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The Gelfand-Kirillov Dimension of Rings with Hopf Algebra Action

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Abstract. Let k be a perfect field, H a irreducible cocommutative Hopf k-algebra and $P(H)$ the space of primitive elements of H, R a k-algebra on which acts locally finitely H and $R\#H$ the associated smash product. Assume that H is almost solvable with $P(H)$ finite-dimensional n and the sequences of divided powers are all infinite. Then the Gelfand-Kirillov dimension of $R\#H$ is $GK(R)+n$.

1. Introduction

It is well known [7], that if δ is a derivation of an algebra R over a field k, then the Gelfand-Kirillov dimension of the polynomial algebra $R[\theta,\delta]$ is equal to $GK(R)+1$, provided R is δ-locally-finite. More generaly, if g is a finite-dimensional k-Lie algebra acting locally finitely on R, then the Gelfand-Kirillov dimension of the differential operator ring $R\#U(g)$ is $GK(R)+dim_k(g)$ where $U(g)$ is the enveloping algebra of g (see [5, Corollary 1.5]). The main objective of this note is to present a generalization of the above mentioned result to the case of a irreducible cocommutative Hopf algebra action. However, we assume that H is amost solvable. Note that $U(g)$ is a irreducible cocommutative Hopf algebra.

The Gelfand-Kirillov dimension of R (see [6] for the basic material), denoted $GK(R)$, is defined as follows (here V^l is the linear span of all products $v_1v_2\cdots v_l$ with $v_1,v_2,\dots,v_l\in V$):

$$GK(R)=\sup\{\limsup_{n\to\infty}(log_n dim_k V^n : V \quad \text{is a finite-dimensional subspace of R})\}.$$

Throughout the paper, k is a field, H is a Hopf k-algebra with comultiplication Δ, counit ϵ and antipode s, and R is an H-module algebra (the action of $h\in H$ shall be denoted by $h.r$), i.e.

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an associative k-algebra with identity which is a left H-module such that the multiplication in R is an H-module map, i.e., $h.(ab) = \sum_{(h)}(h_1.a)(h_2.b)$ for all $h \in H$ and $a, b \in R$. We denote by $R\#H$ the associated smash product. Both R and H are naturally embedded in $R\#H$. The multiplication in $R\#H$ is defined by the rule $(a\#h)(b\#g) = \sum_{(h)} a(h_1.b)\#h_2g$. For further information on Hopf algebras and the ring $R\#H$, the reader is referred to [1, 8 and 10]. We denote by $P(H)$ the space of primitive elements of H. We say that H is cocommutative if $\Delta = \tau \circ \Delta$ where τ is the usual twist map $\tau(a \otimes b) = b \otimes a$. By [8, Corollary 1.5.12], the antipode of a cocommutative Hopf algebra is involutive. We say that H is irreducible if any two nonzero subcoalgebras of H have nonzero intersection.

If H is irreducible cocommutative, then so is any subHopfalgebra of H; if the characteristic of k is 0, then H is the enveloping algebra of $P(H)$.

Let X be an element of $P(H)$. A sequence of divided powers over X of maximum length l possibly infinite is a sequence $X^{(0)} = 1$, $X^{(1)} = X, \ldots, X^{(l)}$ such that $X^{(i)}X^{(j)} = \binom{i+j}{i} X^{(i+j)}$ and $\Delta(X^{(j)}) = \sum_{j'=0}^{j} X^{(j')} \otimes X^{(j-j')}$ for each $i, j \leq l$. It follows routinely from the counitary property that $\epsilon(X^{(l)}) = 0$ for $l > 0$. If k has characteristic 0, then $X^{(n)} = X^n/n!$.

If k is perfect and if H is irreducible with $P(H)$ finite-dimensional n, then by [11, Theorems 2, 3] and [12], H has a basis consisting of ordered monomials ${X_1}^{(i_1)}{X_2}^{(i_2)} \cdots {X_n}^{(i_n)}$; $i_j \in \mathbb{N}$; where $(X_1, X_2, \ldots X_n)$ is a basis for $P(H)$.

Examples 1.1. (1) Let k be of characteristic 0, g a finite-dimensional k-Lie algebra of dimension n and $H = U(g)$. Then H is a irreducible cocommutative Hopf algebra and $P(H) = g$. Furthermore H has a basis consisting of ordered monomials ${X_1}^{(i_1)}{X_2}^{(i_2)} \cdots {X_n}^{(i_n)}$; $i_j \in \mathbb{N}$ as above and the sequences of divided powers are all infinite.

