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Curves in the Lightlike Cone

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Abstract. In this paper, we study curves in the lightlike cone. We first obtain the conformally invariant arc length in the $(n+1)$-dimensional lightlike cone and then characterize some curves in the 2-dimensional lightlike cone and 3-dimensional lightlike cone.

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0. Introduction

In General Relativity, null submanifolds usually appear to be some smooth parts of the achronal boundaries, for example, event horizons of the Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime and their properties are manifested in the proofs of several theorems concerning black holes and singularities. Degenerate submanifolds of Lorentzian manifolds may be useful to study the intrinsic structure of manifolds with degenerate metric and to have a better understanding of the relation between the existence of the null submanifolds and the spacetime metric.([8])

Although much has been known about submanifolds (hypersurfaces) of the pseudo-Riemannian space forms, there are rather few papers on submanifolds (hypersurfaces) of the pseudo-Riemannian lightlike cone. It should be remarked that a simply connected Riemannian manifold of dimension $n \geq 3$ is conformally flat if and only if it can be isometrically immersed as a hypersurface of the lightlike cone ([4], [2]). Moreover, from the relations between the conformal transformation group and the Lorentzian group of the n-dimensional Minkowski space $\mathbf{E}_1^n$, and the submanifolds of the n-dimensional Riemannian sphere $\mathbf{S}^n$ and the submanifolds of the $(n+1)$-dimensional lightlike cone $\mathbf{Q}^{n+1}$, we know that it is important to study submanifolds of the lightlike cone ([17], [9], [11]).

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In this paper, we are concerned with curves in the lightlike cone. For surfaces in the lightlike cone, see [10].

1. Curves in the lightlike cone $\mathbf{Q}^{n+1}$

Let $\mathbf{E}_q^m$ be the m-dimensional pseudo-Euclidean space with the metric

$$\bar{G}(x,y) = <x,y> = \sum_{i=1}^{m-q} x_i y_i - \sum_{j=m-q+1}^{m} x_j y_j,$$

where $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m) \in \mathbf{E}_q^m$. $\mathbf{E}_q^m$ is a flat pseudo-Riemannian manifold of signature $(m-q, q)$.

Let $\mathbf{M}$ be a submanifold of $\mathbf{E}_q^m$. If the pseudo-Riemannian metric $\bar{G}$ of $\mathbf{E}_q^m$ induces a pseudo-Riemannian metric G (respectively, a Riemannian metric, a degenerate quadratic form) on $\mathbf{M}$, then $\mathbf{M}$ is called a timelike (respectively, spacelike, degenerate) submanifold of $\mathbf{E}_q^m$.

Let c be a fixed point in $\mathbf{E}_q^m$ and $r>0$ be a constant. The pseudo-Riemannian sphere is defined by

$$\mathbf{S}_q^n(c,r) = \{x \in \mathbf{E}_q^{n+1} : \bar{G}(x-c, x-c) = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$\mathbf{H}_q^n(c,r) = \{x \in \mathbf{E}_{q+1}^{n+1} : \bar{G}(x-c, x-c) = -r^2\};$$

the pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$\mathbf{Q}_q^n(c) = \{x \in \mathbf{E}_q^{n+1} : \bar{G}(x-c, x-c) = 0\}.$$

It is well-known that $\mathbf{S}_q^n(c,r)$ is a complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$, in $\mathbf{E}_q^{n+1}$ with constant sectional curvature r^{-2}; $\mathbf{H}_q^n(c,r)$ is a complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$, in $\mathbf{E}_{q+1}^{n+1}$ with constant sectional curvature $-r^{-2}$; $\mathbf{Q}_q^n(c)$ is a degenerate hypersurface in $\mathbf{E}_q^{n+1}$. The spaces $\mathbf{E}_q^n$, $\mathbf{S}_q^n(c,r)$ and $\mathbf{H}_q^n(c,r)$ are called pseudo-Riemannian space forms. The point c is called the center of $\mathbf{S}_q^n(c,r)$, $\mathbf{H}_q^n(c,r)$ and $\mathbf{Q}_q^n(c)$. When $c=0$ and $q=1$, we simply denote $\mathbf{Q}_1^n(0)$ by $\mathbf{Q}^n$ and call it the lightlike cone (or simply the light cone) ([12])

