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More on Convolution of Riemannian Manifolds

Dedicated to Professor T. Otsuki on his eighty-fifth birthday

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Abstract. In an earlier paper [1], the author introduced the notion of convolution of Riemannian manifolds. In [1] he also provided some examples and applications of convolution manifolds. In this paper we use tensor product to construct more examples of convolution manifolds and investigate fundamental properties of convolution manifolds. In particular, we study the relationship between convolution manifolds and the gradient of their scale functions. Moreover, we obtain a necessary and sufficient condition for a factor of a convolution Riemannian manifold to be totally geodesic. We also completely classify flat convolution Riemannian surfaces.

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1. Convolution of Riemannian manifolds

Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let f be a positive differentiable function on N_1. The well-known notion of *warped product manifold* $N_1 \times_f N_2$ is defined as the product manifold $N_1 \times N_2$ equipped with the Riemannian metric given by $g_1 + f^2 g_2$. It is well-known that the notion of warped product plays some important roles in differential geometry as well as in physics (cf. [4]). The new notion of convolution of Riemannian manifolds introduced in [1] can be regarded as a natural extension of warped products.

The notion of convolution products is defined as follows: Let N_1 and N_2 be two Riemannian manifolds equipped with metrics g_1 and g_2, respectively. Consider the symmetric tensor field $g_{f,h}$ of type (0,2) on the product manifold $N_1 \times N_2$ defined by

$$g_{f,h} = h^2 g_1 + f^2 g_2 + 2fh df \otimes dh \tag{1.1}$$

for some positive differentiable functions f and h on N_1 and N_2, respectively. We denote the symmetric tensor $g_{f,h}$ by ${}_h g_1 *_f g_2$, which is called the *convolution of* g_1 *and* g_2 (via h and f). The product manifold $N_1 \times N_2$ equipped with ${}_h g_1 *_f g_2$ is called a *convolution manifold*, which is denoted by ${}_h N_1 \bigstar_f N_2$. When the scale functions f, h are irrelevant, we simply denote ${}_h N_1 \bigstar_f N_2$ and ${}_h g_1 *_f g_2$ by $N_1 \bigstar N_2$ and $g_1 * g_2$, respectively.

When ${}_h g_1 *_f g_2$ is a nondegenerate symmetric tensor, it defines a pseudo-Riemannian metric on $N_1 \times N_2$ with index ≤ 1. In this case, ${}_h g_1 *_f g_2$ is called a *convolution metric* and the convolution manifold ${}_h N_1 \bigstar_f N_2$ is called a *convolution pseudo-Riemannian manifold.* If the index of the pseudo-Riemannian metric is zero, ${}_h N_1 \bigstar_f N_2$ is called a *convolution Riemannian manifold.* The author provides in [1] examples and applications of convolution manifolds.

In Section 2 of this paper we provide basic formulas and definitions. In Section 3 we apply tensor product of Euclidean submanifolds to construct more examples of convolution manifolds. In this section we also obtain a necessary and sufficient condition for a convolution of two Riemannian metrics to be a Riemannian metric. Our condition is expressed in terms of the length of gradient of the scale functions of the convolution manifolds. In Sections 4 and 5 we construct examples of submanifolds in Euclidean and in pseudo-Euclidean spaces whose distance function ρ satisfies $|\mathrm{grad}\, \rho| = c \in [0, \infty)$. We also investigate general properties of such submanifolds. In Section 6, we obtain a necessary and sufficient condition for one of the factors of a convolution Riemannian manifold to be totally geodesic. In the last section, we completely classify flat convolution Riemannian surfaces.

2. Preliminaries

Let N be a Riemannian manifold equipped with a Riemannian metric g. The gradient $\mathrm{grad}\, \varphi$ of a function φ on N is defined by $\langle \mathrm{grad}\, \varphi, X \rangle = X\varphi$ for vector fields X tangent to N.

If N is a submanifold of a Riemannian manifold $\tilde{M}$, the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to N and ξ normal to N, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$, σ the second fundamental form, D the normal connection, and A the shape operator of N in $\tilde{M}$. The second fundamental form and the shape operator are related by $\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on M as well as on $\tilde{M}$. A submanifold in a Riemannian manifold is called *totally geodesic* if its second fundamental form vanishes identically, or equivalently, its shape operator vanishes identically.

The *equation of Gauss* of N in $\tilde{M}$ is given by

$$\begin{aligned}\tilde{R}(X,Y;Z,W) =& R(X,Y;Z,W) + \langle \sigma(X,Z), \sigma(Y,W)\rangle \\ &- \langle \sigma(X,W), \sigma(Y,Z)\rangle,\end{aligned} \tag{2.3}$$

for X, Y, Z, W tangent to M, where R and $\tilde{R}$ denote the curvature tensors of N and $\tilde{M}$, respectively.

The covariant derivative $\bar{\nabla}\sigma$ of σ with respect to the connection on $TM \oplus T^{\perp}M$ is defined by

$$(\bar{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.4}$$

The *equation of Codazzi* is

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y,Z) - (\bar{\nabla}_Y \sigma)(X,Z), \tag{2.5}$$

where $(\tilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X,Y)Z$.

Let $\mathbf{E}^m \otimes \mathbf{E}^n$ denote the tensor product of two Euclidean spaces $\mathbf{E}^m$ and $\mathbf{E}^n$. Then $\mathbf{E}^m \otimes \mathbf{E}^n$ is isometric to $\mathbf{E}^{mn}$. The Euclidean inner product $\langle\ ,\ \rangle$ on $\mathbf{E}^m \otimes \mathbf{E}^n$ is given by

$$\langle \alpha \otimes \beta, \gamma \otimes \delta \rangle = \langle \alpha, \gamma \rangle \langle \beta, \delta \rangle, \tag{2.6}$$

where $\langle \alpha, \gamma \rangle$ denotes the Euclidean inner product of $\alpha, \gamma \in \mathbf{E}^m$ and $\langle \beta, \delta \rangle$ the Euclidean inner product of $\beta, \delta \in \mathbf{E}^n$.

We denote $\mathbf{E}^n - \{0\}$ by $\mathbf{E}^n_*$. Let $\mathbf{E}^n_t$ denote the pseudo-Euclidean n-space equipped with a pseudo-Euclidean metric with index t. A pseudo-Euclidean space with index one is known as a Minkowski space-time.