(2) Let k be perfect, G an affine algebraic group over k of dimension n and $H = hyp(G)$ the hyperalgebra of G. Then H is a irreducible cocommutative Hopf algebra and $P(H)$ is the Lie algebra of G. Furthermore H has a basis consisting of ordered monomials ${X_1}^{(i_1)}{X_2}^{(i_2)} \cdots {X_n}^{(i_n)}$; $i_j \in \mathbb{N}$ as above and the sequences of divided powers are all infinite.

This paper accomplishes the following: Let k be perfect, H irreducible cocommutative with $P(H)$ finite-dimensional n and R H-locally finite. If the sequences of divided powers are all infinite and if H is almost solvable, then $GK(R\#H) = GK(R) + n$.

2. The main result

We consider H as a left H-module by the left adjoint action, that is $h.h' = \sum_{(h)} h_1h's(h_2)$. We say that a subHopfalgebra N of H is normal in H if $h.n \in N$ for all $h \in H, n \in N$. Let N be a normal subHopfalgebra of H. There is a natural action of H on $R\#N$ defined by $h.(rn) = \sum_{(h)}(h_1.r)(h_2.n)$.

The bracket product in H is defined by

$$[x, y] = \sum_{x,y} x_1y_1s(x_2)s(y_2) \text{ for } x, y \in H.$$

If I, J are subHopfalgebras of H, $[I, J]$ denotes the subalgebra of H generated by the elements $[x, y]$ with $x \in I$ and $y \in J$; if H is cocommutative, this is a subbialgebra of H.

We will say that I is central in H if $[H, I] = k$. Clearly, I is central in H if and only if $[x, y] = \epsilon(x)\epsilon(y)$ for all $x \in H$ and $y \in I$. If I is central in H, then I is normal in H.

Let G be a connected abelian algebraic group, then G is central in G; so by [14, Corollary 3.4.15], $hyp(G)$ is central in $hyp(G)$; i.e., $hyp(G)$ is a commutative Hopf algebra.

An ideal I of R is H-invariant if $h.I \subseteq I$ for all $h \in H$. Any ideal of $R\#H$ is H-invariant.

We say that R is H-simple, if the only H-invariant ideals of R are (0) and R.

A proper H-invariant ideal Q of R is H-prime if, whenever I and J are H-invariant ideals of R with $IJ \subseteq Q$ then either $I \subseteq Q$ or $J \subseteq Q$.

Any H-invariant prime ideal of R is H-prime. Let $I \subseteq Q$ be H-invariant ideals of R. If Q is H-prime, then Q/I is an H-prime ideal of R/I. We say that the ring R is H-prime if the ideal (0) is H-prime.

If Q is an H-prime ideal of R, then R/Q is an H-prime ring. Any H-simple ring is H-prime. The H-invariant prime ideals of $R\#H$ are precisely its prime ideals. If P is a prime ideal of $R\#H$ then $P \cap R$ is an H-prime ideal of R (see [4, Lemma 1.2]).

We say that R is H-locally finite if every element of R is contained in a finite-dimensional H-stable subspace of R. If H acts trivially on R then R is H-locally finite; in particular, if H is commutative, H is H-locally finite. If R and H are H-locally finite, then $R\#H$ is H-locally finite. By [13, page 259], if $p > 0$ and if H is irreducible cocommutative with $P(H)$ finite-dimensional, then H is the union of its finite-dimensional normal subHopfalgebras; so H is H-locally finite; hence any normal subHopfalgebra of H is H-locally finite. Clearly, R is g-locally finite as in [5, section 1] if and only if R is $U(g)$-locally finite.

Lemma 2.1. *Let G be a connected algebraic group acting rationally on R and $H = hyp(G)$ the hyperalgebra of G. Then R is H-locally finite.*

Proof. Let $a \in R$. Since R is a rational G-module, there exists a finite dimensional G-stable subspace V of R such that $a \in V$. By [14, Corollary 3.4.17], V is also H-stable. □

From now on k is perfect and H is irreducible cocommutative with $P(H)$ finite-dimensional n. So H has a basis consisting of ordered monomials $X_1{}^{(i_1)}X_2{}^{(i_2)}\cdots X_n{}^{(i_n)}$; $i_j \in \mathbb{N}$; where $(X_1, X_2, \ldots X_n)$ is a basis for $P(H)$. This basis will be fixed in the remainder of the paper.