Let $\mathbf{E}_1^{n+2}$ be the $(n+2)$-dimensional Minkowski space and $\mathbf{Q}^{n+1}$ the lightlike cone in $\mathbf{E}_1^{n+2}$. A vector $\alpha \neq 0$ in $\mathbf{E}_1^{n+2}$ is called spacelike, timelike or lightlike, if $<\alpha,\alpha> > 0$, $<\alpha,\alpha> < 0$ or $<\alpha,\alpha> = 0$, respectively. A frame field $\{e_1, \ldots, e_n, e_{n+1}, e_{n+2}\}$ on $\mathbf{E}_1^{n+2}$ is called an asymptotic orthonormal frame field, if

$$\begin{aligned} &<e_{n+1}, e_{n+1}> = <e_{n+2}, e_{n+2}> = 0, \quad <e_{n+1}, e_{n+2}> = 1, \\ &<e_{n+1}, e_i> = <e_{n+2}, e_i> = 0, \quad <e_i, e_j> = \delta_{ij}, \quad i,j = 1, \ldots, n. \end{aligned}$$

A frame field $\{e_1, \ldots, e_n, e_{n+1}, e_{n+2}\}$ on $\mathbf{E}_1^{n+2}$ is called a pseudo orthonormal frame field, if

$$\begin{aligned} &<e_{n+1}, e_{n+1}> = -<e_{n+2}, e_{n+2}> = 1, \quad <e_{n+1}, e_{n+2}> = 0, \\ &<e_{n+1}, e_i> = <e_{n+2}, e_i> = 0, \quad <e_i, e_j> = \delta_{ij}, \quad i,j = 1, \ldots, n. \end{aligned}$$

We assume that the curve

$$x : \mathbf{I} \to \mathbf{Q}^{n+1} \subset \mathbf{E}_1^{n+2}, \quad t \to x(t) \in \mathbf{Q}^{n+1}, \quad t \in \mathbf{I} \subset \mathbf{R}$$

is a regular curve in $\mathbf{Q}^{n+1}$. In the following, we always assume that the curve is regular and $x(t) \not\parallel x'(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$, for all $t \in \mathbf{I}$.

Definition 1.1. *A curve $x(t)$ in $\mathbf{E}_1^{n+2}$ is called a Frenet curve, if for all $t \in \mathbf{I}$, the vector fields $x(t)$, $x'(t)$, $x''(t)$, $\ldots$, $x^n(t)$, $x^{(n+1)}(t)$ are linearly independent and the vector fields $x(t)$, $x'(t)$, $x''(t)$, $\ldots$, $x^{(n+1)}(t)$, $x^{(n+2)}(t)$ are linearly dependent, where $x^{(n)}(t) = \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$.*

Since $<x, x> = 0$ and $<x, \mathrm{d}x> = 0$, $\mathrm{d}x(t)$ is spacelike. Then the induced arc length (or simply the arc length) s of the curve $x(t)$ can be defined by

$$\mathrm{d}s^2 = <\mathrm{d}x(t), \mathrm{d}x(t)>.$$

If we take the arc length s of the curve $x(t)$ as the parameter and denote $x(s) = x(t(s))$, then $x'(s) = \frac{\mathrm{d}x}{\mathrm{d}s}$ is a spacelike unit tangent vector field of the curve $x(s)$. Now we choose the vector field $y(s)$, the spacelike normal space $\mathbf{V}^{n-1}$ of the curve $x(s)$ such that they satisfy the following conditions:

$$<x(s), y(s)> = 1, \quad <x(s), x(s)> = <y(s), y(s)> = <x'(s), y(s)> = 0,$$

$$\mathbf{V}^{n-1} = \{\mathrm{span}_{\mathbf{R}}\{x, y, x'\}\}^{\perp}, \quad \mathrm{span}_{\mathbf{R}}\{x, y, x', V^{n-1}\} = \mathbf{E}_1^{n+2}.$$

Therefore, choosing the vector fields $\alpha_2(s)$, $\alpha_3(s)$, $\ldots$, $\alpha_n(s) \in \mathbf{V}^{n-1}$ suitably, we have the following Frenet formulas

$$(1.1) \qquad \begin{cases} x'(s) = \alpha_1(s) \\ \alpha_1'(s) = \kappa_1(s)x(s) - y(s) + \tau_1(s)\alpha_2(s) \\ \alpha_2'(s) = \kappa_2(s)x(s) - \tau_1(s)\alpha_1(s) + \tau_2(s)\alpha_3(s) \\ \alpha_3'(s) = \kappa_3(s)x(s) - \tau_2(s)\alpha_2(s) + \tau_3(s)\alpha_4(s) \\ \quad \cdots\cdots \\ \alpha_i'(s) = \kappa_i(s)x(s) - \tau_{i-1}(s)\alpha_{i-1}(s) + \tau_i(s)\alpha_{i+1}(s) \\ \quad \cdots\cdots \\ \alpha_{n-1}'(s) = \kappa_{n-1}(s)x(s) - \tau_{n-2}(s)\alpha_{n-2}(s) + \tau_{n-1}(s)\alpha_n(s) \\ \alpha_n'(s) = \kappa_n(s)x(s) - \tau_{n-1}(s)\alpha_{n-1}(s) \\ y'(s) = -\sum_i^n \kappa_i(s)\alpha_i(s), \end{cases}$$

