Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 2, 607-614.

Illumination and Visibility Problems in Terms of Closure Operators

Dedicated in memory to Bernulf Weißbach

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Abstract. The notions of illumination and visibility of convex bodies are well-known in combinatorial and computational geometry. We study closure operators which control illumination and visibility, respectively.

MSC 2000: 52A20 (primary), 52A01, 52A37 (secondary)
Keywords: Convex bodies, illumination, visibility, closure operators, Hadwiger's covering problem

1. Introduction

The notion of *illumination* of a convex body K in $\mathbb{R}^n$ can be considered as the starting point for various interesting problems and as a useful tool in *combinatorial geometry* (cf. Chapter VI and VII in [2] and the survey [7]). This notion was independently introduced by V. Boltyanski [1] and H. Hadwiger [5], motivated by the famous (and still unsettled) question how many light sources are needed to illuminate the whole boundary of K. The conjectured upper bound 2^n, attained when K is an n-dimensional parallelotope, is verified only for special types of convex bodies, e.g. for all centrally symmetric convex bodies in $\mathbb{R}^3$ (as shown by M. Lassak in [6]), or for convex bodies in $\mathbb{R}^n$ whose supporting cones at singular points are not too acute (proved by B. Weißbach in [13]). Variations of the Boltyanski-Hadwiger notion of illumination were considered in [8] and [14].

A modified type of illumination, called *visibility*, was introduced by F. A. Valentine [12], see also [4], [3], and again the survey [7]. Visibility problems play an essential role in computational geometry, e.g. in connection with art gallery questions and the watchman route problem, cf. [10] and [11]. Valentine's notion of visibility can be seen as a weakening of the above mentioned notion of illumination; so already $n+1$ points are sufficient to *see* the whole boundary of a convex body K in $\mathbb{R}^n$.

In this paper we want to present a unified approach to both these notions in terms of *closure operators*, which themselves are tailored to convex sets. In case of visibility, the corresponding closure operator is even new.

In Section 2 (Theorem 2.4) we show already basic connections between the notions of illumination and visibility in view of closure operators, and in Section 3 the new closure operator of the visibility notion is introduced and discussed. Moreover, we prove that those convex sets K, which are compact and for which $E = \mathbb{R}^n \setminus K$ is connected, may be characterized by the fact that either of the two operators controlling illumination and visibility, respectively, is a closure operator, cf. Theorem 3.4. The final Section 4 presents analogous investigations for parallel illumination of convex bodies, surprisingly showing that in this case we cannot have an approach via closure operators (at least not in a canonical way).

One motivation for our investigations is our hope that an extended framework of tools and methods might help to attack more successfully certain longstanding open problems from the combinatorial geometry of convex bodies, such as the Boltyanski-Hadwiger illumination problem (also known as the Gohberg-Markus-Hadwiger covering problem asking for the minimum number of smaller homothets of K sufficient to cover that convex body).

2. Illumination described in terms of a closure operator

For two points $a, b \in \mathbb{R}^n$ with $a \neq b$, $n \geq 1$, let $\overline{ab} := \{a + \lambda \cdot (b-a) \,|\, 0 \leq \lambda \leq 1\}$ denote the closed *line segment* between a and b, while $s(a,b) := \{a + \lambda \cdot (b-a) \,|\, \lambda \geq 0\}$ is written for the *ray* with initial point a and passing through b.

Assume that K is a *convex body*, i.e., a compact, convex set with interior points in $\mathbb{R}^n$. Due to H. Hadwiger [5] we say that a boundary point x of K is *illuminated* by a point $z \in \mathbb{R}^n \setminus K$ if $(s(z,x) \setminus \overline{zx}) \cap \operatorname{int} K \neq \emptyset$. (We note that the analogous notion for parallel illumination was introduced by V. Boltyanski [1], see also [2], Chapter VI, for a historical survey, and [7].) A point set $A \subseteq \mathbb{R}^n \setminus K$ is said to *illuminate* a subset B of the *boundary* ∂K of K if every $x \in B$ is illuminated by at least one element $a \in A$. If A illuminates the whole boundary ∂K of K, we say also that A illuminates the body K.