3. Convolution manifolds

The tensor product of two Euclidean submanifolds have been investigated by F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken in [3]. The following result shows that the notion of convolution manifolds arises very naturally. It also provides us ample examples of convolution manifolds.

Proposition 3.1. *Let $x : (N_1, g_1) \to \mathbf{E}^n_* \subset \mathbf{E}^n$ and $y : (N_2, g_2) \to \mathbf{E}^m_* \subset \mathbf{E}^m$ be isometric immersions of Riemannian manifolds (N_1, g_1) and (N_2, g_2) into $\mathbf{E}^n_*$ and $\mathbf{E}^m_*$, respectively. Then the map*

$$\psi : N_1 \times N_2 \to \mathbf{E}^n \otimes \mathbf{E}^m = \mathbf{E}^{nm};\ (u,v) \mapsto x(u) \otimes y(v),\ \ u \in N_1,\ v \in N_2, \tag{3.1}$$

gives rise to a convolution manifold $N_1 \bigstar N_2$ equipped with

$$_{\rho_2}g_1 *_{\rho_1} g_2 = \rho_2^2 g_1 + \rho_1^2 g_2 + 2\rho_1\rho_2 d\rho_1 \otimes d\rho_2, \tag{3.2}$$

where $\rho_1 = \sqrt{\sum_{j=1}^n x_j^2}$ *and* $\rho_2 = \sqrt{\sum_{\alpha=1}^m y_\alpha^2}$ *denote the distance functions of* x *and* y*, and* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_m)$ *are Euclidean coordinate systems of* $\mathbf{E}^n$ *and* $\mathbf{E}^m$*, respectively.*

Proof. For vector fields X, Y tangent to N_1 and Z, W tangent to N_2, we have

$$d\psi(X) = X\psi = X \otimes x, \quad d\psi(Z) = Z\psi = z \otimes Z. \tag{3.3}$$

Also, it follows from the definitions of gradient of $\rho_1 = |x|$ that

$$\langle X, x \rangle = \frac{1}{2} X \langle x, x \rangle = \rho_1(X\rho_1) = \rho_1 d\rho_1(X). \tag{3.4}$$

Similarly, we have

$$\rho_2 d\rho_2(Z) = \langle Z, y \rangle . \tag{3.5}$$

From (2.6), (3.3), (3.4) and (3.5), we obtain Proposition 3.1. □

Example 3.1. If $y : (N_2, g_2) \to \mathbf{E}^m_* \subset \mathbf{E}^m$ is an isometric immersion such that $y(N_2)$ is contained in the unit hypersphere S^{m-1} of $\mathbf{E}^m$ centered at the origin. Then the convolution $g_1 * g_2$ of g_1 and g_2 on the convolution manifold $N_1 \bigstar N_2$ defined by (3.2) is nothing but the warped product metric: $g_1 + |x|^2 g_2$.

Definition 3.1. *A convolution* ${}_h g_1 * {}_f g_2$ *of two Riemannian metrics* g_1 *and* g_2 *is called degenerate if* $\det({}_h g_1 *_f g_2) = 0$ *holds identically.*

For $X \in T(N_1)$ we denote by $|X|_1$ the length of X with respect to metric g_1 on N_1. Similarly, we denote by $|Z|_2$ for $Z \in T(N_2)$ with respect to metric g_2 on N_2.

Proposition 3.2. *Let* ${}_h N_1 \bigstar {}_f N_2$ *be the convolution of Riemannian manifolds* (N_1, g_1) *and* (N_2, g_2) *via* h *and* f*. Then* ${}_h g_1 *_f g_2$ *is degenerate if and only if we have*
(1) *the length* $|\mathrm{grad}\, f|_1$ *of the gradient of* f *on* (N_1, g_1) *is a nonzero constant, say* c*, and*
(2) *the length* $|\mathrm{grad}\, h|_2$ *of the gradient of* h *on* (N_2, g_2) *is the constant given by* $1/c$*, i.e., the reciprocal of* c*.*

Proof. By a direct computation we have

$$\det({}_h g_1 *_f g_2) = f^{2n_1} h^{2n_2} \big(1 - |\mathrm{grad}\, f|_1^2\, |\mathrm{grad}\, h|_2^2\big), \tag{3.6}$$

where n_1 and n_2 are the dimensions of N_1 and N_2, respectively. Thus the convolution ${}_h g_1 * {}_f g_2$ of g_1 and g_2 is degenerate if and only if $|\mathrm{grad}\, f|_1^2\, |\mathrm{grad}\, h|_2^2 = 1$. Since $|\mathrm{grad}\, f|_1$ and $|\mathrm{grad}\, h|_2$ depend only on N_1 and N_2, respectively, we conclude that $|\mathrm{grad}\, f|_1^2\, |\mathrm{grad}\, h|_2^2 = 1$ holds identically if and only if both statements (1) and (2) of Proposition 3.2 hold. □

The following proposition provides a necessary and sufficient condition for a convolution ${}_h g_1 *_f g_2$ of two Riemannian metrics to be a Riemannian metric.

Proposition 3.3. *Let ${}_hN_1\bigstar{}_fN_2$ be the convolution of Riemannian manifolds (N_1, g_1) and (N_2, g_2) via h and f. Then ${}_hg_1 * {}_fg_2$ is a Riemannian metric on ${}_hN_1\bigstar{}_fN_2$ if and only if we have $|\operatorname{grad} f|_1\,|\operatorname{grad} h|_2 < 1$.*

Proof. Follows from equation (3.6) and the fact that the index of ${}_hg_1 * {}_fg_2$ is at most one. □

4. Examples of submanifolds satisfying $|\operatorname{grad}\boldsymbol{\rho}| = \boldsymbol{c}$

In view of Propositions 3.2 and 3.3, we provide some examples of Riemannian manifolds equipped with a positive function f satisfying $|\operatorname{grad} f| = c$ for some real number $c \geq 0$.

Example 4.1. Let $x : M \to \mathbf{E}^n$ be an isometric immersion such that $x(M)$ is contained in a hypersphere of $\mathbf{E}^n$ centered at the origin. Then the distance function $\rho = |x|$ on M satisfying $|\operatorname{grad}\rho| = 0$. In fact, spherical submanifolds are the only submanifolds in Euclidean space whose distance function has zero gradient.

There exist many submanifolds in Euclidean space whose distance function ρ satisfies $|\operatorname{grad}\rho| = c$ for some real number $c \in (0, 1)$. Here we provide some such examples.

