points in $\mathbf{R}^3$ of intersection points of components (ii) and (iii) and correspond to points on the discriminant variety where the multiplicity of the roots of equation (8) increases.

The whole picture takes on a more symmetrical look if we compactify $\mathbf{R}^3$ to S^3 by adding in the point at infinity. Now the components of the discriminant variety all lie on standard 2-spheres in S^3. They consist of the central axis (component (i)), the $(q-p)$-curves with 2-cusps joined by smooth arcs and one curve $\mathcal{C}_{p,q}$ with $2(q-p)$ cusps joined by smooth arcs. The cusps of the set of $(q-p)$ curves touch those of $\mathcal{C}_{p,q}$.

One particularly interesting case is when $q = p+1$. Then $\mathcal{C}_{p,q}$ has 2 cusps. There are two arcs joining these which rotate through $(p-1)\pi$ as they pass from the top to the bottom of the sphere. This rotation becomes arbitrarily large as $p \to \infty$. The radius of the 2-sphere supporting the arcs is given by

$$\begin{aligned} R_{p,p+1} &= \left(\frac{p+1}{p-1}\right)^2 \frac{(p^2+1)(p^2-1)}{2p^4} \\ &\to \frac{1}{2} \quad \text{as} \quad p \to \infty\,, \end{aligned}$$

which in particular is bounded above and below. Thus as p becomes large, the component $\mathcal{C}_{p,p+1}$ becomes increasingly concentrated on the sphere (Figure 4). Small deformations of the (p,q)-examples yield interesting phenomena (best investigated by computer techniques). For example, if we set $g = z^p$, $h = z^{p+1} + i\varepsilon z^p$ for small $\varepsilon \in \mathbf{R}$, then the set of singular values becomes separated into two components as illustrated in Figure 5. In particular a compact closed loop breaks away. Its corresponding image in $\mathbf{R}^3$ is a continuously embedded curve, approximated by one arc of $\mathcal{C}_{p,p+1}$ together with the part of the planar curve joining the two cusps. Compactified to S^3 (with the complex plane in which the singular values lie also compactified to include ∞) the other 'dual' part of the curve is approximated by the other arc with ends joined by a curve passing through ∞. We do not know whether these closed curves are linked or not.

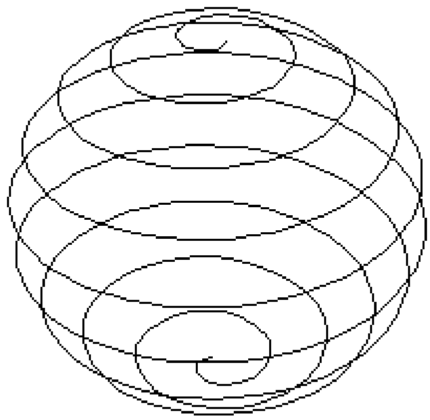


Figure 4.

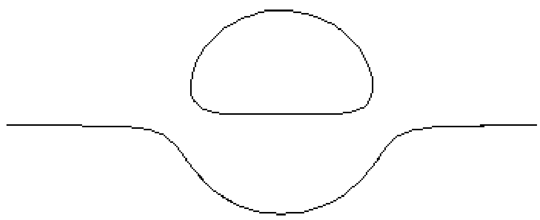


Figure 5

On the other hand, if we set $g = z^p$, $h = z^{p+2} + i\varepsilon z^p$, then the set of singular values breaks into three components as illustrated in Figure 6. It is shown in [3] by purely analytic methods that the compact component corresponds to a continuously embedded closed curve which is knotted.

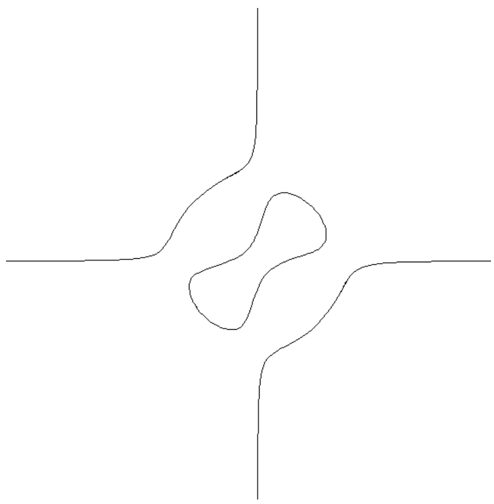


Figure 6.

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References

[1] Baird, P.: *Harmonic morphisms and circle actions on 3- and 4-manifolds*. Ann. Inst. Fourier, Grenoble, **40**(1) (1990), 177–212. Zbl 0676.58023

[2] Baird, P.: *Conformal foliations by circles and complex isoparametric functions on Euclidean 3-space*. Math. Proc. Camb. Phil. Soc., **123** (1998), 273–300. Zbl 0899.58012

[3] Baird, P.: *Knot singularities of harmonic morphisms*. To appear, Proc. Edinburgh Math. Soc.

[4] Baird, P.; Wood, J. C.: *Bernstein theorems for harmonic morphisms from* $\mathbf{R}^3$ *and* S^3. Math. Ann. **280** (1988), 579–603. Zbl 0621.58011

[5] Baird, P.; Wood, J. C.: *Harmonic Morphisms between Riemannian Manifolds*. To appear, London Math. Soc. Monograph, Oxford University Press.

[6] Coolidge, J. L.: *A Treatise on the Circle and the Sphere*. Oxford Univ. Press 1916, Reprint 1971. Zbl 0251.50002

[7] Eisenbud, D.; Neumann, W.: *Three-dimensional Link Theory and Invariants of Plane Curves*. Annals of Mathematics Studies **110**, Princeton Univ. Press 1985. Zbl 0628.57002

[8] Fulton, W.: *Algebraic Curves*. Mathematics Lecture Notes Series, W. A. Benjamin 1969. Zbl 0181.23901

[9] Gudmundsson, S.; Wood, J. C.: *Multivalued harmonic morphisms*. Math. Scand. **73** (1993), 127–155. Zbl 0790.58009

[10] Hoffman, D. A.; Osserman, R.: *The Geometry of the Generalized Gauss Map*. Memoirs of the Amer. Math. Soc. **28**, No. 236 (1980). Zbl 0469.53004

[11] Laguerre, E. N.: *Sur l'emploi des imaginaires dans la géométrie de l'espace*. Nouvelles Annales de Math., Series 2, vol. **xi**, 1872.

[12] Lickorish, W. B. R.: *An Introduction to Knot Theory*. Graduate Texts in Mathematics **175**, Springer 1997. Zbl 0886.57001

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