Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (20024), No. 2, 517-525.

Large Equilateral Triangles Inscribed in the Unit Disk of a Minkowski Plane

Dedicated to the memory of Bernulf Weißbach

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Abstract. We consider a generalization of the problem of finding large equilateral triangles in the unit disks of a normed plane; we accept asymmetric unit disks, i.e. we deal with Minkowski planes. We prove that the unit disk of an arbitrary Minkowski plane permits an inscribed equilateral triangle whose oriented side lengths are at least 1.25. We conjecture that this estimate can be improved up to 1.5. An example shows that a better value is impossible.

MSC 2000: 52A10, 52A21

1. Introduction.

Let z be an interior point of a convex body $C \subset E^n$. We define the *oriented distance* $\delta_{C,z}(a,b)$ of points a and b as $|ab|/|zw|$, where w is the boundary point of C such that the vectors $\overrightarrow{zw}$ and $\overrightarrow{ab}$ have the same orientation. We also use the term *the oriented length* of the segment ab. Here the symbol $|rs|$ denotes the Euclidean distance of the points r and s. We call C the *unit ball* with *origin* z. If $n = 2$ then we call C the *unit disk*. The n-dimensional space with this oriented distance is called the *Minkowski space*, and when $n = 2$ it is called the *Minkowski plane*. Many properties of Minkowski spaces are presented in [5]. When C is centrally symmetric with z as its center, we get an n-dimensional normed space which is also called finite-dimensional Banach space or Minkowski-Banach space.

The most well-known problem concerning Minkowski planes asks if the self-circumference of an arbitrary Minkowski unit disk is at least 6, see [2], [4], [6]. It leads to a variety of questions about "large" inscribed polygons in the unit disk. For an overview of such questions in two-dimensional normed spaces see [10] and [11]. The aim of the present paper

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is to consider a particular question of this nature: how large an equilateral triangle can be inscribed in an arbitrary Minkowski unit disk?

2. Equilateral triangles

Assume that we go in the boundary of a triangle according to the positive (respectively: negative) orientation and that the oriented lengths of the sides are equal. We call such a triangle an *equilateral triangle in the positive* (respectively: *negative*) *orientation*. Pay attention that an equilateral triangle according to an orientation usually is not equilateral after changing its orientation into the opposite. An equilateral triangle in the positive orientation is also called an *equilateral triangle* for simplicity.

We shall describe a *procedure for finding all equilateral triangles* $v_1v_2v_3$ of oriented edge length 1. Assume that the first vertex, v_1, is given. Draw the unit disk C_1 with the origin v_1 (i.e. the translate of the unit disk C by the vector $\overrightarrow{zv_1}$).