To describe illumination more generally, namely in terms of closure operators, we define for every subset M of $\mathbb{R}^n$ and its complement $E = \mathbb{R}^n \setminus M$ the operator $\sigma_M : \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$\sigma_M(A) := A \cup \{b \in E \setminus A \,|\, \exists\, a \in A : \overline{ab} \cap M = \emptyset, s(a,b) \cap M \neq \emptyset\}\,. \tag{2.1}$$

Thus $\sigma_M(A) \setminus A$ consists of those points of $E \setminus A$ which lie in front of M relative to some point $a \in A$.

We have the following proposition which was already proved in [9], see Theorem 2.5 there.

Proposition 2.1. *Assume that $M \subseteq \mathbb{R}^n$ is convex. Then the operator $\sigma_M : \mathcal{P}(E) \to \mathcal{P}(E)$, with $E := \mathbb{R}^n \setminus M$, is a closure operator, i.e.,*

(H0) *σ is increasing:* $\quad$ *$A \subseteq E$ implies $A \subseteq \sigma(A)$,* (H1) *σ is monotone:* $A_1 \subseteq A_2 \subseteq E$ *implies* $\sigma(A_1) \subseteq \sigma(A_2)$,

(H2) *σ is idempotent:* $A \subseteq E$ *implies* $\sigma(\sigma(A)) = \sigma(A)$.

Note that the convexity of M is only used to verify (H2); the axioms (H0) and (H1) hold for all subsets M of $\mathbb{R}^n$.

In the remaining part of this paper, assume again that K is a convex body in $\mathbb{R}^n$. Then the *interior* $K_0 =$ int K of K is also convex; thus, for $E := \mathbb{R}^n \setminus K$ and $E_0 := \mathbb{R}^n \setminus K_0$ we get closure operators

$$\sigma := \sigma_K : \mathcal{P}(E) \to \mathcal{P}(E)\,, \quad \sigma_0 := \sigma_{K_0} : \mathcal{P}(E_0) \to \mathcal{P}(E_0)\,.$$

Our next proposition, which follows trivially from the convexity of K_0, shows that σ_0 is precisely adapted to the illumination problem.

Proposition 2.2. *For a subset A of $\mathbb{R}^n \setminus K$ and a subset B of $\partial K = K \setminus K_0$ the following statements are equivalent*:

(i) *A illuminates B.*

(ii) *$B \subseteq \sigma_0(A)$.*

Still in this section we want to prove an extension of Proposition 2.2 in the case $B = \partial K$. To this end, we consider already now the concept of *visibility* as introduced by F. A. Valentine in [12], see also [4] and [3]. Namely, a point $z \in \mathbb{R}^n \setminus K$ *sees* the point $x \in \partial K$ if $\overline{zx} \cap K = \{x\}$. A point set $A \subseteq \mathbb{R}^n \setminus K$ *sees* a subset B of ∂K if every $b \in B$ is seen from at least one point $a \in A$. If A sees ∂K, we say also that A sees (the whole of) K or that K is visible from A.

Remark 2.3. A subset A of $\mathbb{R}^n \setminus K$, which illuminates a subset B of ∂K, also sees B. □

Theorem 2.4. *For every subset S of $E = \mathbb{R}^n \setminus K$ the following statements are equivalent*:

(i) *S illuminates ∂K.*

(ii) *$\partial K \subseteq \sigma_0(S)$.*

(iii) *For every $x \in \partial K$ there exists some $\delta > 0$ such that $B(x, \delta) \cap \partial K$ is illuminated by some $z \in S$.*

(iv) *For every $x \in \partial K$ there exists some $\delta > 0$ such that $B(x, \delta) \cap \partial K$ is seen by some $z \in S$.*

(v) *There exists some open subset U of $\mathbb{R}^n$ with $K \subseteq U \subseteq K \cup \sigma(S)$.*

(vi) *There exists some open subset U of $\mathbb{R}^n$ with $K \subseteq U \subseteq K_0 \cup \sigma_0(S)$.*

Proof. The relations (i) $\Leftrightarrow$ (ii), (iii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (iv), (vi) $\Rightarrow$ (v), and (vi) $\Rightarrow$ (ii) are either trivial or simple consequences of our considerations above.

