Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 2, 595-605.

Relative Isodiametric Inequalities

Dedicated to the memory of Bernulf Weißbach

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Abstract. We consider subdivisions of bounded convex sets G in two subsets E and $G \setminus E$. We obtain several inequalities comparing the relative volume 1) with the minimum relative diameter and 2) with the maximum relative diameter. In the second case we obtain the best upper estimate only for subdivisions determined by straight lines in planar sets.

MSC 2000: 52A40, 52A10
Keywords: Relative geometric inequalities, relative isodiametric inequalities, area, volume, diameter

1. Introduction

Relative geometric inequalities are inequalities in which we compare relative geometric measures, i.e., functionals that give geometric information on the subsets (E and $G\setminus E$) determined by the subdivision of an original set G.

The first relative geometric inequalities that appeared in the literature were the so called relative isoperimetric inequalities. These inequalities compare the relative area with the relative perimeter:

If G is an open convex set in the Euclidean plane $\mathbb{R}^2$ and E is a subset of G with non-empty interior and rectifiable boundary such that both E and its complement $G\setminus E$ are connected, we define:

- the *relative boundary* of E as $\partial E \cap G$,

- the *relative perimeter* of E, $P(E,G)$, as the length of the relative boundary, and
- the *relative area* of E as:

$$A(E,G) = \min\{A(E), A(G\backslash E)\}.$$

With the above assumptions we say that a relative isoperimetric inequality is an inequality of the type:

$$\frac{A(E,G)}{P(E,G)^{\alpha}} \leq C,$$

where C and α are positive numbers.

This problem can be extended to higher dimensions in the following way: Let G be an open and convex set of $\mathbb{R}^n$. A relative isoperimetric inequality holds if there exist two positive constants α and C such that

$$\frac{\min\{V(E), V(G \setminus E)\}}{P(E,G)^{\alpha}} \leq C, \tag{1}$$

where $V(E)$ is the volume of E and $P(E,G)$ is in this case an appropriate $(n-1)$-dimensional measure of the relative boundary of E.

Many results have been obtained about relative isoperimetric inequalities (see for instance [2],[9]).

There are also results comparing the relative perimeter with other geometric magnitudes different from the relative area. For results comparing the relative perimeter with the relative diameter and the relative inradius see [4].

In this paper we want to study *relative isodiametric inequalities*, in which we compare the relative volume with the relative diameters of a subset of a bounded convex set. First we need to define these notions:

Let $G \subset \mathbb{R}^n$ be a bounded open convex set and $E \subset G$ a subset of G such that E as well as $G \setminus E$ are connected and have non-empty interior. Let $D(.)$ be the diameter functional.

(i) The *relative volume* is the minimum of the volume of E and the volume of its complement,

$$V(E,G) = \min\{V(E), V(G \setminus E)\},$$

(ii) the *minimum relative diameter* is the minimum of the diameter of E and the diameter of its complement,

$$d_m(E,G) = \min\{D(E), D(G \setminus E)\},$$

and

(iii) the *maximum relative diameter* is the maximum of the diameter of E and the diameter of its complement,

$$d_M(E,G) = \max\{D(E), D(G \setminus E)\}.$$

Relative isodiametric inequalities are those that give either an upper or a lower estimate of the ratios:

$$\frac{V(E,G)}{d_m(E,G)^n} \text{ or } \frac{V(E,G)}{d_M(E,G)^n}.$$

We compare the relative volume with the nth-power of the relative diameters (n is the dimension of the ambient space) because as this ratio is invariant under dilatations, we obtain geometric information about the subdivision: The estimates do not depend on the size of the sets but only on their shapes.

We are interested not only in obtaining relative isodiametric inequalities, but also in determining those sets (called maximizers or minimizers) for which the equality sign is attained.

For some of the cases that we are going to consider, we need the classical (absolute) isodiametric inequality:

Among all convex bodies in the n-dimensional Euclidean space with fixed volume, the ball has the smallest diameter:

$$\frac{V}{\omega_n} \le \left(\frac{D}{2}\right)^n,$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the n-dimensional unit ball and Γ is the Euler Gamma function ([1]).