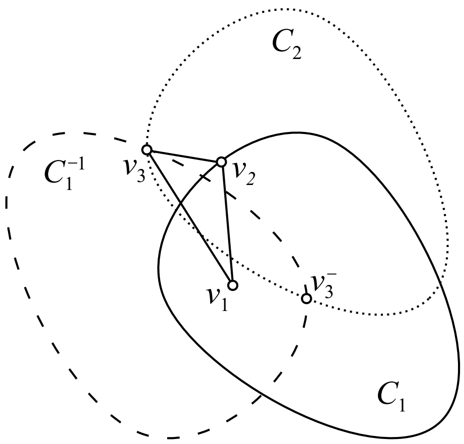

Figure 1

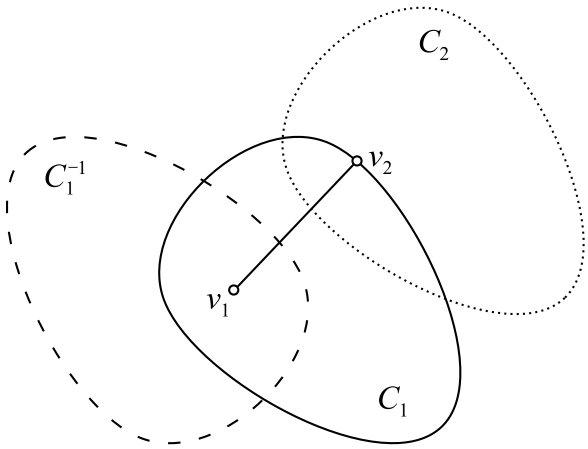

Figure 2

Clearly, all the possible positions of v_2 are on the boundary of C_1, see Figure 1 and Figure 2. In order to find all the required positions of v_3, we draw the unit disk C_2 whose origin is v_2 and the homothetic image C_1^{-1} of C_1 of ratio -1 with homothety center at v_1. Then, $v_1v_2v_3$ is equilateral if and only if $v_3 \in \mathrm{bd}(C_2) \cap \mathrm{bd}(C_1^{-1})$. Beside an equilateral triangle $v_1v_2v_3$, this way we also obtain an equilateral triangle $v_1v_2v_3^-$ of negative orientation (see Figure 1). It may also happen that for a given v_1, v_2 there is no equilateral triangle whose side is v_1v_2 (see Figure 2).

Example. Let s_1, s_2, s_3, s_4 be the four vertices of a square in C and let the origin z be a point on the segment s_1s_3 such that $2|zs_1| = |zs_3|$.

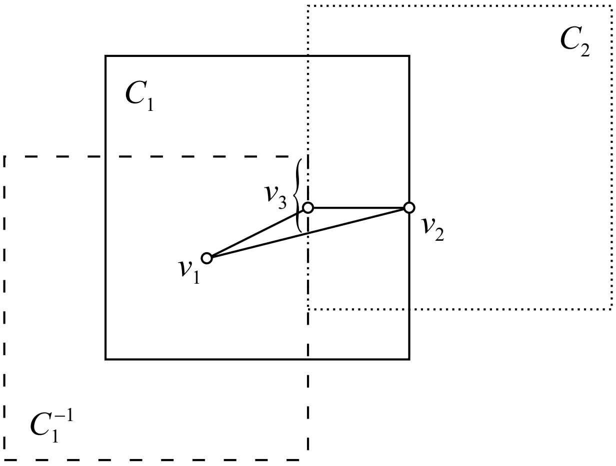

Figure 3

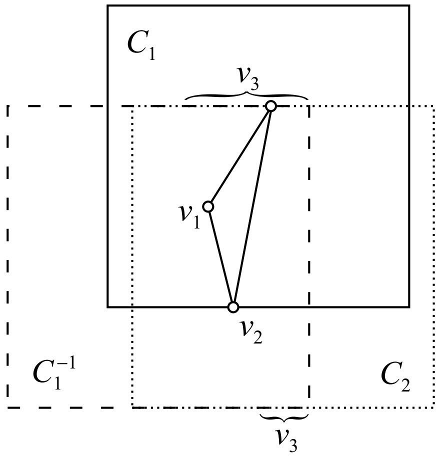

Figure 4

In Figures 3 and 4 we can see how to find v_3 for given points v_1 and v_2 such that $v_1v_2v_3$ is a positively (or negatively) oriented equilateral triangle. Then v_3 must be on the boundaries of both C_2 and C_1^{-1}, i.e. on the segments which are marked by two kinds of broken lines simultaneously. Notice that if in Figure 3 we take v_2 at the right upper corner of C_1, then we get only one possible position of v_3 and that $v_1v_2v_3$ degenerates to a segment: we have $v_3 \in v_1v_2$. On the other hand, if v_2 is at the lower right corner (see Figure 4), v_3 can be taken at every point of $\mathrm{bd}(C_2)$.

We call a direction *admissible* if for the boundary points y_1 and y_2 of C such that $\overrightarrow{y_1y_2}$ is of this direction and such that $z \in y_1y_2$, we have $|y_1z|/|zy_2| \geq \frac{1}{2}$.

Claim 1. *For any two given points v_1 and v_2 there exists a point v_3 such that $v_1v_2v_3$ is an equilateral triangle if and only if $\overrightarrow{v_1v_2}$ is an admissible direction.*

Proof. Without loss of generality we may assume that $\delta_{C,z}(v_1, v_2) = 1$. Thus $v_2 \in \mathrm{bd}(C_1)$. Here and below we stay with the notation introduced when describing the procedure of finding equilateral triangles. Moreover, denote by v the Euclidean midpoint of the segment v_1v_2.