We will say that H is almost solvable if there exists a chain of subHopfalgebras

$$k = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$$

of H such that for each $i \leq n$, H_{i-1} is normal in H_i and the monomials $X_1{}^{(j_1)}X_2{}^{(j_2)}\cdots X_i{}^{(j_i)}$; $j_i \in \mathbb{N}$ form a basis for H_i.

Thus H commutative implies H almost solvable; in particular, if $dim_k(P(H)) = 1$, then H is almost solvable. Let g be as in Examples 0.1 (1), then $U(g)$ is almost solvable if g is solvable in the usual sense. Let G be a connected affine agebraic group, then $hyp(G)$ is amost sovable.

Lemma 2.2. *Let G be a connected affine algebraic group and $H = hyp(G)$. If G is unipotent then H is almost solvable.*

Proof. It is well known that G has a composition series

$$1 = G_0 \subset G_1 \cdots \subset G_{n-1} \subset G_n = G$$

where each G_i is normal in G and each G_i/G_{i-1} is isomorphic to G_a, the one-dimensional additive group. Set $H_i = hyp(G_i)$, then $H_0 = k$ and $H_n = H$. By [14, Corollary 3.4.15], each H_i is a normal subHopfalgebra of H. Since $P(H)$ is nilpotent, there exists an element $X_i \in P(H_i) - P(H_{i-1})$ such that $(X_1, X_2, \ldots, X_{i-1}, X_i)$ is a basis for $P(H_i)$. By [11, Theorems 2, 3] and [12], the monomials ${X_1}^{(j_1)}{X_2}^{(j_2)} \cdots {X_i}^{(j_i)}$; $j_i \in \mathbb{N}$ form a basis for H_i, where the ${X_i}^{(j)}$ are infinite sequences of divided powers over X_i. □

We are now ready to prove the main result of the paper.

Theorem 2.3. *Let k be a perfect field, H a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional n and R an H-locally finite H-module algebra. Assume that the sequences of divided powers are all infinite. Then*

$$GK(R\#H) = GK(R) + n.$$

Proof. Suppose that $n = 1$ and set $g = P(H)$. So H has a basis consisting of ordered monomials $X^{(l)}$, where X is a k-basis of g. Note that R is g-locally finite. By [7], $GK(R\#U(g)) = GK(R) + 1$. So $GK(R\#H) \geq GK(R) + 1$, since $R\#U(g)$ is a subalgebra of $R\#H$. For the reverse inequality, let V be a finite-dimensional subspace of $R\#H$. Using the fact that R is H-locally finite, we see that

$$V \subseteq W + WX^{(1)} + WX^{(2)} + \cdots + WX^{(m)}$$

for some m and some finite-dimensional H-invariant subspace W of R. It is not difficult to show that

$$V^n \subseteq W^n + W^nX + W^nX^2 + \cdots + W^nX^n + W^nX^{(2)} + W^nX^{(3)} + \cdots + W^nX^{(nm)}.$$

So $dim_kV^n \leq (n + nm)(dim_kW^n)$ and we get

$$log_n(dim_kV^n) \leq log_n(dim_kW^n) + log_n(n + nm) = log_n(dim_kW^n) + 1 + log_n(1 + m).$$

This yields the reverse inequality $GK(R\#H) \leq GK(R) + 1$.

For the general case, let

$$k = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$$

be a chain of subHopfalgebras of H such that for each $i \leq n$, H_{i-1} is normal in H_i and the monomials ${X_1}^{(j_1)}{X_2}^{(j_2)} \cdots {X_i}^{(j_i)}$; $j_i \in \mathbb{N}$ form a basis for H_i. Set $R_i = R\#H_i$; so $R_0 = R$ and $R_n = R\#H$. Clearly, $R_{i+1} = R_i\#(k < X_{i+1} >)$ for each $i \leq n-1$, where $k < X_{i+1} >$ is the divided power Hopf algebra spanned by the monomials $X_{i+1}^{(j)}$, this is a subHopfalgebra of H_{i+1}. Now each R_i is $k < X_{i+1} >$-locally finite, since each R_i is H_{i+1}-locally finite. On the other hand, the space of primitive elements of $k < X_{i+1} >$ is the k-vector subspace kX_{i+1} of H_{i+1}. By the previous paragraph, $GK(R_{i+1}) = GK(R_i) + 1$ and the result follows. □

Theorem 1.3 may be applied in the following circumstances:

- k is of characteristic 0, g is a finite-dimensional solvable k-Lie algebra, H is the enveloping algebra of g and R is a g-locally finite $U(g)$-module algebra.

