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On Normal Verbal Embeddings of Some Classes of Groups

Herrn Professor Hermann Heineken, zu seinem 65. Geburtstag

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Abstract. We give a new proof to the criterion for the normal verbal embeddability of groups. The new construction allows us to consider the normal verbal embeddings of soluble groups, of nilpotent groups, and of SN^*-groups into groups of the same type. We also generalize a theorem of Burnside on embeddings of p-groups into commutator subgroups.

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1. Introduction

1.1. The initial examples for normal verbal embeddability

The normal embedding of the group G into the group H is *verbal* for the word set $V \in F_\infty$ if the corresponding isomorphic image $\tilde{G}$ of G lies in the verbal subgroup $V(H)$ of H:

$$G \cong \tilde{G} \triangleleft H \quad \text{and} \quad \tilde{G} \subseteq V(H)$$

(see [13] for background information on verbal subgroups and on varieties of groups). The first examples of groups G which for the given word set V are *not verbal and normal embeddable* – that is, there is no group H possessing a normal verbal embedding of G into H – are constructed by Burnside [3]. He showed that no non-abelian group G with cyclic center, and no non-abelian group G with the property $(G : G') = p^2$ (p is any prime number) can be embedded as a normal subgroup into a finite p-group H such that the isomorphic image $\hat{G}$ of G lies in the commutator subgroup H' (the commutator subgroup H' is, of course, the verbal subgroup $V(H)$ for the word set consisting of a single word: $V = \{[x_1, x_2]\}$). Further, Blackburn described the 2-generator groups that arise as commutator subgroups of 2-generator p-groups [2].

1.2. The problem of Heineken

After an intensive development of the theory of varieties of groups since the sixties it was natural to consider the problem of normal verbal embeddings not merely for the commutator subgroup but also for verbal subgroups for any $V \subseteq F_\infty$. Clearly, non-triviality should be a natural restriction on the word set V (a word set is trivial if it is equivalent to the trivial word). For, if V is trivial, then $V(H) = \{1\}$ for any group H, and our problem has an evident answer in this case: $G = \{1\}$.

In 1992 Heineken [6] posed the problem of normal verbal embeddability for any word w and described the situation for all finite p-groups. The construction of [6] is based on wreath products. Eick modified that construction and found the answer for all finite groups and non-trivial words w [4]. The answer for the case of arbitrary non-trivial word sets and arbitrary groups can be found in our common work [7] (see Theorem 1 in Section 2 in this paper). Since the construction of [6] cannot be applied to the infinite groups (not even to the infinite cyclic group), we had to build in [7] a more complicated construction that embeds the group G into a subgroup $H(G, V)$ of the product $S_M \cdot \mathrm{Aut}(G)$, where S_M is the group of all permutations on the group $M = \mathrm{Hol}\,(G \times K)$, and where K is a group with the property $K \notin \mathfrak{V} = \mathrm{var}\,(F_\infty/V(F_\infty)))$.

1.3. The aim of this work

The disadvantage of the construction in [7] is that the structure of the subgroup $H(G, V)$ in $S_M \cdot \mathrm{Aut}(G)$ is not clearly imaginable, and that the properties of $H(G, V)$ are very far from the properties of the initial group G[1]. In [10, 11, 12] we have developed constructions for verbal normal and subnormal embeddings that we will use here.

The first aim of this paper is to present another, with the method of [7] not connected construction which allows us to find a shorter and more effective proof to Main Theorem in [7]. Our argument generalizes the elegant idea of [6], and is based not on wreath products, but on something similar, namely on the construction $W(G, D, A)$ (see Section 2 for definitions).

Secondly, the new construction allows us to answer the following question: For a given word set V when is a group of a *given class* (a soluble group, a nilpotent group, or an SN^*-group) normal and verbal embeddable into a group of the same class? This question is natural after we answered the similar question for abelian groups in [7, Theorem 2]. The normal verbal embeddings of soluble groups into soluble groups were considered in [7] only for the simplest situations where V consists of x^n, of $[x_1, x_2]$, or of $\delta_n(x_1, \dots, x_{2^n})$ [7, Theorem 3].

[1] For abelian (even for cyclic) groups G and $\mathrm{Aut}(G)$ the group $H(G, V)$ constructed in [7] can be insoluble.

And the normal verbal embeddings of nilpotent groups into nilpotent groups were considered in [7] for the case when V consists of $[x_1, x_2]$ only [7, Theorem 3].

In Section 2 we present the construction of the new proof for Main Theorem [7] (Theorem 1 in Section 2 of the current paper), and we prove the criterion for normal verbal embeddings of soluble groups into soluble groups (Theorem 2).