where $\alpha_2(s), \ldots, \alpha_n(s) \in \mathbf{V}^{n-1}$, $<\alpha_i, \alpha_j> = \delta_{ij}$, $i, j = 1, 2, \ldots, n$. The functions $\kappa_1(s), \ldots,$ $\kappa_n(s)$, $\tau_1(s), \ldots, \tau_{n-1}(s)$ are called *cone curvature functions* of the curve $x(s)$. The frame

$$\{x(s), y(s), \alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)\}$$

is called the asymptotic orthonormal frame on $\mathbf{E}_1^{n+2}$ along the curve $x(s)$ in $\mathbf{Q}^{n+1}$.

Let $x(s)$ be a curve in $\mathbf{Q}^{n+1}$. From $<x(s), x(s)> = 0$ we know that $\tilde{x}(s) = e^{\sigma(s)}x(s)$ is also a curve in $\mathbf{Q}^{n+1}$, where $\sigma(s) : \mathbf{I} \to \mathbf{R}$ is a differentiable function. Since $<\mathrm{d}\tilde{x}, \mathrm{d}\tilde{x}> = e^{2\sigma}<\mathrm{d}x, \mathrm{d}x>$, we know that the curves $\tilde{x}(s)$ and $x(s)$ are conformal. Denoting by $\tilde{s}$ the arc length of the curve $\tilde{x}(s)$, we have

$$\tilde{x}' = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}} = (\frac{\mathrm{d}(e^\sigma x)}{\mathrm{d}s})(\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}}) = (\sigma' e^\sigma x + e^\sigma x')(\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}}). \tag{1.2}$$

The relations $<\tilde{x}', \tilde{x}'> = <x', x'> = 1$ and $<x, x'> = 0$ yield that

$$\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}} = e^{-\sigma}. \tag{1.3}$$

We may choose $\tilde{s}$ such that $\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}} > 0$. By a direct calculation, we get

$$\tilde{x}' = \sigma' x + x', \tag{1.4}$$

$$\tilde{x}'' = (\sigma'' x + \sigma' x' + x'')(\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}}) = e^{-\sigma}(\sigma'' x + \sigma' x' + x''), \tag{1.5}$$

$$<\tilde{x}'', \tilde{x}''> = e^{-2\sigma}({\sigma'}^2 - 2\sigma'' + <x'', x''>), \tag{1.6}$$

$$\tilde{x}''' = e^{-\sigma}[e^{-\sigma}(\sigma'' x + \sigma' x' + x'')]' = e^{-2\sigma}[(\sigma''' - \sigma'\sigma'')x + (2\sigma'' - {\sigma'}^2)x' + x'''], \tag{1.7}$$

and

$$\begin{aligned} <\tilde{x}''', \tilde{x}'''> &= e^{-4\sigma}[({\sigma'}^2 - 2\sigma'')^2 + 2(2\sigma'' - {\sigma'}^2)<x', x'''> + <x''', x'''>] \\ &= e^{-4\sigma}[({\sigma'}^2 - 2\sigma'')^2 + 2({\sigma'}^2 - 2\sigma'')<x'', x''> + <x''', x'''>]. \end{aligned} \tag{1.8}$$

From these relations we obtain

$$<\tilde{x}''', \tilde{x}'''> - <\tilde{x}'', \tilde{x}''>^2 = e^{-4\sigma}(<x''', x'''> - <x'', x''>^2). \tag{1.9}$$

From (1.1), by a direct calculation, we get

$$<x''', x'''> - <x'', x''>^2 = (\kappa_2 + \tau_1')^2 + (\kappa_3 + \tau_1\tau_2)^2 + \sum_{i=4}^n \kappa_i^2.$$

Also by a direct calculation we know that $\kappa_2 + \tau_1' \neq 0$ for the Frenet curve $x(s)$ and $n \geq 2$. Therefore, when $n \geq 2$, for any Frenet curves $x(s)$ and $\tilde{x}(\tilde{s})$, we have

$$\sqrt{<\tilde{x}''', \tilde{x}'''> - <\tilde{x}'', \tilde{x}''>^2}<\mathrm{d}\tilde{x}, \mathrm{d}\tilde{x}> = \sqrt{<x''', x'''> - <x'', x''>^2}<\mathrm{d}x, \mathrm{d}x>. \tag{1.10}$$

Theorem 1.1. *Let $x : \mathbf{I} \to \mathbf{Q}^{n+1} \subset \mathbf{E}_1^{n+2}$ $(n \geq 2)$ be a Frenet curve in $\mathbf{Q}^{n+1}$ with the induced arc length parameter s. Then*

$$\sqrt{<x''', x'''> - <x'', x''>^2}<\mathrm{d}x, \mathrm{d}x>$$

is a conformal invariant.