In the plane we can write this inequality as follows:

Let C be a planar closed convex domain with area A and diameter D. Then,

$$A \le \frac{1}{4}\pi D^2$$

where the inequality holds if and only if C is a circle ([1]).

2. Relative isodiametric inequalities concerning the relative volume and the minimum relative diameter of a subset of a bounded convex set

The aim of this section is to maximize and minimize the ratio of the relative volume and the nth-power of the minimum relative diameter of a subset E of G.

We begin with minimizing the given ratio, and in this case we have to consider two different cases. First, we are going to study general subdivisions of G and later we are going to consider the special case in which the subdivision is obtained by a hyperplane cut.

Proposition 1. *Let G be an open bounded convex set and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_m(E,G)^n} \ge 0$$

is the optimal lower estimate.

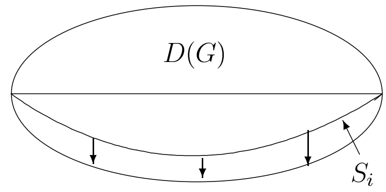


Figure 1

Proof. For any bounded convex set G we can consider a sequence of hypersurfaces $\{S_i\}_{i=1}^{\infty}$ as close as we want to the boundary of G, such that both ends of a diameter of G belong to S_i $\forall i \in \mathbb{N}$, and the regions E_i bounded by S_i have volume decreasing to zero.

Then, we conclude that:

$$\lim_{i\to\infty} \frac{V(E_i,G)}{d_m(E_i,G)^n} = \frac{0}{D(G)^n} = 0. \qquad \square$$

Proposition 2. *Let G be an open bounded convex set and E a subset of G obtained by a hyperplane cut. Then,*

$$\frac{V(E,G)}{d_m(E,G)^n} \geq 0 \tag{2}$$

is the optimal lower estimate.

Proof. We can distinguish two cases:

Case 1: G is strictly convex:

Let x be a regular point in the boundary of G, ∂G. Let $\nu(x)$ be the outward unit normal vector of ∂G at x. Let $T_x\partial G$ be the tangent space of ∂G at x.

Let Π_0 be a hyperplane parallel to $T_x\partial G$ intersecting G and let E_0 be the intersection of G with the upper half-space determined by Π_0 and containing $\nu(x)$. We can choose Π_0 and E_0 so that $V(E_0) \leq \frac{V(G)}{2}$.

Applying the Schwarz symmetrization with respect to the line determined by $\nu(x)$ to E_0 we obtain a new set E_0' of revolution with the same volume as E_0; the image of the relative boundary of E_0 under Schwarz symmetrization is an $(n-1)$-dimensional ball with radius r_0. This symmetrization does not increase the diameter, so $D(E_0) \geq D(E_0')$.

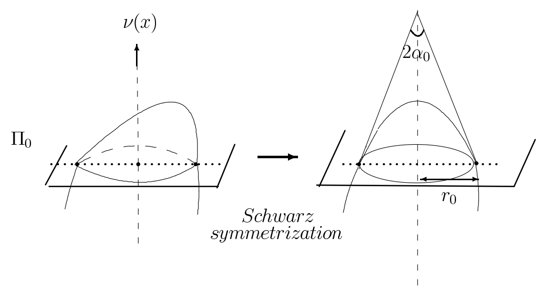


Figure 2

Then,

$$\frac{V(E_0,G)}{d_m(E_0,G)^n} = \frac{V(E_0)}{D(E_0)^n} \leq \frac{V(E_0')}{(2r_0)^n}. \tag{3}$$

We can choose a sequence of hyperplanes $\{\Pi_i\}_{i=1}^{\infty}$ parallel to Π_0 and such that the intersection of E_0' with the half-spaces determined by Π_i and containing $\nu(x)$ be a contractive sequence of convex sets $\{E_i'\}_{i=1}^{\infty}$ so that $\lim_{i\to\infty} V(E_i') = 0$ and such that the supporting hyperplanes at the $(n-2)$-spheres determined by the intersection of Π_i with the boundary of E_0' provide a sequence of cones $\{C_i\}_{i=1}^{\infty}$ whose angles at their vertices are $2\alpha_i$. We can choose this sequence so that $\lim_{i\to\infty}\alpha_i = \frac{\pi}{2}$.