(i) $\Rightarrow$ (iii): Assume $x \in \partial K$, and choose $z \in S$ and $y \in K_0$ with $x \in \overline{zy}$. Suppose that $r > 0$ satisfies $B(y, r) \subseteq K_0$. Then for every $x' \in \partial K$ with $||x - x'|| < \frac{||z-x||}{||z-y||} \cdot r$ there exists some $y' \in B(y, r)$ with $x' \in s(z, y')$. Thus x' is illuminated by z.

(iv) $\Rightarrow$ (ii): Assume $x \in \partial K$, and choose $\delta > 0$ and $z \in S$ such that $B(x, \delta) \cap \partial K$ is seen from z. It suffices to prove that $s(z, x) \cap K_0 \neq \emptyset$. Otherwise we would have $s(z, x) \cap K =$

$s(z,x) \cap \partial K = \{x\}$, because x is the unique point in $s(z,x) \cap \partial K$ which is seen from z. Now assume that $y \in K_0$ is arbitrary, and for $\eta > 0$ put $w_\eta := x + \eta \cdot (x - z)$, see Figure 1. If η is small enough, we get $x' \in B(x,\delta) \cap \partial K$ for some $x' \in \overline{yw_\eta} \setminus \{y\}$. On the other hand, we have $\overline{xy} \cap \overline{zx'} \neq \emptyset$. This means that x' is not seen from z, a contradiction to $x' \in B(x,\delta) \cap \partial K$.

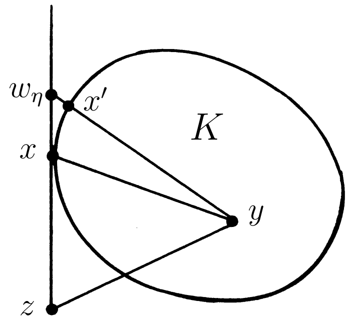

Figure 1

(i) $\Rightarrow$ (vi): For every $x \in \partial K$, choose some $y = y_x \in K_0$ and some $z = z_x \in S$ with $x \in \overline{zy}$. Furthermore, choose $r = r_x > 0$ with $B(y,r) \subseteq K_0$ as well as $w = w_x \in \overline{zx} \setminus \{z,x\}$ such that for

$$r' = r'_x := \frac{||z-w||}{||z-y||} \cdot r$$

we have $B(w,r') \subseteq E$, see Figure 2. Then we have $B(w,r') \subseteq \sigma_0(\{z\})$. Therefore, the set $U_x =$ conv $(B(w,r') \cup B(y,r))$ is an open subset of the convex set $K_0 \cup \sigma_0(\{z\}) =$ conv $(K_0 \cup \{z\})$, see [9], Proposition 2.6 i). Now put

$$U := K \bigcup \bigcup_{x \in \partial K} U_x = K_0 \bigcup \bigcup_{x \in \partial K} U_x \,.$$

U is an open subset of $\mathbb{R}^n$ with $K \subseteq U \subseteq K_0 \cup \sigma_0(S)$.