For an arbitrary parameter t of the curve $x(s(t)) = x(t)$, we have

$$\begin{aligned}
x' &= (\frac{\mathrm{d}x}{\mathrm{d}t})(\frac{\mathrm{d}t}{\mathrm{d}s}) = \dot{x}(\frac{\mathrm{d}t}{\mathrm{d}s}),\\
x'' &= \ddot{x}(\frac{\mathrm{d}t}{\mathrm{d}s})^2 - <\dot{x}, \ddot{x}>\dot{x}(\frac{\mathrm{d}t}{\mathrm{d}s})^4,\\
x''' &= \dddot{x}(\frac{\mathrm{d}t}{\mathrm{d}s})^3 - 3<\dot{x}, \ddot{x}>\ddot{x}(\frac{\mathrm{d}t}{\mathrm{d}s})^5 + \dot{x}(\frac{\mathrm{d}^3t}{\mathrm{d}s^3}),\\
(\frac{\mathrm{d}t}{\mathrm{d}s}) &= (<\dot{x}, \dot{x}>)^{-\frac{1}{2}},\\
(\frac{\mathrm{d}^2t}{\mathrm{d}s^2}) &= -<\dot{x}, \ddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^4,\\
(\frac{\mathrm{d}^3t}{\mathrm{d}s^3}) &= -<\ddot{x}, \ddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^5 + 4<\dot{x}, \ddot{x}>^2(\frac{\mathrm{d}t}{\mathrm{d}s})^7 - <\dot{x}, \dddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^5.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(1.11)\qquad <x''', x'''> - <x'', x''>^2 &= <\dddot{x}, \dddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^6 + 9<\dot{x}, \ddot{x}>^2<\ddot{x}, \ddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^{10}\\
&\quad - 6<\dot{x}, \ddot{x}><\ddot{x}, \dddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^8 - <\dot{x}, \dddot{x}>^2(\frac{\mathrm{d}t}{\mathrm{d}s})^8\\
&\quad + 6<\dot{x}, \ddot{x}>^2<\dot{x}, \dddot{x}>(\frac{\mathrm{d}t}{\mathrm{d}s})^{10}\\
&\quad - 9<\dot{x}, \ddot{x}>^4(\frac{\mathrm{d}t}{\mathrm{d}s})^{12}\\
&= \left\| \frac{<\dot{x}, \dot{x}>\dddot{x} - 3<\dot{x}, \ddot{x}>\ddot{x} - <\dot{x}, \dddot{x}>\dot{x}}{<\dot{x}, \dot{x}>^{\frac{5}{2}}} \right\|^2 - 9\frac{<\dot{x}, \ddot{x}>^4}{<\dot{x}, \dot{x}>^6}.
\end{aligned}$$

Therefore we have

Theorem 1.2. *Let $x : \mathbf{I} \to \mathbf{Q}^{n+1} \subset \mathbf{E}_1^{n+2}$ $(n \geq 2)$ be a Frenet curve with an arbitrary parameter t. Then*

$$(1.12)\qquad \sqrt{\left\| \frac{<\dot{x}, \dot{x}>\dddot{x} - 3<\dot{x}, \ddot{x}>\ddot{x} - <\dot{x}, \dddot{x}>\dot{x}}{<\dot{x}, \dot{x}>^{\frac{5}{2}}} \right\|^2 - 9\frac{<\dot{x}, \ddot{x}>^4}{<\dot{x}, \dot{x}>^6}}<\mathrm{d}x, \mathrm{d}x>$$

is a conformal invariant.