Assume that the direction $\overrightarrow{v_1v_2}$ is admissible. Then $v \in C_2$ and $v \in C_1^{-1}$. This and the convexity of C_1 and C_2 implies $\mathrm{bd}(C_2) \cap \mathrm{bd}(C_1^{-1}) \neq \emptyset$. This guarantees the existence of v_3 such that $v_1v_2v_3$ is equilateral.

Assume that $\overrightarrow{v_1v_2}$ is not admissible. Then $\delta_{C,z}(v_2, v_1) > 2$. Since $\delta_{C,z}(a, b)$ satisfies the triangle inequality, as established by Minkowski [9], there is no v_3 such that $\delta_{C,z}(v_2, v_3) = 1$ and $\delta_{C,z}(v_3, v_1) = 1$ simultaneously. Thus for every v_3 the triangle $v_1v_2v_3$ is not equilateral. □

From this proof we can also see that *if C is strictly convex, then v_3 is unique.* Of course, if C is not strictly convex it may happen that v_3 is not unique like in Example.

Assume that a planar convex body C is given. Claim 1 leads to the question how to determine the set of those positions of the origin z in the interior of C such that in the obtained Minkowski plane for every v_1 and v_2 we can construct an equilateral triangle

$v_1v_2v_3$ for every v_1 and v_2. Claim 2 provides the answer; its proof, which is similar to that of Claim 1, is left to the reader.

Claim 2. *Denote by C_H the set of those points s such that the image of C under the homothety with center s and ratio $-\frac{1}{2}$ is contained in C. There exists an equilateral triangle $v_1v_2v_3$ for every given v_1 and v_2 if and only if $z \in C_H$.*

3. Equilateral triangles inscribed in the unit disk

Papers [1], [3], [7], and [8] consider large equilateral polygons, and particular triangles, inscribed in the unit disk of a two-dimensional normed space. Currently the best estimate is given in [8], where it is shown that every such unit disk permits an inscribed equilateral triangle whose side length is at least $1 + \frac{1}{3}\sqrt{3} \approx 1.577$. A conjecture from [3] and [7] is that in every such unit disk we can inscribe an equilateral triangle of sides length at least $1+\frac{1}{2}\sqrt{2} \approx 1.707$. This value cannot be improved when the unit disk is the regular octagon (see [3] and [7]). We intend to estimate how large equilateral triangle can be inscribed in an arbitrary Minkowski unit disk C.

Conjecture. *The unit disk of every Minkowski plane permits an inscribed equilateral triangle with oriented side length at least* 1.5.

For the Minkowski plane whose unit disk is considered in the Example the value 1.5 cannot be enlarged. This follows from the shape of the equilateral triangles $v_1v_2v_3$ established in the Example. Just observe that in the Example we always have $\overrightarrow{v_1v_2} = \overrightarrow{zw}$ or $\overrightarrow{v_2v_3} = \overrightarrow{zw}$ or $\overrightarrow{v_3v_1} = \overrightarrow{zw}$ for a point $w \in s_2s_3 \cup s_3s_4$. Hence if a positive homothetic image of $v_1v_2v_3$ is inscribed in the unit disk (square) C, then the ratio of this homothety is at most 1.5. This implies that every equilateral triangle inscribed in the unit disk C has sides of oriented length at most 1.5.

The question how large equilateral triangles can be inscribed in the unit disk of a Minkowski plane does not seem to be easy. Some of the reasons are that only equilateral triangles with sides of admissible directions exist, that their homothetic copies cannot always be inscribed in the unit disk, and that for a given boundary point of C sometimes in C we cannot inscribe an equilateral triangle with a vertex at this point (for instance, the last two situations may happen in the space considered in the Example).

When we say that a point q is *on the opposite side of a line $L \subset E^2$* than a point $p \notin L$, we mean that the segment pq has non-empty intersection with L. We say that q is *strictly on the opposite side of L* if, additionally, $q \notin L$.

Theorem. *In every unit disk C one can inscribe an equilateral triangle of oriented side length at least* 1.25.

Proof. According to the earlier introduced notation, the symbol z denotes the origin. Since $z \in \operatorname{int}(C)$, by continuity arguments we see that there are points $b, e \in \operatorname{bd}(C)$ such that z is the midpoint of be (that is $\overrightarrow{bz} = \overrightarrow{ze}$). Denote by g the midpoint of bz, and by h the midpoint of ze. Again, continuity arguments guarantee that there are points $a, c, d, f \in \operatorname{bd}(C)$ such that g is the midpoint of ac and that h is the midpoint of df.