Let r_i be the radius of the $(n-1)$-ball which is the basis of the cone C_i. For each of these cones C_i

$$\frac{V(E_i')}{(2r_i)^n} \leq \frac{V(C_i)}{(2r_i)^n} = \frac{\frac{1}{n}V(B(r_i)^{n-1})r\cot\alpha_i}{(2r_i)^n} = \frac{\pi^{\frac{(n-1)}{2}}\cot\alpha_i}{n2^n\Gamma(\frac{n-1}{2}+1)}. \tag{4}$$

Taking the limit when $i\to\infty$,

$$\lim_{i\to\infty}\frac{\pi^{\frac{(n-1)}{2}}\cot\alpha_i}{n2^n\Gamma(\frac{n-1}{2}+1)} = 0. \tag{5}$$

Then, by (3),(4) and (5) we obtain that

$$0 \leq \lim_{i\to\infty}\frac{V(E_i,G)}{d_m(E_i,G)^n} \leq 0,$$

and so the inequality (2) is best possible.

Case 2: G is not strictly convex:
If G is not strictly convex there exists a straight line segment t in the boundary of G (Figure 3). If we consider a sequence of hyperplanes Π_i parallel to t so that the volume of the subsets E_i determined by the intersections of G with Π_i decreases to zero, then,

$$\lim_{i\to\infty}\frac{V(E_i,G)}{d_m(E_i,G)^n} \leq \frac{0}{(\text{length}(t))^n} = 0. \qquad \square$$

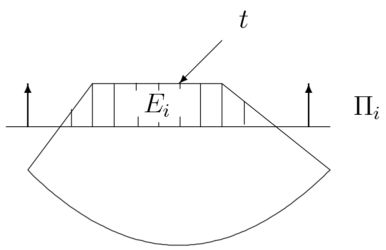


Figure 3

The following proposition provides an upper bound for the ratio of the relative volume and the nth-power of the minimum relative diameter.

Proposition 3. *Let G be an open bounded convex set and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_m(E,G)^n} \leq \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})2^n}\,.$$

Proof. Let $B^n(r)$ be a ball with radius r contained in G (Figure 4) such that:

$$V(B^n(r)) \leq \frac{V(G)}{2} \Longleftrightarrow r \leq \left(\frac{V(G)}{2\omega_n}\right)^{1/n}.$$

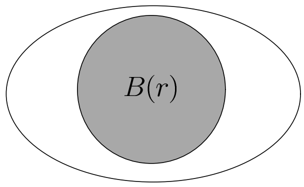


Figure 4

As a consequence of the isodiametric inequality ([1]), for any subset E of G:

$$\frac{V(E)}{D(E)^n} \leq \frac{V(B^n(r))}{(2r)^n},$$

where equality holds if and only if $E = B^n(r)$. Then,

$$\frac{V(E,G)}{d_m(E,G)^n} \leq \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})2^n}. \qquad \square$$

3. Relative isodiametric inequalities concerning the relative volume and the maximum relative diameter of a subset of a bounded convex set

The aim of this section is to maximize and minimize the ratio of the relative volume and the maximum relative diameter of a subset E of G.

Proposition 4. *Let G be an open bounded convex set in $\mathbb{R}^n$ and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_M(E,G)^n} \geq 0$$

is the best possible lower estimate.

Proof. Let G be an open bounded convex set in the Euclidean space. We can suppose without loss of generality that $0 \in G$. Let us consider the sequence $\{E_i\}_{i=2}^{\infty}$ where each $E_i = \frac{1}{i}G$.