2. Curves in the lightlike cone $\mathbf{Q}^2$

In this section, we consider curves in the lightlike cone $\mathbf{Q}^2$. For a curve $x : \mathbf{I} \to \mathbf{Q}^2 \subset \mathbf{E}^3_1$, from (1.1) we have

$$\left\{ \begin{array}{l} x'(s) = \alpha(s) \\ \alpha'(s) = \kappa(s)x(s) - y(s) \\ y'(s) = -\kappa(s)\alpha(s), \end{array} \right. \tag{2.1}$$

where s is an arc length parameter of the curve and $x(s)$, $y(s)$, $\alpha(s)$ satisfy

$$<x,x> = <y,y> = <x,\alpha> = <y,\alpha> = 0, \quad <x,y> = <\alpha,\alpha> = 1.$$

For an arbitrary parameter t of the curve $x(t)$, the cone curvature function κ is given by

$$\kappa(t) = \frac{<\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}>^2 - <\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}><\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t}>}{2<\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t}>^5}. \tag{2.2}$$

Since $<y(s), y(s)> = 0$, we can take $\tilde{x}(s) = y(s)$. Then $\tilde{x}(s)$ is also a curve in $\mathbf{Q}^2$. We call $\tilde{x}(s)$ *the associated curve* of the curve $x(s)$. We denote by $\tilde{s}$ the arc length of the curve $\tilde{x}(s)$. Then, marking by "tilde" the quantities of the curve $\tilde{x}(s)$, we have

$$\tilde{x}' = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}} = \tilde{\alpha} = (\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}})y' = (\frac{\mathrm{d}s}{\mathrm{d}\tilde{s}})(-\kappa\alpha). \tag{2.3}$$

Let $\tilde{\alpha} = \varepsilon\alpha = \pm\alpha$, then

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = -\varepsilon\kappa. \tag{2.4}$$

From (2.1), (2.3), (2.4) and $\tilde{x} = y$ we get

$$\left\{ \begin{array}{l} \varepsilon(\kappa x - y) = (\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s})(\tilde{\kappa}\tilde{x} - \tilde{y}) \\ \qquad\qquad\quad = -\varepsilon\kappa(\tilde{\kappa}y - \tilde{y}). \end{array} \right. \tag{2.5}$$

Therefore, if $\kappa \neq 0$, we have

$$\tilde{x} = y, \quad \tilde{\alpha} = \varepsilon\alpha, \quad \tilde{y} = x, \quad \tilde{\kappa} = \frac{1}{\kappa}. \tag{2.6}$$

Changing the parameter if necessary, we may assume that $\varepsilon = 1$. From (2.4) and (2.6) we have

$$\left\{ \begin{array}{l} \mathrm{d}\tilde{s} = -\kappa \mathrm{d}s \\ \mathrm{d}\tilde{\kappa} = -\kappa^{-2}\mathrm{d}\kappa. \end{array} \right.$$

Then

$$\tilde{\kappa}^{-\frac{3}{2}}\frac{\mathrm{d}\tilde{\kappa}}{\mathrm{d}\tilde{s}} = \kappa^{\frac{3}{2}}\kappa^{-3}\frac{\mathrm{d}\kappa}{\mathrm{d}s} = \kappa^{-\frac{3}{2}}\frac{\mathrm{d}\kappa}{\mathrm{d}s} \tag{2.7}$$

for any curve $x(s)$ and its associated curve $\tilde{x}(s)$ in $\mathbf{Q}^2$ with $\kappa \neq 0$.

If the curves $x(s)$ and $\tilde{x}(\tilde{s})$ have the same cone curvature functions $\kappa(s)$ and $\tilde{\kappa}(\tilde{s})$, assume that $\kappa^{-\frac{3}{2}}\frac{\mathrm{d}\kappa}{\mathrm{d}s} = -2a = \text{constant} \neq 0$. Then we obtain that $\kappa = (as+c)^{-2}$, where $c \in \mathbf{R}$.

Theorem 2.1. *Let $x : \mathbf{I} \to \mathbf{Q}^2$ be a curve with the cone curvature function $\kappa = cs^{-2}$ for some nonzero constant c and arc length parameter s, then $x(s)$ can be written as follows*:

$$x(s) = a_1 s + a_2 s^{(1+\sqrt{1+2c})} + a_3 s^{(1-\sqrt{1+2c})} \tag{i}$$

for $c \neq -\frac{1}{2}$, where a_1, a_2, $a_3 \in \mathbf{E}_1^3$;

$$x(s) = a_1 s + a_2 s \log s + a_3 s \log^2 s \tag{ii}$$

for $c = -\frac{1}{2}$, where a_1, a_2, $a_3 \in \mathbf{E}_1^3$.