Example 4.2. For any real numbers a, c with $0 \leq a < c < 1$, the curve

$$\gamma(s) = \left(\sqrt{c^2 - a^2}\, s \sin\Big(\frac{\sqrt{1-c^2}}{\sqrt{c^2-a^2}} \ln s\Big), \sqrt{c^2 - a^2}\, s \cos\Big(\frac{\sqrt{1-c^2}}{\sqrt{c^2-a^2}} \ln s\Big), as\right) \tag{4.1}$$

in $\mathbf{E}^3$ is a unit speed curve satisfying $|\operatorname{grad}\rho| = c$. A direct computation shows that the curvature function κ of the space curve γ is given by

$$\kappa(s) = \frac{\sqrt{(1-a^2)(1-c^2)}}{\sqrt{(c^2-a^2)}\, s}. \tag{4.2}$$

When $a = 0$, (4.1) defines a planar curve which satisfies the condition $|\operatorname{grad}\rho| = c$ and whose curvature κ equals to $\sqrt{(1-c^2)}/cs$.

Example 4.3. Let $\gamma(s_j) : I \to E^{n_i}$, $j = 1, \ldots, k$, be k unit speed curves in Euclidean spaces which satisfy the condition: $|\operatorname{grad}\rho_j| = c$, $\rho_j = |\gamma_j|$, for some constant c. Then the product immersion

$$x : I^k \to \mathbf{E}^{n_1+\cdots+n_k}; (s_1, \ldots, s_k) \mapsto (\gamma_1(s_1), \ldots, \gamma_k(s_k)) \tag{4.3}$$

is an isometric immersion satisfying the condition $|\operatorname{grad}\rho| = c$, too.

There exist many space-like submanifolds in pseudo-Euclidean spaces whose distance function ρ satisfies the condition: $|\operatorname{grad}\rho| = c$ for some real number $c > 1$ or $c < 1$. Here we provide some such examples.

Example 4.4. Let a, c be two real numbers satisfying $c > 1$ and $c > a \geq 0$. We put $b = \sqrt{c^2 - a^2}$. Then the curve

$$\gamma(s) = \left(\frac{1}{2}s^{1-\sqrt{c^2-1}/b}\left(s^{2\sqrt{c^2-1}/b} - b^2\right), \frac{1}{2}s^{1-\sqrt{c^2-1}/b}\left(s^{2\sqrt{c^2-1}/b} + b^2\right), as\right) \tag{4.4}$$

in $\mathbf{E}^3_1$ is a unit speed space-like curve which satisfies the condition: $|\text{grad}\,\rho| = c > 1$, $\rho = |\gamma|$. Here, the Minkowski metric on $\mathbf{E}^3_1$ is given by $g = -dx_1^2 + dx_2^2 + dx_3^2$.

Example 4.5. For any real numbers a and c with $0 < c < 1$, the curve

$$\gamma(s) = \left(as, \sqrt{a^2+c^2}\, s \sin\left(\frac{\sqrt{1-c^2}}{\sqrt{a^2+c^2}} \ln s\right), \sqrt{a^2+c^2}\, s \cos\left(\frac{\sqrt{1-c^2}}{\sqrt{a^2+c^2}} \ln s\right)\right)$$

in $\mathbf{E}^3_1$ is a unit speed space-like curve satisfying the condition: $|\text{grad}\,\rho| = c < 1$.

Example 4.6. Let $\gamma(s_j) : I \to \mathbf{E}^{n_i}_{t_i}$, $j = 1, \ldots, k$ be k unit speed space-like curves in pseudo-Euclidean spaces which satisfy the condition: $|\text{grad}\,\rho_j| = c > 1$, $\rho_j = |\gamma_j|$. Then the product immersion

$$x : I^k \to \mathbf{E}^{n_1+\cdots+n_k}_{t_1+\cdots+t_k}; (s_1, \ldots, s_k) \mapsto (\gamma_1(s_1), \ldots, \gamma_k(s_k)) \tag{4.5}$$

is a space-like submanifold satisfying the condition $|\text{grad}\,\rho| = c > 1$, where the pseudo-Euclidean space $\mathbf{E}^{n_1+\cdots+n_k}_{t_1+\cdots+t_k}$ is given by $\mathbf{E}^{n_1}_{t_1} \oplus \cdots \oplus \mathbf{E}^{n_k}_{t_k}$.

5. Convolution of Euclidean submanifolds

Let $x : M \to \mathbf{E}^n_*$ be an isometric immersion. We denote the position vector function of M in $\mathbf{E}^n$ by also x. At each point on M, we decompose the position vector x into $x = x^T + x^\perp$, where x^T and $x^\perp$ are the tangential and normal components of x at the point, respectively. Hence we have $|x|^2 = |x^T|^2 + |x^\perp|^2$.

In views of Propositions 3.2 and 3.3, we give the following.

Lemma 5.1. *Let $x : M \to \mathbf{E}^n$ be an isometric immersion. Then the distance function $\rho = |x|$ satisfies $|\text{grad}\,\rho| = c$ for some constant c if and only if we have $|x^T| = c|x|$. In particular, if $|\text{grad}\,\rho| = c$ holds, then $c \in [0, 1]$.*

Proof. Let $e_1, \ldots, e_{n_1}$ be a local orthonormal frame field on M. Then the gradient of ρ is given by $\text{grad}\,\rho = \sum_{j=1}^{n-1}(e_j\rho)e_j$. Since $e_j\rho = \langle e_j, x\rangle / |x|$, we find

$$|\text{grad}\,\rho|^2 = \sum_{j=1}^{n_1} \frac{\langle e_j, x\rangle^2}{|x|^2}. \tag{5.1}$$

Therefore, the condition $|\text{grad}\,\rho| = c$ holds for some constant c if and only if we have $|x^T| = c|x|$.

In particular, since $|x^T| \leq |x|$, the condition $|\text{grad}\,\rho| = c$ implies $c \leq 1$. □

Definition 5.1. *By a cone in $\mathbf{E}^n$ with vertex at the origin we mean a ruled submanifold generated by a family of lines passing through the origin. A submanifold of $\mathbf{E}^n$ is called a conic submanifold with vertex at the origin if it is contained in a cone with vertex at the origin.*

The following result provides a very simple geometric characterization of conic submanifolds.