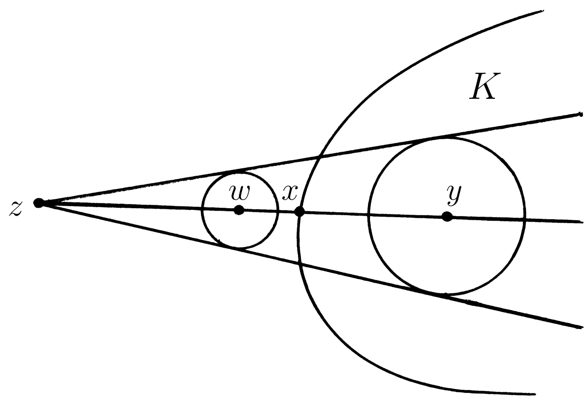

Figure 2

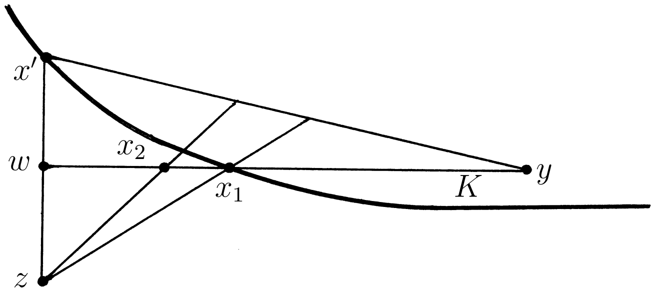

Figure 3

(v) $\Rightarrow$ (vi): We prove that, if $U \subseteq \mathbb{R}^n$ is open and satisfies (v), then U satisfies (vi), too. Namely, assume that $x \in U \setminus K_0$. We must prove that $x \in \sigma_0(S)$. By (v) we have $x \in \partial K$ or $x \in \sigma(S)$. (In Figure 3 we have $x_1 \in \partial K$ and $x_2 \in \sigma(S)$.) Assume that $y \in K_0$ is arbitrary. Then there exists some $w \in U \setminus (K \cup \{x\})$ with $x \in \overline{wy}$, because U is open. (v) implies $w \in \sigma(S)$; thus there exists some $x' \in \partial K$ with $w \in \overline{zx'}$ for some suitable $z \in S$. Clearly, we have $\overline{yx'} \setminus \{x'\} \subseteq K_0$, and hence $x \in \sigma_0(\{z\}) \subseteq \sigma_0(S)$. $\square$

3. Visibility described in terms of a closure operator

In contrast to Proposition 2.2, the closure operator $\sigma = \sigma_K : \mathcal{P}(E) \to \mathcal{P}(E)$ is not precisely adapted to the visibility problem, because the boundary points of the convex body $K \subseteq \mathbb{R}^n$ do not belong to $E = \mathbb{R}^n \setminus K$. Thus they cannot lie in $\sigma_K(A)$ for a subset $A \subseteq E$. However, for any closed subset M of $\mathbb{R}^n$ we define an operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \text{int } M) \to \mathcal{P}(\mathbb{R}^n \setminus \text{ int } M)$ which in case $M = K$ precisely controls visibility: For closed $M \subseteq \mathbb{R}^n, A \subseteq \mathbb{R}^n \setminus M$ and $B \subseteq \partial M = M \setminus \text{ int } M$ put

$$\hat{\sigma}_M(A \cup B) := \sigma_M(A) \cup B \cup \{x \in \partial M \,|\, \overline{zx} \cap M = \{x\} \text{ for some } z \in A\}\,. \tag{3.1}$$

By the definitions it is clear that for all $A \subseteq F_0 := \mathbb{R}^n \setminus \text{int } M$ we have

$$\hat{\sigma}_M(A) = A \cup \{b \in F_0 \setminus A \,|\, \exists\, a \in A\,,\ \exists\, x \in \partial M : b \in \overline{ax},\, \overline{ax} \cap M = \{x\}\}\,. \tag{3.2}$$

By the definition of visibility ($\overline{zx} \cap K = \{x\}$, see the passage after Proposition 2.2 above) we have the following trivial

Proposition 3.1. *For a subset A of $\mathbb{R}^n \setminus K$ and a subset B of $\partial K = K \setminus K_0$ the following statements are equivalent:*

(i) *A sees B.*

(ii) *$B \subseteq \hat{\sigma}_K(A)$.*

To prove that $\hat{\sigma}_M$ is a closure operator in case M equals the convex body K, we show first the following

Lemma 3.2. *For a boundary point $x \in \partial K$ of the convex body K and a subset A of $\mathbb{R}^n \setminus K$ the following statements are equivalent:*

(i) *There exists some* $z \in A$ *with* $\overline{zx} \cap K = \{x\}$.