Assume that the triangle abz has positive orientation (in this order), and that points a and f are in one half-plane bounded by the straight line through b and z.

Observe that the choice of the points g, a, b and c implies that $abcz$ is a parallelogram with center g. This and $\overrightarrow{bz} = \overrightarrow{ze}$ imply that the triangle abz is equilateral. Analogously, $defz$ is a parallelogram with center h and the triangle dez is equilateral. Both triangles have sides of unit oriented length. Our intention is to show that a homothetic image of ratio at least 1.25 of at least one of those triangles is inscribed in C.

We make the assumption that

$$\sphericalangle bza \leq \sphericalangle ezd, \tag{1}$$

and we intend to show that a homothetic image of the triangle abz of a ratio at least 1.25 is inscribed in C. (If $\sphericalangle ezd \leq \sphericalangle bza$, similar argument leads to the conclusion that a homothetic image of the triangle dez of ratio at least 1.25 is inscribed in C.)

Assume that $\sphericalangle bza = 90°$ and that $|ab| = 2$. This is possible because we can apply a convenient affine transformation. It matters here the obvious property that the oriented length does not change under affine transformations. Some of the arguments of this proof are simpler when we consider the Cartesian coordinate system Oxy in which $z = (0,0)$, $a = (-1,-1)$ and $b = (1,-1)$. As a consequence, $c = (2,0)$, $e = (-1,1)$, $g = (\frac{1}{2}, -\frac{1}{2})$ and $h = (-\frac{1}{2}, \frac{1}{2})$. By *horizontal lines* we mean lines parallel to the Ox-axis. By the *level* of such a line we mean the y-coordinate of points of this line. We say that a point p_2 is *above the level* (respectively: *below the level*) of a point p_1 provided the y-coordinate of p_2 is greater (respectively: smaller) than the y-coordinate of p_1.

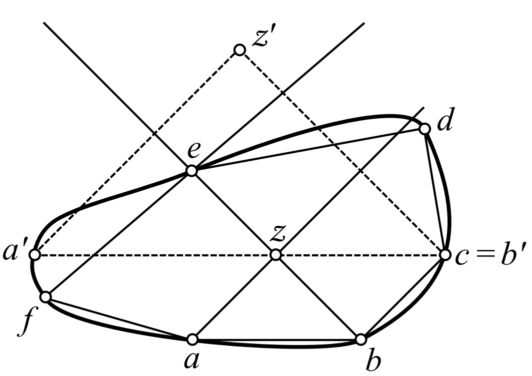

Figure 5

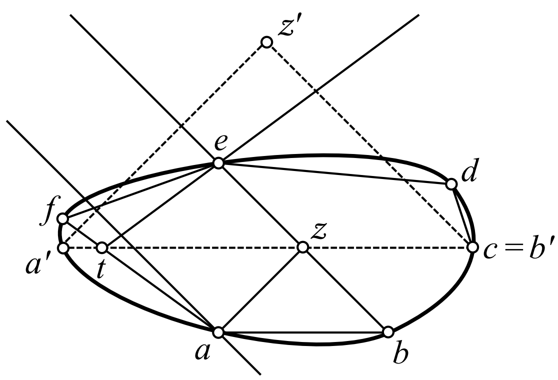

Figure 6

Consider the positive homothetic copy $a'b'z'$ of the triangle abz such that $b' = c$ and that $a' \in \mathrm{bd}\,(C)$. Since $abcz$ is a parallelogram, $z \in a'b'$.

Since $b' = c$ is on the opposite side of the straight line containing bz than f, from the definition of the triangle $a'b'z'$ we see that

$$z' \text{ is on the opposite side of the straight line containing } bz \text{ than } f. \tag{2}$$

Below in Cases 1 and 2 we show that $z' \notin \mathrm{int}\,(C)$.

Case 1, when the segments ac and fd are disjoint.

Subcase 1.1, when f is not over the level of z.