Proof. From (2.1) and $\kappa = cs^{-2}$ we have

$$s^3 x'''(s) - 2csx'(s) + 2cx(s) = 0.$$

Solving this equation, we obtain

$$\begin{aligned} x(s) &= a_1 s + a_2 s^{(1+\sqrt{1+2c})} + a_3 s^{(1-\sqrt{1+2c})}, \quad c \neq -\frac{1}{2}, \\ x(s) &= a_1 s + a_2 s \log s + a_3 s \log^2 s, \quad c = -\frac{1}{2}, \end{aligned}$$

a_1, a_2, $a_3 \in \mathbf{E}_1^3$. $\square$

Using a pseudo orthonormal frame on the curve $x(s)$ and denoting by $\bar{\kappa}$, $\bar{\tau}$, β and γ the curvature, the torsion, the principal normal and the binormal of the curve $x(s)$ in $\mathbf{E}_1^3$, we have

$$\begin{aligned} x' &= \alpha, \\ \alpha' &= \kappa x - y = \bar{\kappa}\beta, \end{aligned}$$

where $\kappa \neq 0$, $<\beta,\beta> = \varepsilon = \pm 1$, $<\alpha,\beta> = 0$, $\varepsilon\kappa<0$. Then we get

$$\bar{\kappa} = \sqrt{-2\varepsilon\kappa}, \quad \varepsilon\beta = \frac{\kappa x - y}{\sqrt{-2\varepsilon\kappa}};$$

$$\begin{aligned}\varepsilon(-\bar{\kappa}\alpha+\bar{\tau}\gamma) &= (\frac{\kappa}{\sqrt{-2\varepsilon\kappa}}x-\frac{1}{\sqrt{-2\varepsilon\kappa}}y)' \\ &= (\frac{-\varepsilon\sqrt{-2\varepsilon\kappa}}{2}x-\frac{1}{\sqrt{-2\varepsilon\kappa}}y)' \\ &= \frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}x-\frac{\varepsilon\sqrt{-2\varepsilon\kappa}}{2}x'-\frac{\varepsilon\kappa'}{\sqrt{(-2\varepsilon\kappa)^3}}y-\frac{1}{\sqrt{-2\varepsilon\kappa}}y'.\end{aligned}$$

Therefore

$$\varepsilon\bar{\tau}\gamma=\frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}x-\frac{\varepsilon\kappa'}{\sqrt{(-2\varepsilon\kappa)^3}}y=\frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}(x+\frac{1}{\kappa}y).$$

Choosing $\gamma=\sqrt{\frac{-\varepsilon\kappa}{2}}(x+\frac{1}{\kappa}y)$, we obtain

$$\bar{\kappa}=\sqrt{-2\varepsilon\kappa},\quad \bar{\tau}=-\frac{1}{2}(\frac{\kappa'}{\kappa})=-(\frac{\bar{\kappa}'}{\bar{\kappa}}). \tag{2.8}$$

Theorem 2.2. *The curve $x:\mathbf{I}\to\mathbf{Q}^2$ is a planar curve if and only if the cone curvature function κ of the curve $x(s)$ is constant.*

Proof. This is immediate from (2.8). □

Theorem 2.3. *Let $x:\mathbf{I}\to\mathbf{Q}^2$ be a curve with cone curvature function $\kappa=$ constant and arc length parameter s, then $x(s)$ can be written as follows:*

(i) $$x(s)=a_1s^2+a_2s+a_3,\quad a_1,a_2,a_3\in\mathbf{E}^3_1;$$

for $\kappa=0$, the curve is a parabola;

(ii) $$x(s)=a_1\sinh(\sqrt{2\kappa})s+a_2\cosh(\sqrt{2\kappa})s+a_3;$$

for $\kappa>0$, the curve is a hyperbola;

(iii) $$x(s)=a_1\sin(\sqrt{-2\kappa})s+a_2\cos(\sqrt{-2\kappa})s+a_3;$$

for $\kappa<0$, the curve is an ellipse.

Proof. From (2.1) and $\kappa=$constant, we have

$$x'''(s)=2\kappa x'(s).$$

Solving this equation, we obtain that

$$\begin{aligned}x(s)&=a_1s^2+a_2s+a_3,\quad \kappa=0,\\ x(s)&=a_1\sinh(\sqrt{2\kappa})s+a_2\cosh(\sqrt{2\kappa})s+a_3,\quad \kappa>0,\\ x(s)&=a_1\sin(\sqrt{-2\kappa})s+a_2\cos(\sqrt{-2\kappa})s+a_3,\quad \kappa<0,\end{aligned}$$

where a_1, a_2, $a_3 \in \mathbf{E}_1^3$. $\square$

Theorem 2.4. *Let $x : \mathbf{I} \to \mathbf{Q}^2$ be a curve whose tangent vector field intersects a fixed vector in $\mathbf{E}_1^3$ at a constant angle, then $x(s)$ can be written as one of the curves given by Theorem 2.1 and Theorem 2.3.*