In Section 3 we consider the similar question for nilpotent groups. For this case we build a few modifications of our construction (Theorems 3, 4, and 5). As a consequence of this we get the following generalization of the theorem of Burnside cited at the beginning:

Let G be a p-group of finite exponent and let V be any non-trivial word set. Then there exists a p-group H with normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if for a Sylow p-subgroup L of $\mathrm{Aut}(G)$ *holds:* $V(L) \supseteq \mathrm{Inn}(G)$ (Theorem 6 in Section 3).

In Section 4 we consider the normal verbal embeddings of SN^*-groups, that is, groups with (finite or infinite) subnormal soluble ascending series (Theorem 7).

Finally, in Section 5 we consider a few more "economical" normal verbal embedding constructions for the most "well-known" words.

My work at the Universität Würzburg, Germany (1997-98) was a great stimulus for me to see the importance of methods of varieties of groups in embedding constructions for groups. For that nice possibility, and also for very warm hospitality in Würzburg I am very much thankful to Professor Dr. Hermann Heineken and to the members of the Lehrstuhl für Mathematik I. Also, I am very much thankful to the referee for valuable remarks.

2. Construction of the normal verbal embedding, embeddings of soluble groups

2.1. The main results

We restate here the main Theorem of [7] as:

Theorem 1. *Let G be an arbitrary group and $V \subseteq F_\infty$ be a non-trivial word set. Then there exists a group $H = H(G, V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if* $V(\mathrm{Aut}(G)) \supseteq \mathrm{Inn}(G)$.

And for embeddings of soluble groups:

Theorem 2. *Let G be an arbitrary soluble group and $V \subseteq F_\infty$ be a non-trivial word set. Then there exists a soluble group $H = H(G, V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if there is a soluble subgroup B of* $\mathrm{Aut}(G)$ *such that* $V(B) \supseteq \mathrm{Inn}(G)$.

Remark 1. As we will see, *if the group G of Theorem* 1 *(or the soluble group G of Theorem* 2*) is finite or finitely generated, the corresponding group H can be constructed to be finite or finitely generated, respectively. And if the group G is infinite, the corresponding group H can be constructed to have the same cardinality as G.*

In order to avoid unnecessary repetitions, we will build one construction for both theorems, and then we will consider the soluble groups as a special case.

2.2. The construction of the group $W(G, D, A)$

Definition 1. *Let G and D be arbitrary groups and let A be a subgroup of* $\mathrm{Aut}(G)$ *containing the group of inner automorphisms: $InnG \le A \le \mathrm{Aut}(G)$. Define $W(G, D, A)$ to be the extension of the Cartesian product $\prod_{d\in D} G$ of copies of the group G by means of the direct product $D \times A$ defined by the operation:*

$$\varphi^{(h,f)}(d) \stackrel{\text{def}}{=} f(\varphi^{h^{-1}}(d)) \stackrel{\text{def}}{=} f(\varphi(dh)),$$

$$\text{for all } \varphi \in \textstyle\prod_{d\in D} G, \text{ and all } (h, f) \in D \times A.$$

And if the subgroup A of $\mathrm{Aut}(G)$ *is not given explicitly, we take $A = \mathrm{Aut}(G)$.*

This is the construction that we called in the introduction *"not wreath products, but something similar"*. As it is easy to see, the group D functions on the copies of the group G in the same way as the "active" group D of wreath product $G \,\mathrm{Wr}\, D$ functions on the copies of the "passive" group G in the base subgroup G^D. For the element $g \in G$ we denote by f_g the inner automorphism of $W(G, D, A)$ induced by g (each automorphism of G can be continued on $W(G, D, A)$). Further denote by φ_g the diagonal element corresponding to $g \in G$ in the Cartesian product $\prod_{d\in D} G$, that is, $\varphi_g(d) = g$ for all $d \in D$. Now we embed the group G in $W(G, D, A)$:

$$g \mapsto g_W \stackrel{\text{def}}{=} (1, f_g)\varphi_{g^{-1}}, \quad g \in G.$$

It is easy to see that this map is an injection and that:

$$\begin{aligned}(1, f_g)\varphi_{g^{-1}} \cdot (1, f_{g'})\varphi_{g'^{-1}} &= (1, f_g)(1, f_{g'}) \cdot (\varphi_{g^{-1}})^{(1,f_{g'})}\varphi_{g'^{-1}} \\ &= (1, f_{gg'})\varphi_{(gg')^{-1}},\end{aligned}$$

because

$$\begin{aligned}\left[(\varphi_{g^{-1}})^{(1,f_{g'})}\varphi_{g'^{-1}}\right](d) &= g'^{-1}\varphi_{g^{-1}}(d)\; g' \cdot \varphi_{g'^{-1}}(d) \\ &= g'^{-1}g^{-1}\, g'g'^{-1} = (gg')^{-1} = \varphi_{gg'}(d).\end{aligned}$$

Set $G_W = \{g_W |\, g \in G\}$ to be the image of G in $W(G, D, A)$. As we said, this embedding modifies the argument of [6] (see also Remark 3).