In this subcase the segment $a'z$ intersects the segment ef (see Figure 5). From (1) and since the points b, z, e are collinear, we conclude that the straight line through a and z intersects the segment de. Those facts and the fact that $defz$ is a parallelogram imply that z' is on the opposite side of the line containing ef than z. Remember that (2) holds true. Hence e belongs to the triangle fzz'. If we suppose that $z' \in \operatorname{int}(C)$, then by the convexity of C all the points of the triangle fzz' beside f are in $\operatorname{int}(C)$. Since in Subcase 1.1 we have $e \neq f$, we see that $e \in \operatorname{int}(C)$. This contradicts $e \in \operatorname{bd}(C)$. Thus $z' \notin \operatorname{int}(C)$.

Subcase 1.2, when f is over the level of z.

In this subcase the segments $a'z$ and af intersect at a point t.

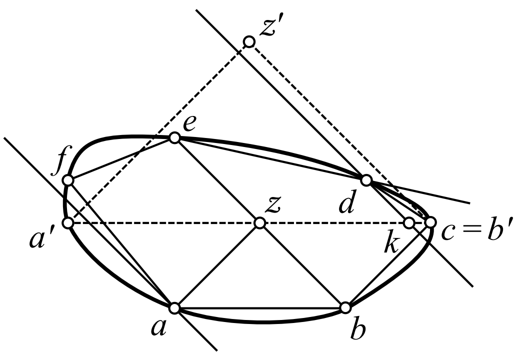

Figure 7

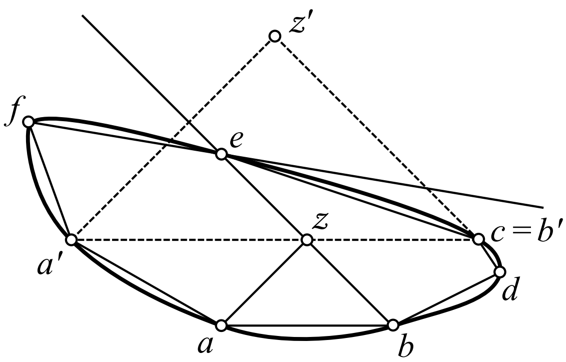

Figure 8

Subcase 1.2.1, when f is on the opposite side of the line through a parallel to bz than z.

Since $a'z$ and af intersect, a' is on the opposite side of the straight line containing et than z (see Figure 6). This and $|tz| \geq 2$ imply that z' is on the opposite side of the line containing et than z. From this fact, statement (2), and the convexity of C we conclude that $z' \notin \operatorname{int}(C)$ (we provide an argument similar to Subcase 1.1).

Subcase 1.2.2, when f is not on the opposite side of the line through a parallel to bz than z.

Since d is symmetric to f with respect to h, we see that d and z are on one side of the straight line through c parallel to bz (see Figure 7). This and $b' = c$ imply that z' is on the opposite side than z of the line through d parallel to bz. Moreover, since $a'b'z'$ is a homothetic enlarged copy of abz and since the sum of levels of a and e is 0, we see that z' is over the level of e.

As a consequence of this and of the earlier statement that z' is on the opposite side than z of the line through d parallel to bz, we conclude that z' is also on the opposite side of the straight line containing the segment de. Denote by k the point of intersection of cz with the straight line through d parallel to bz. From the convexity of C we obtain that $z' \notin \operatorname{int}(C)$ (we provide an argument like in Subcase 1.1; this time instead of points f, e, and z we take points e, d, and k, respectively).

Case 2, when segments df and ac intersect.

Remember that (2) holds true. Of course, z' is over the level of e. From those facts and from the fact that in Case 2 point f is not below the level of e we conclude that z' is on the opposite side of the line containing ef than z (see Figure 8). This, (2) and the convexity of C imply that $z' \notin \operatorname{int}(C)$ (we use an argument like in Subcase 1.1).

As promised, we have established that $z' \notin \operatorname{int}(C)$.

Below by H we denote the hexagon $abcdef$ (respectively, $abdcef$) when the assumption of Case 1 (respectively, of Case 2) is fulfilled.

We intersect C by all horizontal lines which are between the lines containing segments ab and $a'b'$. Every such line (possibly besides the line containing ab) intersects the boundary of C in two points a'' and b''. Here the notation is chosen in order to have $\overrightarrow{ab} = \overrightarrow{a''b''}$. The intersection of C and the line containing ab can be a boundary segment a^+b^+ of C. Then in this level we take all segments $a''b''$, where $a'' = \lambda a^+ + (1-\lambda)a$ and $b'' = \lambda b^+ + (1-\lambda)b$ for $0 \leq \lambda \leq 1$.