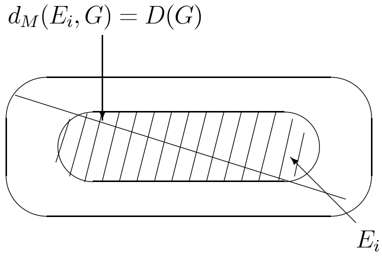


Figure 5

If we compute the ratio of the relative volume and the nth-power of the maximum relative diameter, the limit of this ratio is 0 when $i \to \infty$. In fact, the relative volume decreases to 0 and $d_M(E_i, G) = D(G \setminus E_i)$ is the diameter of G for all i:

$$\lim_{i\to\infty} \frac{V(E_i, G)}{d_M(E_i, G)^n} = \frac{0}{D(G)^n} = 0. \qquad \square$$

Proposition 5. *Let G be an open bounded convex set and E a subset of G obtained by a hyperplane cut. Then,*

$$\frac{V(E, G)}{d_M(E, G)^n} \geq 0$$

is the best possible lower estimate.

Proof. We consider a sequence of hyperplanes $\{\Pi_i\}_{i=1}^{\infty}$ parallel to a diameter of G, and such that the volume of the subsets E_i determined by the intersections of G with Π_i decreases to zero (Figure 6). Then we can see that:

$$\lim_{i\to\infty} \frac{V(E_i, G)}{d_M(E_i, G)^n} = \lim_{i\to\infty} \frac{V(E_i, G)}{D(G)^n} = 0. \qquad \square$$

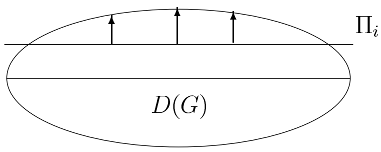


Figure 6

The problem of maximizing the ratio of the relative area and the maximum relative diameter is attached to the so-called "fencing problems". Such problems consider dividing a region into two parts of equal volume (area) by a hypersurface (continuous curve); such hypersurfaces are called fences.

We are going to state two results about fencing problems that we shall use in the isodiametric inequality which we are going to prove.

Theorem A. ([7]) *Let K be a planar, bounded, and centrally symmetric convex set with area A. For every subdivision of K into two parts $(E, K \setminus E)$ of equal area by a continuous curve, the maximum relative diameter satisfies the following inequality:*

$$d_M(E, K) \geq C\sqrt{A},$$

where $C \cong 0.8815$.

The equality is attained for the subdivision of the shaded optimal body described in the Figure 7.

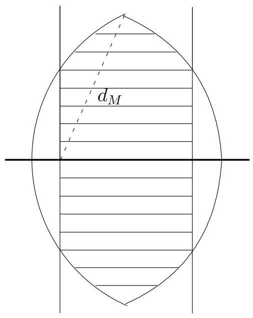


Figure 7

Theorem B. ([8]) *In the class of planar convex sets with area A the minimum of d_M, with respect to straight line cuts, is attained on a centrally symmetric convex set.*

Proposition 6. *Let G be a planar bounded convex set and E a subset of G obtained by a straight line cut, then:*

$$\frac{A(E, G)}{d_M(E, G)^2} \leq 1.2869... \, .$$

Proof. Let l be the straight line dividing G into two regions E and $G \setminus E$, and suppose that $A(E) \leq A(G \setminus E)$. Let us consider two different cases: 1) $d_M(E, G) = D(G \setminus E)$ and 2) $d_M(E, G) = D(E)$.

1) If $d_M(E, G) = D(G \setminus E)$ and $A(E) < A(G \setminus E)$, we translate l till another line l' determining a new division of G into two other regions E' and $G \setminus E'$ in such a way that one of the two following situations occurs:

1.1) $A(E') = A(G \setminus E')$ and $d_M(E', G) = D(G \setminus E')$.