Proof. Let a be a constant vector in $\mathbf{E}_1^3$ and $<\alpha, a> = l$, l is a constant. If $l = 0$, by $<\alpha, a> = 0$ we have

$$0 = <\alpha', a> = <\kappa x - y, a> = \kappa <x, a> - <y, a>,$$
$$0 = \kappa' <x, a> + \kappa <x', a> - <y', a> = \kappa' <x, a>.$$

Therefore we get that $\kappa =$ constant. The curves are given by Theorem 2.3. Let $l \neq 0$ and $\kappa \neq$ constant. From $<\alpha, a> = l$ we get

$$\kappa' <x, a> + \kappa <x', a> - <y', a> = \kappa' <x, a> + 2\kappa l = 0 \tag{2.9}$$

and

$$\kappa'' <x, a> + \kappa' l + 2\kappa' l = 0. \tag{2.10}$$

By (2.9) and (2.10) we obtain

$$2\kappa\kappa'' - 3{\kappa'}^2 = 0.$$

Therefore $\kappa = c_1(s + c_2)^{-2}$, where $c_1 \neq 0$ and c_2 are constants. The curves are given by Theorem 2.1. $\square$

Theorem 2.5. *Let $x : \mathbf{I} \to \mathbf{Q}^2$ be a curve in $\mathbf{Q}^2$ with arc length parameter s. Then $x(s)$ is a Bertrand curve in $\mathbf{E}_1^3$ if and only if the cone curvature function $\kappa(s)$ of the curve $x(s)$ is equal to*:

(i) *a nonzero constant*; *or*
(ii) *the function $-\varepsilon(a + e^{bs})^{-2}$, where a and b are constant, $ab \neq 0$, $\varepsilon = \pm 1$.*

Proof. Let $x : \mathbf{I} \to \mathbf{Q}^2$ be a Bertrand curve in $\mathbf{E}_1^3$. Then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of the curve $x(s)$ satisfy

$$l\bar{\kappa} + \mu\bar{\tau} = 1 \tag{2.11}$$

for some constants $l \neq 0$ and μ ([13]). The constant $\mu = 0$ yields $\bar{\kappa} =$ constant $\neq 0$. From (2.8) we know that the cone curvature function κ of the curve $x(s)$ is also a nonzero constant.

Now assume that $\mu \neq 0$, $\kappa \neq$ constant. Then (2.8) and (2.11) yield that

$$l\bar{\kappa} - \mu(\frac{\bar{\kappa}'}{\bar{\kappa}}) = 1.$$

Solving this equation we get

$$\bar{\kappa} = \sqrt{-2\varepsilon\kappa} = (l + e^{\frac{s}{\mu}+c})^{-1},$$

where c is a constant. By a parameter transformation, the cone curvature function $\kappa(s)$ of the curve $x(s)$ can be written as in (ii) of Theorem 2.5.

Let $x : \mathbf{I} \to \mathbf{Q}^2$ be a curve with cone curvature function $\kappa = \text{constant} \neq 0$. From (2.8) we know that $\bar{\kappa}$ is also a nonzero constant. Hence the curve is a Bertrand curve.

Now we consider the curve $x : \mathbf{I} \to \mathbf{Q}^2$ with the cone curvature function $\kappa(s) = -\varepsilon(a + e^{bs})^{-2}$. Then from (2.8) and a direct calculation we know that there are constants $l = \frac{a}{\sqrt{2}}$, $\mu = \frac{1}{b}$, such that

$$l\bar{\kappa} + \mu\bar{\tau} = 1.$$

Then the curve $x(s)$ is a Bertrand curve in $\mathbf{E}_1^3$. □

3. Curves in the lightlike cone $\mathbf{Q}^3$

In this section, we consider curves in the lightlike cone $\mathbf{Q}^3$. Let $x : \mathbf{I} \to \mathbf{Q}^3 \subset \mathbf{E}_1^4$ be a curve in $\mathbf{Q}^3$, then from (1.1) we have

$$\begin{cases} x'(s) = \alpha_1(s) \\ \alpha_1'(s) = \kappa_1(s)x(s) - y(s) + \tau(s)\alpha_2(s) \\ \alpha_2'(s) = \kappa_2(s)x(s) - \tau(s)\alpha_1(s) \\ y'(s) = -\kappa_1(s)\alpha_1(s) - \kappa_2(s)\alpha_2(s). \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $x : \mathbf{I} \to \mathbf{Q}^3 \subset \mathbf{E}_1^4$ be a curve with arc length parameter s so that the cone curvature functions κ_1, κ_2 and τ are constant, then it can be written as follows*:

(i) $$x(s) = a_1 s^2 + a_2 s + a_3 + a_4$$

for $\kappa_2 = \tau = \kappa_1 = 0$;