- k is perfect, G is a connected unipotent affine algebraic group acting rationally on R and H is the hyperalgebra of G.

- k is perfect, G is a connected abelian affine algebraic group acting rationally on R and H is the hyperalgebra of G.

- k is perfect, H is a divided powers Hopf algebra (with $dimP(H) = 1$) acting on R such that R is an H-locally finite H-module algebra.

As an application of Theorem 1.3 we shall show some results concerning incomparability and prime length. In the remainder of this section, R will be noetherian of finite Gelfand-Kirillov dimension and all the smash products are noetherian. We denote by *dim* the classical Krull dimension and by *H-dim* its H-invariant version; i.e. the maximal length of a chain of H-prime ideals of R. We have $H\text{-}dim(R\#H) = dim(R\#H)$. If R is H-locally finite, the H-prime ideals of R are prime [2, Proposition 1.3]; so $H\text{-}dim(R) \leq dim(R)$.

Corollary 2.4. *Let k be a perfect field, H a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional n, R an H-locally finite H-module algebra and $A = R\#H$. Assume that the sequences of divided powers are all infinite. Let P be a prime ideal of A such that $P \cap R = 0$. Then $ht(P) \leq n$. If R is H-simple, then $dim(A) \leq n$.*

Proof. Since $R = R/(P \cap R)$ is a subalgebra of A/P, we have $GK(R) \leq GK(A/P)$. Theorem 1.3 implies that $GK(A) - GK(A/P) \leq n$. By [6, Proposition 3.16], $ht(P) \leq n$. If R is H-simple, $ht(Q) \leq n$ for any prime ideal Q of A. □

The next result bounds $dim(R\#H)$ in terms of H-dim(R). Although, the bound is surely not sharp.

Proposition 2.5. *Let k be a perfect field, H a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional n, R an H-locally finite H-module algebra and $A = R\#H$. Assume that the sequences of divided powers are all infinite. Suppose that $P_0 \subset P_1 \subset \cdots \subset P_{n+1}$ is a strictly increasing chain of prime ideals of A, then $P_0 \cap R \subset P_{n+1} \cap R$ and $dim(A) < (n+1)(H - dim(R) + 1)$.*

Proof. Suppose that $P_0 \cap R = P_{n+1} \cap R = I$. By [4, Lemma 1.2], I is an H-prime ideal of R and $IA = AI$ is an ideal of A. By [2, Proposition 1.3], I is a prime ideal of R. One can show that $A/IA \simeq (R/I)\#H$. Set $\bar{R} = R/I$ and $\bar{A} = A/IA$. In $\bar{A}$, we have a strictly increasing chain of prime ideals $\overline{P_0} \subset \overline{P_1} \subset \cdots \subset \overline{P_{n+1}}$ of length $n+1$ such that $\overline{P_0} \cap \bar{R} = \overline{P_{n+1}} \cap \bar{R} = \bar{I} = 0$; where $\overline{P_i}$'s denote the natural images of P_i's in $\bar{A}$. It follows that $ht(\overline{P_{n+1}}) \geq n+1$. By Corollary 1.4, $ht(\overline{P_{n+1}}) \leq n$ and we get a contradiction.

Let $P_0 \subset P_1 \subset \cdots \subset P_s$ be a strictly increasing chain of prime ideals of A. By the preceding paragraph,

$$P_0 \cap R \subset P_{n+1} \cap R \subset P_{2(n+1)} \cap R \subset P_{3(n+1)} \cap R \subset \cdots$$

is a strictly increasing chain of H-invariant prime ideals of R. Since this chain can contain at most $(1 + H\text{-}dim(R))$ H-invariant prime ideals, we conclude that $s < (n+1)(H\text{-}dim(R)+1)$. □

Proposition 1.5 may be applied to the smash product $R\#U(g)$, where k is of characteristic 0, R is noetherian of finite Gelfand-Kirillov dimension and g is a finite dimensional solvable k-Lie algebra. For related work, see [3] and [9, Corollary 4.4].

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