Proposition 5.2. *Let $x : M \to \mathbf{E}^n_* \subset \mathbf{E}^n$ be an isometric immersion. Then x is a conic submanifold with vertex at the origin if and only if the distance function $\rho = |x|$ satisfies the condition: $|\text{grad}\,\rho| = 1$.*

Proof. Assume that $x : M \to \mathbf{E}^n$ satisfies $|\text{grad}\,\rho| = 1$. Then we have $x^T = x$. Hence $e_1 = x/|x|$ is a unit vector field tangent to M. Thus, we obtain $\tilde{\nabla}_{e_1} x = e_1$ and $\tilde{\nabla}_{e_1} x = \tilde{\nabla}_{e_1}(\rho e_1) = (e_1\rho)e_1 + \rho\tilde{\nabla}_{e_1} e_1$. Therefore , we find $\tilde{\nabla}_{e_1} e_1 = 0$. Hence, the integral curves of e_1 are lines in $\mathbf{E}^n$. Moreover, from the fact that the position vector is always tangent to the submanifold, we also know that the lines given by the integral curves of e_1 must pass through the origin. Consequently, x is a conic submanifold with vertex at the origin.

The converse follows from Lemma 5.1. □

The following result provides us a necessary and sufficient condition for the convolution of two Euclidean submanifolds to be degenerate.

Proposition 5.3. *Let $x : (N_1, g_1) \to \mathbf{E}^n_* \subset \mathbf{E}^n$ and $y : (N_2, g_2) \to \mathbf{E}^m_* \subset \mathbf{E}^m$ be isometric immersions of Riemannian manifolds (N_1, g_1) and (N_2, g_2) into $\mathbf{E}^n_*$ and $\mathbf{E}^m_*$, respectively, and let $\rho_1 = |x|$ and $\rho_2 = |y|$ be the distance functions of x and y. Then the convolution ${}_{\rho_2}g_1 *_{\rho_1} g_2$ is degenerate if and only if both x and y are conic submanifolds with vertex at origin.*

Proof. Let $x : (N_1, g_1) \to \mathbf{E}^n_* \subset \mathbf{E}^n$ and $y : (N_2, g_2) \to \mathbf{E}^m_* \subset \mathbf{E}^m$ be isometric immersions of Riemannian manifolds (N_1, g_1) and (N_2, g_2) into $\mathbf{E}^n_*$ and $\mathbf{E}^m_*$, respectively. If ${}_{\rho_2}g_1 *_{\rho_1} g_2$ is degenerate, then Proposition 3.2 implies that both $|\text{grad}\,\rho_1|_1$ and $|\text{grad}\,\rho_2|_2$ are nonzero constants satisfying $|\text{grad}\,\rho_1|_1\,|\text{grad}\,\rho_2|_2 = 1$. Hence, by applying Lemma 5.1, we obtain $|\text{grad}\,\rho_1|_1 = |\text{grad}\,\rho_2|_2 = 1$. Thus, we get $x^T = x$ and $y^T = y$. Therefore, by applying Proposition 5.2, we conclude that x and y are both conic submanifolds with vertex at the origin.

The converse follows from Proposition 3.2 and Proposition 5.2. □

Definition 5.2. *An immersion $x : M \to \mathbf{E}^n$ is said to be transversal at a point $p \in M$ if and only if the position vector $x(p)$ is not tangent to M at p, that is $x(p) \notin dx(T_pM)$. If x is transversal at every point of M, then the immersion x is said to be transversal.*

Corollary 5.4. [3] *Let $x : (N_1, g_1) \to \mathbf{E}^n_* \subset \mathbf{E}^n$ and $y : (N_2, g_2) \to \mathbf{E}^m_* \subset \mathbf{E}^m$ be isometric immersions of Riemannian manifolds (N_1, g_1) and (N_2, g_2) into $\mathbf{E}^n_*$ and $\mathbf{E}^m_*$, respectively. If either x or y is transversal, then $x \otimes y : N_1 \times N_2 \to \mathbf{E}^n \otimes \mathbf{E}^m$ is an immersion.*

Proof. Follows from Proposition 5.2 and Proposition 5.3. □

For curves in a Euclidean space, we have the following.

Lemma 5.5. *Let $\gamma : I \to \mathbf{E}^n$ be a unit speed curve and $c \in (0,1)$. Then, up to translations of the arclength function s, we have*
(a) $|\mathrm{grad}\,\rho| = c \iff |\gamma(s)| = cs$.
(b) *If $n = 2$ and $|\mathrm{grad}\,\rho| = c$, then the curvature function κ of γ satisfies $\kappa^2(s) = (1-c^2)/c^2(s^2+b)$ for some constant b.*

Proof. Let $\gamma : I \to \mathbf{E}^n$ be a unit speed curve and let $\rho(s) = |\gamma(s)|$ be the distance function of γ. Then
$$\mathrm{grad}\,\rho = \frac{d\rho}{ds}\gamma'(s) = \frac{\langle \gamma(s), \gamma'(s)\rangle}{|\gamma(s)|}\gamma'(s) = \frac{\langle \gamma(s), \gamma(s)\rangle'}{2|\gamma(s)|}\gamma'(s). \tag{5.2}$$
Hence, we have $|\mathrm{grad}\,\rho| = c$ for some constant $c \in (0,1)$ if and only if we have $\langle \gamma(s), \gamma(s)\rangle' = 2c\,\langle \gamma, \gamma\rangle^{1/2}$. The later condition is equivalent to $\rho'(s) = c$. Thus the condition $|\mathrm{grad}\,\rho| = c$ holds if and only if we have $|\gamma(s)| = cs + b$ for some constant b. After a suitable reparametrization of the arclength function s, we have $b = 0$. Hence we obtain $|\gamma(s)| = cs$. This proves statement (a).