2.3. The characteristic subgroups in G and the normal subgroups in $W(G, D, A)$

Lemma 1. *A subgroup $K \le G$ is characteristic in G if and only if its image K_W is normal in $W(G, D, A)$.*

Proof. Let K be characteristic in G and let $g_W \in K_W$. For an arbitrary $h \in D$ holds

$$g_W^{(h,1)} = (1, f_g)^{(h,1)}{\varphi_{g^{-1}}}^{(h,1)} = (1, f_g)\varphi_{g^{-1}}$$

because

$${\varphi_{g^{-1}}}^{(h,1)}(d) = {\varphi_{g^{-1}}}^{h^{-1}}(d) = \varphi_{g^{-1}}(dh) = g^{-1} = \varphi_{g^{-1}}(d).$$

Further for an arbitrary $f \in A$ holds

$$g_W^{(1,f)} = (1, f_g)^{(1,f)} \varphi_{g^{-1}}^{(1,f)} = (1, f_{f(g)}) \varphi_{f(g)^{-1}}$$

because $(f_g)^f = f_{f(g)}$ and $(\varphi_{g^{-1}})^f(d) = f(g^{-1})$. Finally, for arbitrary $\psi \in \prod_{d \in D} G$ holds

$$\begin{aligned} g_W^{\psi} &= \psi^{-1}(1, f_g)\varphi_{g^{-1}}\psi \\ &= (1, f_g)(\psi^{-1})^{(1,f_g)}\varphi_{g^{-1}}\psi = (1, f_g)\varphi_{g^{-1}} \end{aligned}$$

because

$$\begin{aligned} \left[(\psi^{-1})^{(1,f_g)}\varphi_{g^{-1}}\psi\right](d) &= (\psi^{-1})^{(1,f_g)}(d)\ \varphi_{g^{-1}}(d)\ \psi(d) \\ &= f_g\left(\psi^{-1}(d)\right)\ g^{-1}\ \psi(d) \\ &= g^{-1}\ \psi^{-1}(d)\ g\ g^{-1}\ \psi(d) = g^{-1} = \varphi_{g^{-1}}(d). \end{aligned}$$

Thus, K_W is normal in $W(G, D, A)$. And if K_W is normal in $W(G, D, A)$, then $(g_W)^{(1,f)} \in K_W$ and, thus, $g^f = f(g) \in K$ for an arbitrary automorphism $f \in \mathrm{Aut}(G)$ and for an arbitrary element $g \in K$. □

The presented construction allows us to embed the given group G in a bigger group $W(G, D, A)$ in such a way that the structure of the characteristic subgroups of G is connected with the structure of the normal subgroups of $W(G, D, A)$ in G. Additionally, the group $W(G, D, A)$ can have many pregiven properties because this construction inherits some of the helpful properties of wreath products.

2.4. The verbal embedding of G in $W(G, D, A)$

Let now V be a non-trivial word set and let $\mathfrak{V} = \mathrm{var}\,(F_\infty/V(F_\infty))$ be the corresponding variety of groups. We take $D = D(V)$ to be such a group that:

1. D is soluble,
2. D does not belong to the variety $\mathfrak{V}$,
3. D is torsion-free.

Such a group D can always be chosen, and it can even be nilpotent because the finite p-groups generate the variety of all groups, and it is sufficient to consider the corresponding finite p-group $P \notin \mathfrak{V}$ and to take the free nilpotent group (of finite rank) the factor of which is P. For the sequel let $D = D(V)$ be the group obtained here.

Lemma 2. *Let G be a group and V be a non-trivial word set. Then, if*

$$V(\mathrm{Aut}(G)) \supseteq \mathrm{Inn}(G),$$

then the normal subgroup G_W of $W(G, D, A)$ lies in the verbal subgroup $V(W(G, D, A))$.