(ii) $$x(s) = a_1 \sinh(\sqrt{2\kappa_1})s + a_2 \cosh(\sqrt{2\kappa_1})s + a_3 + a_4$$

for $\kappa_2 = \tau = 0,\ \kappa_1 > 0$;

(iii) $$x(s) = a_1 \sin(\sqrt{-2\kappa_1})s + a_2 \cos(\sqrt{-2\kappa_1})s + a_3 + a_4$$

for $\kappa_2 = \tau = 0,\ \kappa_1 < 0$;

(iv) $$x(s) = a_1 \sinh ls + a_2 \cosh ls + a_3 \sin \mu s + a_4 \cos \mu s$$

for $\kappa_2 \neq 0$;

where a_1, a_2, a_3, $\alpha_4 \in \mathbf{E}_1^4$, $l > 0$, $\mu > 0$, $\pm l$ *and* $\pm\sqrt{-1}\mu$ *are the real roots and the imaginary roots of the equation:* $t^4 - (2\kappa_1 - \tau^2)t^2 - \kappa_2^2 = 0$.

Proof. From (3.1) and that κ_1, κ_2 and τ are constant, we have

$$\begin{aligned} x'' &= \kappa_1 x - y + \tau\alpha_2, \\ x''' &= \kappa_2\tau x + (2\kappa_1 - \tau^2)x' + \kappa_2\alpha_2, \\ x'''' &= \kappa_2^2 x + (2\kappa_1 - \tau^2)x'', \end{aligned}$$

that is,

$$\left\{ \begin{aligned} x''' - (2\kappa_1 - \tau^2)x' &= 0, \quad \text{for } \kappa_2 = 0 \\ x'''' - (2\kappa_1 - \tau^2)x'' - \kappa_2^2 x &= 0, \quad \text{for } \kappa_2 \neq 0. \end{aligned} \right. \tag{3.2}$$

From (3.1) we know that $\kappa_2 = 0$ yields that $\tau = 0$. Then $x(s)$ is a curve in $\mathbf{Q}^2$ with $\kappa_1 =$ constant. Therefore, solving (3.2), we obtain that
(1) when $\kappa_2 = \tau = 0$:

$$\begin{aligned} x(s) &= a_1 s^2 + a_2 s + a_3 + a_4, \quad 2\kappa_1 = 0; \\ x(s) &= a_1 \sinh(\sqrt{2\kappa_1})s + a_2 \cosh(\sqrt{2\kappa_1})s + a_3 + a_4, \quad 2\kappa_1 > 0; \\ x(s) &= a_1 \sin(\sqrt{-2\kappa_1})s + a_2 \cos(\sqrt{-2\kappa_1})s + a_3 + a_4, \quad 2\kappa_1 < 0; \end{aligned}$$

(2) when $\kappa_2 \neq 0$:

$$x(s) = a_1 \sinh ls + a_2 \cosh ls + a_3 \sin \mu s + a_4 \cos \mu s;$$

where a_1, a_2, a_3, $\alpha_4 \in \mathbf{E}_1^4$ and

$$l^2 = \frac{\sqrt{(2\kappa_1 - \tau^2)^2 + 4\kappa_2^2} + (2\kappa_1 - \tau^2)}{2}$$

$$\mu^2 = \frac{\sqrt{(2\kappa_1 - \tau^2)^2 + 4\kappa_2^2} - (2\kappa_1 - \tau^2)}{2}.$$

This completes the proof of the theorem. □

Remark. By a transformation in $\mathbf{E}_1^4$, the curve (i) of Theorem 3.1 can be written as

$$x(s) = a(1, s, \frac{1}{2}s^2, \frac{1}{2}s^2 + 1), \quad \text{where } 0 \neq a \in \mathbf{R};$$

the curve (ii) of Theorem 3.1 can be written as

$$x(s) = (a, b, \sqrt{a^2 + b^2}\sinh(\sqrt{2\kappa_1})s, \sqrt{a^2 + b^2}\cosh(\sqrt{2\kappa_1})s), \quad \text{where } a^2 + b^2 \neq 0;$$

the curve (iii) of Theorem 3.1 can be written as

$$x(s) = (\sqrt{a^2-b^2}\sin(\sqrt{-2\kappa_1})s, \sqrt{a^2-b^2}\cos(\sqrt{-2\kappa_1})s, b, a), \quad \text{where } a^2-b^2>0;$$

the curve (iv) of Theorem 3.1 can be written as

$$x(s) = a(\sin(\mu s), \cos(\mu s), \sinh(ls), \cosh(ls)), \quad \text{where } 0 \neq a \in \mathbf{R}.$$

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