Suppose $n = 2$ and $|\mathrm{grad}\,\rho| = c \in (0,1)$. Then, by applying Proposition 5.2, we obtain $\langle \gamma, T\rangle^2 = c^2\langle \gamma, \gamma\rangle$, $T = \gamma'$. Differentiating this equation with respect to arclength function s yields
$$\kappa\,\langle \gamma, N\rangle = c^2 - 1, \tag{5.3}$$
where N is a unit normal vector field of γ. Thus, by applying Frenet's formula and (5.3), we obtain
$$\langle \gamma, T\rangle = \frac{\kappa'}{\kappa^2}\langle \gamma, N\rangle = \frac{\kappa'}{\kappa^3}(c^2-1). \tag{5.4}$$
Differentiating (5.4) with respect to s and applying (5.3) give
$$\left(\frac{\kappa'}{\kappa^3}\right)' = \frac{c^2}{c^2-1}. \tag{5.5}$$
Therefore, by solving (5.5), we get
$$\frac{1}{\kappa^2} = \frac{c^2}{1-c^2}\big((s+a)^2 + b\big), \tag{5.6}$$
where a and b are the integrating constants. Thus, we may obtain statement (b) after applying a suitable translation in s. □

Theorem 5.6. *Let $\gamma : I \to \mathbf{E}^n$ be a unit speed curve in the Euclidean n-space. Then $|\mathrm{grad}\,\rho| = c$ holds for a constant c if and only if one of the following three cases occurs:*
(1) *$\gamma(I)$ is contained in a hypersphere centered at the origin.*
(2) *$\gamma(I)$ is an open portion of a line through the origin.*

(3) $\gamma(s) = csY(u)$, $c \in (0,1)$, *where* $Y = Y(u)$ *is a unit speed curve in the unit hypersphere of* $\mathbf{E}^n$ *centered at the origin and* $u = (\sqrt{1-c^2}/c)\ln s$.

Proof. If $c = 0$ or $c = 1$, we have case (1) or case (2), respectively. So, let us assume that $|\mathrm{grad}\,\rho| = c$ holds for some $c \in (0,1)$. In this case, Lemma 5.5 implies that

$$\gamma(s) = csY(s) \tag{5.7}$$

for some $Y(s)$ with $|Y(s)| = 1$. From (5.7) we get

$$\gamma'(s) = cY(s) + csY'(s) \tag{5.8}$$

which implies $1 = |\gamma'(s)|^2 = c^2(1 + s^2|Y'(s)|^2)$. Thus, $|Y'(s)| = \sqrt{1-c^2}/(cs)$. Hence, if we put $u = (\sqrt{1-c^2}/c)\ln s$, then $Y(u)$ is a unit speed curve in the unit hypersphere of $\mathbf{E}^n$ centered at the origin. Thus, we obtain case (3).

The converse can be verified easily. □

The same proof as for statement (a) of Lemma 5.5 also gives the following.

Lemma 5.7. *Let* $\gamma : I \to \mathbf{E}^n_t$ *be a unit speed space-like curve in a pseudo-Euclidean space with index* t. *Then* $|\mathrm{grad}\,\rho| = c$ *holds for a constant* c *if and only if, up to translations of the arclength function* s, *we have* $|\gamma(s)| = cs$.

Theorem 5.8. *Let* $\gamma : I \to \mathbf{E}^n_t$ *be a unit speed space-like curve in a pseudo-Euclidean* n*-space. Then* $|\mathrm{grad}\,\rho| = c$ *holds for a constant* c *if and only if one of the following three cases occurs:*

(1) $\gamma(I)$ *is contained in* $\{w \in \mathbf{E}^n_t : \langle w, w\rangle = \alpha\}$, *for some real number* $\alpha \neq 0$,

(2) $\gamma(I)$ *is an open portion of a space-like line through the origin.*

(3) $\gamma(s) = csY(u)$ *for some* $c \neq 0, 1$, *where* $Y = Y(u)$ *is a unit speed curve in* $\{w \in \mathbf{E}^n_t : \langle w, w\rangle = \epsilon_1\}$, $(\epsilon_1 = 1$ *or* $-1)$, $u = (\sqrt{1-\epsilon_1\epsilon_2 c^2}/c)\ln s$, *and* $\epsilon_2 = 1$ *or* -1, *according to* $Y(u)$ *is a space-like or time-like unit speed curve.*

This theorem can be proved in a way similar to the proof of Theorem 5.6.

Remark 5.1. (Added on August 29, 2001) Recently, the author has completely classified Riemannian submanifolds in a pseudo-Euclidean space whose distance function ρ satisfies the condition: $|\mathrm{grad}\,\rho| = c$ holds for a constant c. For the details, see [2].

6. Total geodesy of convolution Riemannian manifolds

Let us assume that (N_1, g_1) and (N_2, g_2) are Riemannian manifolds and f and h are positive functions on N_1 and N_2 satisfying $0 < |\mathrm{grad}\,f|_1\,|\mathrm{grad}\,h|_2 < 1$. Then by Proposition 3.3, we know that the convolution $g_{f,h} = {}_h g_1 *_f g_2$ of g_1, g_2 via f and h is a Riemannian metric on $N_1 \times N_2$; thus ${}_hN_1 \bigstar {}_fN_2 = (N_1 \times N_2, g_{f,h})$ is a convolution Riemannian manifold.

If we put

$$\begin{aligned} \mathcal{D}_1 &= T(N_1), \quad \mathcal{F}_1 = \{X \in \mathcal{D}_1 : Xf = 0\}, \\ \mathcal{D}_2 &= T(N_2), \quad \mathcal{F}_2 = \{Z \in \mathcal{D}_2 : Zh = 0\}. \end{aligned} \tag{6.1}$$

then $\mathcal{D}_1, \mathcal{D}_2, \mathcal{F}_1$ and $\mathcal{F}_2$ can be regarded as distributions on ${}_hN_1\bigstar{}_fN_2$ in a natural way.

It follows from the definition of $g_{f,h}$ that the distribution $\mathcal{F}_1$ is a normal subbundle of $\{u\}\times N_2$ in ${}_hN_1\bigstar{}_fN_2$ for $u \in N_1$. Similarly, $\mathcal{F}_2$ is a normal subbundle of $N_1\times\{v\}, v \in N_2$, in ${}_hN_1\bigstar{}_fN_2$.

Theorem 6.1. *Let ${}_hN_1\bigstar{}_fN_2$ be a convolution Riemannian manifold. Then, for every $v \in N_2$, we have*

(1) *$N_1\times\{v\}$ is $\mathcal{F}_2$-totally geodesic in ${}_hN_1\bigstar{}_fN_2$, that is, the shape operator A^1_Z of $N_1\times\{v\}$ in ${}_hN_1\bigstar{}_fN_2$ vanishes identically for every $Z \in \mathcal{F}_2$,*

(2) *$N_1 \times \{v\}$ is a totally geodesic submanifold of ${}_hN_1\bigstar{}_fN_2$ if and only if*

$$\left\langle \nabla_X Y, \operatorname{grad} h^2 \right\rangle = |\operatorname{grad} h|_2^2 \left\langle \nabla_X Y, \operatorname{grad} f^2 \right\rangle \tag{6.2}$$

holds for any vector fields X, Y tangent to $N_1 \times \{v\}$.