For every segment $a''b''$ defined above denote by z'' the point such that $a''b''z''$ is a positive homothetic image of abz. When $a''b'' = ab$, we have $z'' = z \in \operatorname{int}(C)$, and when $a''b'' = a'b'$, we have $z'' = z' \notin \operatorname{int}(C)$ (we have established this in Cases 1 and 2). Consequently, by continuity arguments, there is a position of the triangle $a''b''z''$, denoted below by $a_C b_C z_C$, which is inscribed in C.

In particular, in H we can inscribe a homothetic image $a_H b_H z_H$ of abz in H. From (1) we see that $a_H \in af$. Obviously, in Case 1 we have $b_H \in bc$. We leave it for the reader to show that in Case 2 we have $b_H \in bd$ (hint: since d is symmetric to f with respect to h and since H is convex, d is on the opposite side of the line through z parallel to ce than z).

Denote by r_H and r_C the homothety ratios of the homotheties which transform abz into $a_H b_H z_H$ and into $a_C b_C z_C$, respectively. We intend to show that $r_H \leq r_C$.

Consider the translation τ such that $\tau(a_H)\tau(b_H)$ is a segment of the form $a''b''$ considered earlier. Clearly, $\tau(a_H), \tau(b_H) \in \operatorname{bd}(C)$. Moreover, observe that $\tau(z_H)$ and z_H are on the same sides of the lines containing ez and az. We can see that $\tau(z_H)$ is in the quadrangle $zuz_H w$ (the order of these points agrees with the orientation) enclosed by four lines: the lines containing ez and az, and the lines through z_H parallel to af and to bc in Case 1 (respectively bd in Case 2). Of course, $z, z_H, w \in C$. It is clear that in Case 2 also $u \in C$. In Case 1, from the fact that $defz$ is a parallelogram it follows that lines containing af and de intersect at a point v such that $e \in dv$ and $f \in av$. It implies that also in Case 1 we have $u \in C$. Thus the quadrangle $zuz_H w$ is contained in C. As a consequence, $\tau(z_H) \in C$. We can see that $\tau(a_H)\tau(b_H)\tau(z_H)$ is contained in C. Hence the level of points a_C, b_C is not below the level of points $\tau(a_H)$, $\tau(b_H)$. So $|a_C b_C| \geq |\tau(a_H)\tau(b_H)| = |a_H b_H|$. As a consequence, $r_H \leq r_C$.

We can see that in order to show that the triangle $a_C b_C z_C$ is an image of the triangle abz under a homothety with a ratio at least 1.25 it is sufficient to show that we can inscribe a homothetic copy $a_H b_H z_H$ of the triangle abz of ratio 1.25 in the hexagon H.

Denote by d_0 the point of intersection of the segment dz with the segment ec in Case 1 (respectively, with bc in Case 2). Moreover, denote by f_0 the new specific position of f which is a consequence of exchanging the position of d into d_0; from the fact that $defz$ (and thus also d_0ef_0z) is a parallelogram it follows that f_0 is in the segment ef. The case when we deal with the pentagon $abcef_0 = abcd_0ef_0$ in Case 1 (respectively, with $abcef_0 = abd_0cef_0$ in Case 2) is a special case of our general situation when we have the hexagon $abcdef$ (respectively, $abdcef$). We see that the pentagon $abcef_0$ is a subset of the hexagon H. As a consequence, it is sufficient to show that a homothetic copy of the triangle abz of a ratio at least 1.25 is inscribed in the pentagon $abcef_0$.

Again let us reduce the preceding situation in Case 1 to a more specific case. We mean the case when $\sphericalangle bza = \sphericalangle ezd_0$. Since d_0ef_0z is a parallelogram, we see that the resulting specific position f_1 of f_0 is in the segment f_0z. As a consequence, the pentagon $P = abcef_1$ is a subset of the pentagon $abcef_0$ (see Figure 9).