Then $A(E', G) = A(E') > A(E) = A(E, G)$ and $d_M(E', G) < d_M(E, G)$ (Figure 8) and E' determines a fencing problem. Hence,

$$\frac{A(E, G)}{d_M(E, G)^2} \leq \frac{A(E')}{d_M(E', G)^2} \leq 1.2869...,$$

where the last inequality follows from theorems A and B.

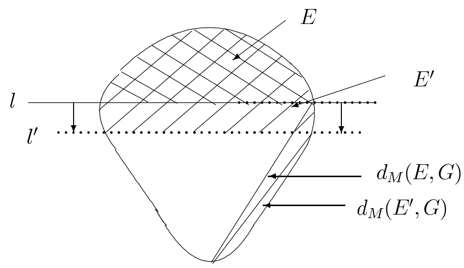
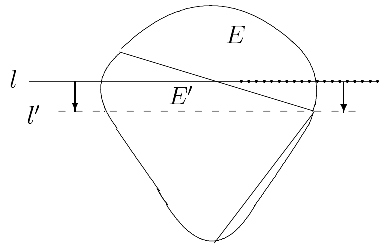


Figure 8



$d_M(E',G) = D(E') = D(G \setminus E')$

Figure 9

1.2) $A(E') < A(G \setminus E')$ and $d_M(E',G) = D(E') = D(G \setminus E')$.

In this case we have $A(E,G) = A(E) \leq A(E') = A(E',G)$ and $d_M(E,G) \geq d_M(E',G)$, so

$$\frac{A(E,G)}{d_M(E,G)^2} = \frac{A(E)}{D(G \setminus E)^2} \leq \frac{A(E')}{D(E')^2}, \tag{6}$$

but $D(E') = D(G \setminus E')$, so we have also

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(G \setminus E')}{D(G \setminus E')^2}. \tag{7}$$

We consider now the intersection points P and Q of l' with ∂G.

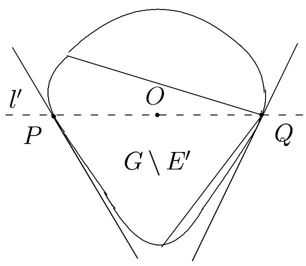


Figure 10

Let E'' be either E' or $G \setminus E'$, where the supporting lines at P and Q make internal angles whose sum is smaller or equal than π. Let us consider the symmetric set of E'' with respect to the middle point O of the segment PQ. Let this set be E'''.

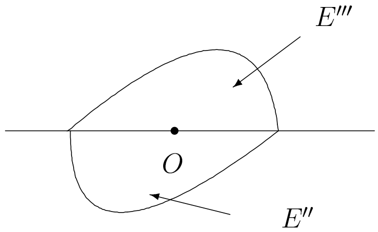


Figure 11

$E'' \cup E'''$ is a centrally symmetric convex set where the area is $2A(E'')$ (Figure 11). It is easy to see that

$$D(E''') = D(E'') = d_M(E', G) = D(E') = D(G \setminus E').$$

Then, from inequalities (6) and (7),

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E'')}{D(E'')^2} \leq 1.2869...,$$

where the last inequality holds as a consequence of Theorem A.

2) Suppose that $d_M(E, G) = D(E)$.

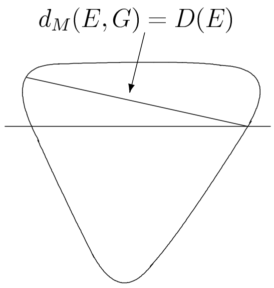


Figure 12

Then,

$$\frac{A(E,G)}{d_M(E,G)^2} = \frac{A(E)}{D(E)^2},$$

and also

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(G \setminus E)}{D(G \setminus E)^2}.$$

Let E' be either E or $G \setminus E$, where the supporting lines at P and Q realize internal angles whose sum is smaller or equal than π. By a similar argument to that used in the case 1.2 we conclude that

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E')}{D(E')^2} \leq 1.2869... \qquad \square$$

The n-dimensional version of this proposition seems to be very challenging.

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Received December 3, 2003