Proof. Let X, Y be any two vector fields in $\mathcal{D}_1$ and Z be a vector field in $\mathcal{F}_2$. Then we have

$$[X, Z] = \nabla_X Z - \nabla_Z X = 0. \tag{6.3}$$

Let $\langle\ ,\ \rangle$ denote the inner product of ${}_hN_1\bigstar{}_fN_2$. Then

$$Z\left\langle X, Y\right\rangle = Z(h^2 g_1(X,Y)) = 2\Big(\frac{Zh}{h}\Big)\left\langle X, Y\right\rangle = 0. \tag{6.4}$$

On the other hand, from (6.3), we find

$$\begin{aligned} Z\left\langle X, Y\right\rangle &= \left\langle \nabla_Z X, Y\right\rangle + \left\langle X, \nabla_Z Y\right\rangle \\ &= \left\langle \nabla_X Z, Y\right\rangle + \left\langle X, \nabla_Y Z\right\rangle \\ &= -\left\langle Z, \nabla_X Y\right\rangle - \left\langle \nabla_Y X, Z\right\rangle \\ &= -2\left\langle Z, \sigma^1(X,Y)\right\rangle \\ &= -2\left\langle A^1_Z X, Y\right\rangle, \end{aligned} \tag{6.5}$$

where σ^1 is the second fundamental form of $N_1 \times \{v\}$ in ${}_hN_1\bigstar{}_fN_2$. Combining (6.4) and (6.5), we conclude that $N_1 \times \{v\}$ is $\mathcal{F}_2$-totally geodesic in ${}_hN_1\bigstar{}_fN_2$. This proves statement (1).

From (1.1) it follows that

$$V_2 = \operatorname{grad} h - \Big(\frac{f}{h}\,|\operatorname{grad} h|_2^2\Big)\operatorname{grad} f \tag{6.6}$$

is a normal vector field of $N_1 \times \{v\}$ in ${}_hN_1\bigstar{}_fN_2$, where $\operatorname{grad} f$ and $\operatorname{grad} h$ are regarded as vector fields on ${}_hN_1\bigstar{}_fN_2$. The vector field V_2 is perpendicular to the normal subbundle

$\mathcal{F}_2$. Clearly, the normal bundle of $N_1 \times \{v\}$ in ${}_hN_1\bigstar_f N_2$ is spanned by $\mathcal{F}_2$ and V_2. Hence, by applying statement (1), we know that $N_1 \times \{v\}$ is a totally geodesic submanifold of ${}_hN_1\bigstar_f N_2$ if and only if we have

$$0 = \langle \operatorname{grad} h, \nabla_X Y\rangle - \Big(\frac{f}{h}\,|\operatorname{grad} h|_2^2\Big)\,\langle \operatorname{grad} f, \nabla_X Y\rangle\,. \tag{6.7}$$

for vector fields X, Y tangent to $N_1 \times \{v\}$. Clearly, condition (6.7) is nothing but condition (6.2). ☐

Similarly,

$$V_1 = \operatorname{grad} f - \Big(\frac{h}{f}\,|\operatorname{grad} f|_2^2\Big)\operatorname{grad} h \tag{6.8}$$

is a normal vector field of $\{u\} \times N_2$ in ${}_hN_1\bigstar_f N_2$. Moreover, V_1 is orthogonal to the normal subbundle $\mathcal{F}_1$ and the normal bundle of $\{u\} \times N_2$ in ${}_hN_1\bigstar_f N_2$ is spanned by $\mathcal{F}_1$ and V_1.

In general, V_1 and V_2 are not perpendicular to each other, since

$$\langle V_1, V_2\rangle = fh\,|\operatorname{grad} f|_1^2\,|\operatorname{grad} h|_2^2\big(\,|\operatorname{grad} f|_1^2\,|\operatorname{grad} h|_2^2 - 1\big). \tag{6.9}$$

When the convolution ${}_hg_1 *_f g_2$ is a Riemannian metric, Proposition 3.3 implies $|\operatorname{grad} f|_1^2\,|\operatorname{grad} h|_2^2 < 1$. In this case, V_1 and V_2 are orthogonal if and only if either f or h is a constant function.

We may apply Theorem 6.1 to obtain the following.

Theorem 6.2. *Let ${}_hI\bigstar_f N$ be the convolution Riemannian manifold of an open interval I of the real line and a Riemannian $(n-1)$-manifold N via f and h. Then $I \times \{v\}$ is a geodesic in ${}_hI\bigstar_f N$ for every $v \in N$ if and only if one of the following two cases occurs:*
(1) h is a constant function on N.
(2) Up to translations of I, $f(s)$ is given by $\sqrt{s^2 + c}$ for some constant c.

Proof. The proof of this theorem bases on Theorem 6.1 and the following.

Lemma 6.3. *Let ${}_hI\bigstar_f N$ be the convolution Riemannian manifold of an open interval I and a Riemannian $(n-1)$-manifold N via f and h. Denote by s an arclength function of I. Then we have*

$$\langle\, \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \operatorname{grad} f^2 \,\rangle = 0, \tag{6.10}$$

$$\langle\, \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \operatorname{grad} h^2 \,\rangle = 2(f'^2 + ff'' - 1)h^2\,|\operatorname{grad} h|^2. \tag{6.11}$$

Proof of Lemma 6.3. Let $\{u_2, \ldots, u_n\}$ be a local coordinate system on N. Suppose $g_2 = \sum_{j,k=2}^n \tilde g_{jk} du_j du_k$. Let (g^{ab}) be the inverse matrix of (g_{ab}), g_{ab} the coefficients of g. And denote by $(\tilde g_{jk})$ the inverse matrix of $(\tilde g_{jk})$.