(ii) *There exists some* $z' \in \sigma_K(A)$ *with* $\overline{z'x} \cap K = \{x\}$.

Proof. (i) $\Rightarrow$ (ii): This implication is trivial in view of $A \subseteq \sigma_K(A)$.

(ii) $\Rightarrow$ (i): Choose some $z \in A$ with $z' \in \sigma_K(\{z\})$ and assume, without loss of generality, that $z' \neq z$. Then there exists some $x' \in \partial K$ with $z' \in \overline{zx'}$ and $\overline{zx'} \cap K = \{x'\}$. We prove that $\overline{zx} \cap K = \{x\}$. Otherwise there would exist some $y \in \overline{zx} \cap K$ with $y \neq x$. Since K is convex, we get even $\overline{yx'} \subseteq K$. On the other hand, we have $\overline{yx'} \cap \left(\overline{z'x} \setminus \{x\}\right) \neq \emptyset$ and thus also $K \cap \left(\overline{z'x} \setminus \{x\}\right) \neq \emptyset$, which contradicts (ii). $\square$

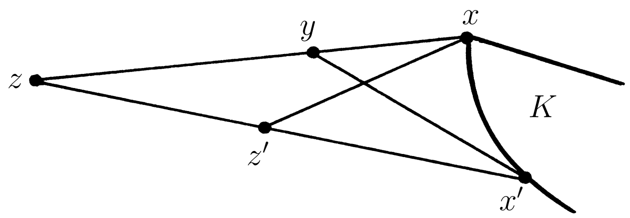

Figure 4

In Figure 4 one sees that Lemma 3.2 is not a trivial corollary of Proposition 2.1. Now we can prove

Theorem 3.3. *The operator* $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M) \to \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M)$, *where* $M \subseteq \mathbb{R}^n$ *is closed, satisfies*

(H0) $\hat{\sigma}_M$ *is increasing*: $A \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $A \subseteq \hat{\sigma}_M(A)$.

(H1) $\hat{\sigma}_M$ *is monotone*: $A_1 \subseteq A_2 \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $\hat{\sigma}_M(A_1) \subseteq \hat{\sigma}_M(A_2)$.

If M *equals a convex body* $K \subset \mathbb{R}^n$, *then we have also*

(H2) $\hat{\sigma}_M$ *is idempotent*: $A \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $\hat{\sigma}_M(\hat{\sigma}_M(A)) = \hat{\sigma}_M(A)$.

Thus $\hat{\sigma}_K$ *is a closure operator for every convex body* K *in* $\mathbb{R}^n$.

Proof. By (3.2), (H0) and (H1) are trivial for arbitrary closed $M \subseteq \mathbb{R}^n$. To prove (H2) for $M = K$, assume that $A \subseteq \mathbb{R}^n \setminus K$ and $B \subseteq \partial K$. Then we obtain by (3.1), Lemma 3.2, and the fact that σ_K is a closure operator:

$$\begin{aligned}
\hat{\sigma}_K\left(\hat{\sigma}_K(A \cup B)\right) &= \hat{\sigma}_K\left(\sigma_K(A) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in A\}\right) \\
&= \sigma_K\left(\sigma_K(A)\right) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in \sigma_K(A)\} \\
&= \sigma_K(A) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in A\} \\
&= \hat{\sigma}_K(A \cup B)\,. \qquad \square
\end{aligned}$$

We close this section by proving the following result, which shows that, under certain conditions, convex subsets M of $\mathbb{R}^n$ may be characterized by the fact that either of the operators σ_M and $\hat{\sigma}_M$ is a closure operator.