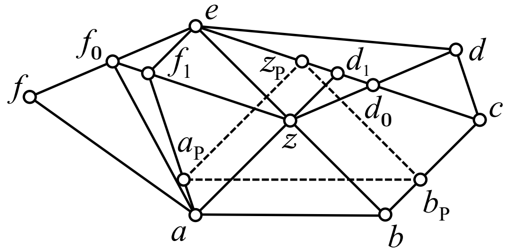

Figure 9

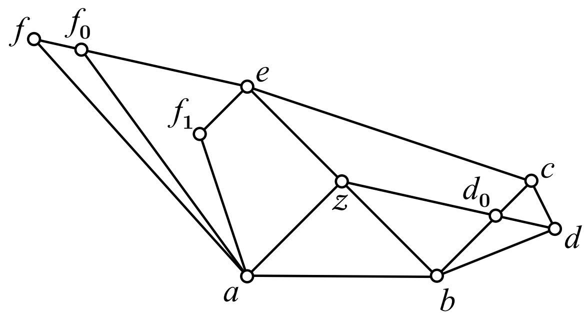

Figure 10

Since ef_1 is parallel to az and since f_1z is parallel to ec, we easily obtain that $f_1 = (-\frac{3}{2}, \frac{1}{2})$. It is easy to show that also in Case 2 the pentagon $P = abcef_1$ is a subset of the pentagon $abcef_0$ (see Figure 10). We conclude that it is sufficient to show that a homothetic copy of the triangle abz of ratio 1.25 is inscribed in $abcef_1$.

The intersection point $p = (-\frac{1}{2}, -\frac{5}{2})$ of the straight lines containing af_1 and bc is the center of the homothety which transforms the triangle abz into the required triangle $a_Pb_Pz_P$ with $z_P \in ec$. A simple calculation shows that $z_P = (\frac{1}{8}, \frac{5}{8})$ and, as a consequence, that the ratio of this homothety is 1.25. □

The example of the pentagon P considered at the end of the proof of the Theorem shows that by the approach of this proof we cannot improve the obtained estimate 1.25.

It is clear that the unit disk of the Euclidean plane permits an inscribed equilateral triangle of edge lengths $\sqrt{3}$. Claim 3 shows that if we move the origin z from the center of the disk, we still can inscribe an equilateral triangle of sides of at least the same oriented length of sides.

Claim 3. *Consider the Minkowski plane whose unit disk is the usual circular disk with z at any interior point of the disk. The unit disk permits an inscribed equilateral triangle whose oriented side lengths are at least $\sqrt{3}$.*

Proof. Consider the disk $x^2 + y^2 \leq 1$ with $z = (0, \lambda)$, where $\lambda \in [0, 1)$. Put $o = (0, 0)$. Construct an equilateral triangle according to the procedure given in the proof of the Theorem using the same notation.

Of course, $b = (\sqrt{1-\lambda^2}, \lambda)$ and thus $g = (\frac{1}{2}\sqrt{1-\lambda^2}, \lambda)$. The equation of the straight line through g perpendicular to the segment og is $y = -\frac{\sqrt{1-\lambda^2}}{2\lambda}x + \frac{3\lambda^2+1}{4\lambda}$. The intersection of this line with the circle $x^2 + y^2 = 1$ gives point $a = \Big(\frac{1}{2}\sqrt{1-\lambda^2} + \lambda\sqrt{\frac{3-3\lambda^2}{3\lambda^2+1}},\ \lambda -$

$\frac{1}{2}(1-\lambda^2)\sqrt{\frac{3}{3\lambda^2+1}}\Big)$. Having the coordinates of the vertices of the triangle abz we can find the perpendicular bisectors of the segments az and bz, and their point of intersection which is the center r_0 of the circle circumscribed about abz. We omit an easy but time consuming calculation which leads to the conclusion that $r_0 = \left(\frac{1}{2}\sqrt{1-\lambda^2},\ \lambda - \frac{1}{6}\sqrt{9\lambda^2+3}\right)$. Thus we easily get $|r_0 z| = \frac{1}{3}\sqrt{3}$. This implies that a positive homothetic copy of abz of ratio $\sqrt{3}$ is inscribed in the circle $x^2+y^2=1$. Obviously, this homothetic copy is an equilateral triangle of sides of oriented length $\sqrt{3}$. □

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Received August 1, 2003