The convolution metric on the ${}_hI\bigstar{}_fN$ is given by

$$g = h^2(u_2,\dots,u_n)ds^2 + f^2(s)\sum_{j=2}^{n}\sum_{j,k=2}^{n}\tilde g_{jk}du_jdu_k + 2ff'hds\otimes\sum_{j=2}^{n}h_jdu_j, \tag{6.12}$$

where $h_j = \partial h/\partial u_j$. Then, by a direct computation, we find

$$\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = (f'^2+ff''-1)h\left\{\sum_{j=2}^{n}g^{1j}h_j\frac{\partial}{\partial s} + \sum_{\ell,t=2}^{n}g^{\ell t}h_\ell\frac{\partial}{\partial u_t}\right\}, \tag{6.13}$$

where

$$\Delta = \det(g_{ab}) = f^{2n-2}h^2\mathcal{G}\big(1-f'^2\,|\mathrm{grad}\,h|_2^2\big), \quad \mathcal{G}=\det(\tilde g_{jk}). \tag{6.14}$$

From the definition of gradient, we have

$$\mathrm{grad}\,f^2 = 2ff'\frac{\partial}{\partial s}, \quad \mathrm{grad}\,h^2 = 2h\sum_{j,k=2}^{n}\tilde g^{jk}h_k\frac{\partial}{\partial u_j}. \tag{6.15}$$

From (6.12), (6.13) and (6.15), we obtain

$$\begin{aligned}&\langle\, \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \mathrm{grad}\,f^2\,\rangle\\ &= 2ff'(f'^2+ff''-1)h^2\left\{h\sum_{j=2}^{n}h_jg^{1j} + ff'\sum_{j,k=2}^{n}g^{jk}h_jh_k\right\}\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}&\langle\, \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}, \mathrm{grad}\,h^2\,\rangle = 2(f'^2+ff''-1)h^2\times\\ &\left\{\sum_{j,k,\ell=2}^{n}g^{1j}g_{1\ell}\tilde g^{k\ell}h_jh_k + \sum_{j,k,\ell,t=2}^{n}\tilde g^{jk}g^{\ell t}g_{tj}h_kh_\ell\right\}.\end{aligned} \tag{6.17}$$

Equation (6.11) now follows from(6.17) and $\sum_{t=2}^{n}g_{tj}g^{\ell t} = \delta_j^\ell - g_{1j}g^{1\ell}$.

Since equation (6.10) is independent of the choice of coordinate system on N, we only need to verify the equation at any arbitrary point $v\in N$. For simplicity, we choose a local normal coordinate system $\{u_2,\dots,u_n\}$ on N about v. At $v\in N$ we have

$$\begin{aligned}g^{11} &= \frac{f^{2n-2}}{\Delta},\\ g^{1j} &= -\frac{f^{2n-3}f'hh_j}{\Delta},\\ g^{jj} &= \frac{f^{2n-4}h^2}{\Delta}\Big(1-f'^2\sum_{k\neq j}h_k^2\Big),\\ g^{jk} &= \frac{f^{2n-4}f'^2h^2h_jh_k}{\Delta}, \quad 2\le j\neq k\le n,\end{aligned} \tag{6.18}$$

From (6.18) we find

$$\begin{aligned} & h\sum_{j=2}^{n} h_j g^{1j} + ff' \sum_{j,k=2}^{n} g^{jk} h_j h_k \\ & = h\sum_{j=2}^{n} h_j g^{1j} + ff' \sum_{j=2}^{n} g^{jj} h_j^2 + ff' \sum_{j\neq k} g^{jk} h_j h_k = 0 \end{aligned} \tag{6.19}$$

at v. Combining (6.16) and (6.19) gives equation (6.10). This proves Lemma 6.3.

By applying Theorem 6.1 and Lemma 6.3, we know that $I \times \{v\}$ is a geodesic in ${}_hI\bigstar{}_fN$ for every $v \in N$ if and only if we have

$$(f'^2 + ff'' - 1)\,|\mathrm{grad}\, h|^2 = 0, \tag{6.20}$$

which implies either h is constant on N or $f(s)$ on I satisfies the following differential equation:

$$1 = f'^2 + ff''. \tag{6.21}$$

By solving (6.21) we obtain $f(s) = \sqrt{s^2 + c_1 s + c_2}$ for some constants c_1 and c_2. Hence, after applying a suitable translation on s, we have $f(s) = \sqrt{s^2 + c}$ for some real number c.

The converse is easy to verify. □

7. When ${}_hI_1\bigstar{}_fI_2$ is a flat surface?

Let α be a positive number and k a nonzero real number. We define functions $F_{k,\alpha}$ and $H_{k,\beta}$ of one variable by

$$F_{k,\alpha}(u) = \int_0^u \frac{dx}{\sqrt{1 + kx^{2/\alpha}}}, \tag{7.1}$$

$$H_{k,\alpha}(u) = \int_0^u \frac{x^{1/\alpha}dx}{\sqrt{k + x^{2/\alpha}}}. \tag{7.2}$$

Denote by $F_{k,\alpha}^{-1}$ and $H_{k,\alpha}^{-1}$ the inverse functions of $F_{k,\alpha}$ and $H_{k,\alpha}$, respectively.

In this section, we completely classify flat convolution Riemannian surfaces.

Theorem 7.1. *Let ${}_hI_1\bigstar{}_fI_2$ be a convolution Riemannian surface of two open intervals of the real line. Then ${}_hI_1\bigstar{}_fI_2$ is a flat surface if and only if one of the following seven cases occurs:*

(1) *f and h are positive constants.*

(2) *f is a positive constant and $h = h(t)$ is a positive linear function or h is a positive constant and $f = f(s)$ is a positive linear function.*

(3) *$f = s+b$ for some constant b and $h(t)$ is a positive function with $h'(t) \neq 1$ or $h = t+d$ for some constant d and $f(s)$ is a positive function with $f'(s) \neq 1$.*

(4) *$f = as + b, h = ct + d$, where a, b, c, d are constants satisfying $0 < a^2c^2 < 1$.*

(5) $f(s) = F^{-1}_{k,\alpha}(s)$ *and* $h(t) = H^{-1}_{m,-\alpha-1}(t)$ *for some nonzero constants* k, m *and positive number* α.
(6) $f(s) = H^{-1}_{k,-\alpha}(s)$ *and* $h(t) = F^{-1}_{m,-\alpha-1}(t)$ *for some nonzero constants* k, m *and real number* $\alpha < -1$.
(7) $f(s) = H^{-1}_{k,-\alpha}(s)$ *and* $h(t) = H^{-1}_{m,-\alpha-1}(t)$ *for some nonzero constants* k, m *and number* α *satisfying* $0 > \alpha > -1$.