Theorem 3.4. *Assume that* $M \subseteq \mathbb{R}^n$ *is compact and that* $E = \mathbb{R}^n \setminus M$ *is connected. Then the following three statements are equivalent*:

(i) *M is convex.*
(ii) *The operator $\sigma_M : \mathcal{P}(E) \to \mathcal{P}(E)$ is a closure operator.*
(iii) *The operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \mathrm{int}M) \to \mathcal{P}(\mathbb{R}^n \setminus \mathrm{int}M)$ is a closure operator.*

Proof. (i) $\Leftrightarrow$ (ii) is Theorem 2.10 in [9].
(i) $\Rightarrow$ (iii) is Theorem 3.3 above.
(iii) $\Rightarrow$ (ii): Assume that $A \subseteq \mathbb{R}^n \setminus M$. Since $\hat{\sigma}_M$ satisfies (H2), we get

$$\sigma_M(\sigma_M(A)) \subseteq \hat{\sigma}_M(\hat{\sigma}_M(A)) = \hat{\sigma}_M(A) \subseteq \sigma_M(A) \cup \partial M\,.$$

However, $\sigma_M(\sigma_M(A))$ is contained in the open set $\mathbb{R}^n \setminus M$, which does not intersect ∂M. This means that $\sigma_M(\sigma_M(A)) = \sigma_M(A)$, as claimed. $\square$

4. Illumination by directions

Throughout this section assume that we have $0 \in \mathrm{int}\ K$ for a convex body K in $\mathbb{R}^n$. Due to [1], a point $x \in \partial K$ is said to be *illuminated by the direction* $u \in S^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ if the ray $s(x, x+u)$ with initial point x and direction u meets the interior of K. Therefore it seems natural to consider the operator $\tau_0 : \mathcal{P}(S^{n-1}) \to \mathcal{P}(S^{n-1})$ given by

$$\tau_0(A) := \left\{ b \in S^{n-1} \mid \exists\, u \in A, \exists\, x \in \partial K : \frac{1}{||x||} \cdot x = b, s(x, x+u) \cap \mathrm{int}\ K \neq \emptyset \right\}, \tag{5.1}$$

which is adapted to the problem of illumination by directions. Note that $0 \in \mathrm{int}\ K$ implies that for every $b \in S^{n-1}$ there exists a unique element $x \in \partial K$ with $\frac{1}{||x||} \cdot x = b$. However, τ_0 is not increasing, since for $u \in S^{n-1}$ and $x \in \partial K$ with $\frac{1}{||x||} \cdot x = u$ we have $s(x, x+u) \cap K = \{x\}$ and thus $s(x, x+u) \cap \mathrm{int}\ K = \emptyset$. This means $u \notin \tau_0(\{u\})$. Therefore let us look at the modified operator $\tau_1 : \mathcal{P}(S^{n-1}) \to \mathcal{P}(S^{n-1})$ given by

$$\tau_1(A) := \left\{ b \in S^{n-1} \mid \exists\, u \in A, \exists\, x \in \partial K : \frac{1}{||x||} \cdot x = b,\ s(x, x-u) \cap \mathrm{int}\ K \neq \emptyset \right\}. \tag{5.2}$$

Clearly, we have $\tau_1(A) = \tau_0(-A)$ for $A \subseteq S^{n-1}$. Now $0 \in \mathrm{int}\ K$ implies that τ_1 is an increasing operator which is also monotone and in some sense analogous to σ_0 by relating $u \in S^{n-1}$ to points $x \in \mathbb{R}^n$ for which $||x||$ is large and $\frac{1}{||x||} \cdot x = u$.

However, τ_1 is in general not a closure operator, though σ_0 is. If, for instance, K is a ball with 0 as its center, then we have $\tau_1(\{u\}) = \{v \in S^{n-1} | \cos(\sphericalangle(u,v)) > 0\}$ for all $u \in S^{n-1}$. Thus we get $\tau_1(\tau_1(\{u\})) = S^{n-1} \setminus \{-u\}$ and $\tau_1(\tau_1(\tau_1(\{u\}))) = S^{n-1}$, whence τ_1 is not idempotent.

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Received December 15, 2003