Proof. That G_W is normal in $W(G,D,A)$ follows from Lemma 1. Let $a \in V(D)$ be an element of infinite order, and let T be a left transversal of $\langle a \rangle$ in D. For $g \in G$ and for $d = ta^i$ ($t \in T$, $i \in \mathbb{Z}$) we define the element τ_g as:

$$\tau_g(d) = \tau_g(ta^i) = g^i.$$

We have

$$\begin{aligned}(a^{-1},1)^{\tau_g} &= {\tau_g}^{-1}(a^{-1},1)\ \tau_g \\ &= (a^{-1},1)({\tau_g}^{-1})^{(a^{-1},1)}\tau_g = (a^{-1},1)(\varphi_{g^{-1}})^{-1}\end{aligned}$$

because

$$({\tau_g}^{-1})^{(a^{-1},1)}(d)\ \tau_g(d) = {\tau_g}^{-1}(d\,a^{-1})\ \tau_g(d) = g = (\varphi_{g^{-1}}(d))^{-1}.$$

Since both $(a^{-1},1)^{\tau_g}$ and $(a^{-1},1)$ belong to $V(W(G,D,A))$, we get that the element $(\varphi_{g^{-1}})^{-1}$ and, thus, the element g_W belongs to $V(W(G,D,A))$. □

Lemma 3. *Let G be a soluble group and V be a non-trivial word set. If the group of automorphisms $\mathrm{Aut}(G)$ has a soluble subgroup B such that $V(B) \supseteq \mathrm{Inn}(G)$ holds, then the group $W(G,D,B)$ is soluble and its normal subgroup G_W lies in the verbal subgroup $V(W(G,D,B))$.*

Proof. The proof is very similar to the proof of the previous lemma. Instead of $A = \mathrm{Aut}(G)$ we should take the group B. □

Remark 2. We can notice that, if the solubility lengths of the groups G, D, B are l_G, l_D, l_B correspondingly, then the solubility length of the group $W(G,D,B)$ is at most $l_W = l_G + \max(l_D, l_B)$.

The following lemmas prove necessity of the conditions of Theorem 1 and of Theorem 2.

Lemma 4. *Let G be a group and V be a non-trivial word set. Only then there exists a group H with a normal subgroup $\hat{G}$ such that $\hat{G}$ is isomorphic to G and lies in $V(G)$, when $V(\mathrm{Aut}(G)) \supseteq \mathrm{Inn}(G)$.*

Lemma 5. *Let G be a group and V be a non-trivial word set. Only then there exists a soluble group H with a normal subgroup $\hat{G}$ such that $\hat{G}$ is isomorphic to G and lies in $V(H)$, when $\mathrm{Aut}(G)$ contains a soluble subgroup B such that $V(B) \supseteq \mathrm{Inn}(G)$.*

Proof of Lemmas 4 *and* 5. This proof coincides with the proof of Lemma 2 in [7] or Lemma 2 in [6]. We briefly outline the proof because this condition is very frequently used in this paper and we would like to show its origin. The operation of elements of H on $\hat{G}$ defines an isomorphism between $H/C_H(\hat{G})$ and a subgroup of $\mathrm{Aut}(\hat{G})$. The image of $\hat{G}C_H(\hat{G})/C_H(\hat{G})$ under this isomorphism is $\mathrm{Inn}(\hat{G})$. Thus, if $V(\mathrm{Aut}(G))$ does not contain $\mathrm{Inn}(G)$ (that is, if $V(\mathrm{Aut}(\hat{G}))$ does not contain $\mathrm{Inn}(\hat{G})$), then holds

$$V(H/C_H(\hat{G})) = V(H)C_H(\hat{G})/C_H(\hat{G}) \not\supseteq \hat{G}C_H(\hat{G})/C_H(\hat{G}).$$

And if H is a soluble group, the corresponding subgroup of $\mathrm{Aut}(\hat{G})$ also is soluble. □

Remark 3. A comparison of the group $W(G, D, A)$ with the construction of [6] shows that $W(G, D, A)$ generalizes and modifies the concept of [6] for the case of infinite groups. In [6] the group H is the extension of the wreath product $G \operatorname{Wr} S$ with the group $\operatorname{Aut}(G)$, where S is a suitably chosen finite p-group whose verbal subgroup[2] $V(S)$ is contained in the center of S and is of exponent $m = p^k = \exp G$.

To conclude this section it remains to prove Remark 1 on cardinalities of our groups.

Proof of Remark 1. That for the finite group G the corresponding group $H = H(G, V)$ can be chosen finite follows from construction of [6] (in this case there is no need in our group $W(G, D, A)$). And that for the soluble finite group G the finite group $H = H(G, V)$ can be chosen to be soluble follows from the properties of the construction in [6] (we will consider them in the next section).