Proof. Without loss of generality, we may assume that I_1 and I_2 contains the zero. Suppose that ${}_hI_1\bigstar{}_fI_2$ is a flat convolution Riemannian surface. Then, according to Proposition 3.3, we have $f'^2h'^2 \neq 1$. Since the convolution metric on ${}_hI_1\bigstar{}_fI_2$ is given by

$$g = h^2(t)ds^2 + f^2(s)dt^2 + 2fhdf \otimes dh, \tag{7.3}$$

a direct computation gives

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} &= \frac{(1 - f'^2 - ff'')h'}{f^2(1 - f'^2h'^2)} \Big(ff'h' \frac{\partial}{\partial s} - h \frac{\partial}{\partial t} \Big), \\ \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} &= \frac{(1 - f'^2)h'}{(1 - f'^2h'^2)h} \frac{\partial}{\partial s} + \frac{(1 - h'^2)f'}{(1 - f'^2h'^2)f} \frac{\partial}{\partial t} \\ \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\frac{(1 - h'^2 - hh'')f'}{h^2(1 - f'^2h'^2)} \Big(f \frac{\partial}{\partial s} - f'hh' \frac{\partial}{\partial t} \Big). \end{aligned} \tag{7.4}$$

The equations in (7.4) and a straightforward long computation imply that the Riemann curvature tensor of ${}_hI_1\bigstar{}_fI_2$ satisfies

$$R\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\Big)\frac{\partial}{\partial s} = A(s,t)\frac{\partial}{\partial s} + B(s,t)\frac{\partial}{\partial t}, \tag{7.5}$$

where

$$\begin{aligned} A &= -f'h' \frac{(1 - f'^2)hh'' + (1 - h'^2 - hh'')ff''}{(1 - f'^2h'^2)^2 fh}, \\ B &= \frac{(1 - f'^2)hh'' + (1 - h'^2 - hh'')ff''}{(1 - f'^2h'^2)^2 f^2}. \end{aligned} \tag{7.6}$$

Thus ${}_hI_1\bigstar{}_fI_2$ is a flat convolution Riemannian manifold if and only if

$$(1 - f'^2)hh'' + (1 - h'^2 - hh'')ff'' = 0 \tag{7.7}$$

holds.

Case (1)*:* h is constant. In this case, (7.7) reduces to $f'' = 0$. Thus, $f = as + b$ for some constants a, b. Hence, f is either a nonzero constant or a linear function in s.

Case (2)*:* f is constant and h is nonconstant. In this case, (7.7) reduces to $h'' = 0$. Thus, $h(t)$ is a linear function in t.

Case (3)*:* f, h are non-constant and $h'' = 0$. In this case, $h = ct + d$ for some constants $c \neq 0, d$. Hence, equation (7.7) implies $(1 - c^2)f'' = 0$. Therefore, either we have $c^2 = 1$ or $f(s)$ is a linear function.

Case (3.i)*:* If $c^2 = 1$, then without loss of generality we may assume $c = 1$; and hence $h = t + d$. From (7.7) we know that $f = f(s)$ can be any positive function with $f' \neq 1$.

Case (3.ii)*:* If $f(s)$ is a linear function, then we may put $f = as + b$ and $h = ct + d$ for some constants a, b, c, d. Since f and h are nonconstant, a and c are nonzero. In order that the convolution defines a Riemannian metric, we must have $a^2c^2 < 1$, according to Proposition 3.3.

Case (4)*:* f and h are non-constant and $f'' = 0$. In this case, we have either $f = s + b$ for some constant b and $h(t)$ any positive function with $h'(t) \neq 1$ or both $f(s)$ and $h(t)$ are linear functions such that $f'^2h'^2 \neq 1$.

Case (5)*:* $f'' \neq 0$ and $h'' \neq 0$. In this case, (7.7) implies

$$\frac{f'^2 - 1}{ff''} = \frac{1 - h'^2 - hh''}{hh''} = \alpha, \tag{7.8}$$

for some constant α. Hence, we find

$$\alpha ff'' - f'^2 + 1 = 0, \tag{7.9}$$
$$\beta hh'' - h'^2 + 1 = 0, \tag{7.10}$$

where $\alpha + \beta = -1$.

If $\alpha = 0$, then (7.9) reduces to $f'^2 = 1$ which is a contradiction, since $f'' \neq 0$. Therefore, we have $\alpha \neq 0$. Similarly, (7.10) implies $\alpha \neq -1$.

In order to solve equation (7.9). Let us put $v = f'(s)$. Then (7.9) can be written as

$$\frac{vdv}{v^2 - 1} = \frac{df}{\alpha f}. \tag{7.11}$$

So, after integrating both sides of (7.11) we find

$$\frac{df}{ds} = \pm\sqrt{1 + kf^{2/\alpha}} \tag{7.12}$$

for some constant $k \neq 0$.

Case (5.i)*:* $\alpha > 0$. In this case, solving (7.12) yields $F_{k,\alpha}(f) = \pm(s + c_1)$ for some constant c_1. Thus, we obtain $f(s) = F^{-1}_{k,\alpha}(\pm(s + c_1))$. Without loss of generality, we may assume $f(s) = F^{-1}_{k,\alpha}(s)$, by applying a reparametrization of s if necessary.

Since $\alpha > 0$, we have $\beta = -1 - \alpha < -1$. So, after solving (7.10) for h' in the same was as f', we obtain

$$\frac{dh}{dt} = \pm\frac{\sqrt{m + h^{2/|\beta|}}}{h^{1/|\beta|}}, \quad |\beta| = 1 + \alpha, \tag{7.13}$$

for some constant m. Since $h'' \neq 0$, we have $m \neq 0$. Thus, from (7.13), we find $H_{m,1+\alpha}(h) = \pm(t+c_2)$ for some constant c_2. Therefore, we find $h(t) = H^{-1}_{m,1+\alpha}(\pm(t+c_2))$. Without loss of generality, we may assume $h(t) = H^{-1}_{m,1+\alpha}(t)$, by applying a reparametrization of s if necessary.

Case (5.ii)*:* $\alpha < -1$. In this case, we have $\beta > 0$. Thus, we may solve (7.9) and (7.10) in a way similar to Case (5.i) to obtain $f(s) = H^{-1}_{k,-\alpha}(s)$, $h(t) = F^{-1}_{m,-\alpha-1}(t)$ for some nonzero constants k and m.

Case (5.iii)*:* $0 > \alpha > -1$. In this case, $\beta = -1-\alpha < 0$. Thus, we may solve (7.9) and (7.10) as solving h' in Case (5.i) to obtain $f(s) = H^{-1}_{k,-\alpha}(s)$, $h(t) = H^{-1}_{m,-\alpha-1}(t)$ for some nonzero constants k and m.

The converse follows from straightforward computation. □

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