Let now the group G of Theorem 1 (or of Theorem 2) be finitely generated: $G = \langle g_i |\ i \in I\rangle$ ($\operatorname{card}(I) < \infty$). Then $\hat{G} = \langle \hat{g}_i |\ i \in I\rangle$ and since $\hat{G}$ lies in $V(H)$, holds

$$\hat{g}_i = \left(w_1^{(i)}(h_{11}^{(i)}, \dots, h_{1q_1}^{(i)})\right)^{\delta_1^{(i)}} \cdots \left(w_u^{(i)}(h_{u1}^{(i)}, \dots, h_{uq_u}^{(i)})\right)^{\delta_u^{(i)}} \quad \text{for all } i \in I,$$

where $w_1^{(i)}, \dots, w_u^{(i)} \in V$; $\delta_u^{(i)} = \pm 1$; $h_{jk}^{(i)} \in H$ ($j = 1, \dots, u$; $k = 1, \dots, q_i$). Clearly, the finitely generated subgroup $H_1 = \langle h_{jk}^{(i)} |\ h_{jk}^{(i)};\ j = 1, \dots, u;\ k = 1, \dots, q_i;\ i \in I\rangle \le H$ contains the subgroup $\hat{G}$ in its verbal subgroup $V(H_1)$, and $\hat{G}$ is normal in H_1.

Finally, let G be an infinite (not necessarily finitely generated) group. G has a set of generators $\{g_i |\ i \in I\}$ of the same cardinality as G. The construction that we just built above shows that it is possible to chose such a subgroup $H_1 \le H$ that H_1 contains $\hat{G}$ in its verbal subgroup $V(H_1)$, and is of the same cardinality as G. □

3. The case of nilpotent groups, generalization of a theorem of Burnside

3.1. A necessary condition

For the case of normal verbal embeddings of nilpotent groups into nilpotent groups we have to built a slightly more complicated construction to be able to deal with relatively restricted situations. Since nilpotence is a stronger property than solubility, we can expect that the conditions of Lemma 4 and Lemma 5 need to be strengthened. Indeed:

Lemma 6. *Let G be a nilpotent group and V be a non-trivial word set. Only then there exists a nilpotent group H with a normal subgroup $\hat{G}$ such that $\hat{G}$ is isomorphic to G and lies in $V(G)$, when* $\operatorname{Aut}(G)$ *contains a nilpotent subgroup N such that $V(N) \supseteq \operatorname{Inn}(G)$, and the extension of G with N (as semidirect product) is nilpotent.*

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[2] [6] considers the situation not for an arbitrary word set V but for one word w only. However, the argument of [6] is true for any word set, too.

Proof. The existence of the nilpotent subgroup $N \subseteq \mathrm{Aut}(G)$ can be proved in analogy with the proof of Lemma 5: as N one can take the corresponding subgroup $N \cong H/C_H(G) \leq \mathrm{Aut}(G)$. The group $K = \langle G, N\rangle$ is nilpotent. For, since the group H is nilpotent, it has a central series containing the subgroup G. The images of the members of that series form a central series in K because the operation of N on them is induced by operation of H on the corresponding preimages. □

3.2. The case of finite nilpotent groups

It turns out that for finite nilpotent groups the condition of Lemma 6 already is sufficient:

Theorem 3. *Let G be an arbitrary finite nilpotent group and $V \subseteq F_\infty$ be a non-trivial word set. Then there exists a (finite) nilpotent group $H = H(G,V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if there is a nilpotent subgroup N of* $\mathrm{Aut}(G)$ *such that $V(N) \supseteq \mathrm{Inn}(G)$, and if the split extension of G by N is nilpotent.*

Proof. The finite nilpotent group G is the direct product of its Sylow subgroups $G_{p_1}, \dots, G_{p_n}$. Set $p_i^{k_i} = \exp G_{p_i}$, $i = 1, \dots, n$. According to Lemma 7 in [5] or Lemma 1 in [6], there is a finite p-group S_i whose verbal subgroup[3] $V(S_i)$ is of exponent $p_i^{k_i}$ and lies in the center of S_i. Let now $G_{1\,p_i} = G_{p_i} \mathrm{Wr}\, S_i$ and $G_{p_i} \to G_{1\,p_i}$ be the embedding $g \mapsto \varphi_g \in G_{p_i}^{S_i}$ $(i = 1, \dots, n)$, where as above: $\varphi_g(s) = g$, $s \in S_i$. Then holds $\langle \varphi_g |\ g \in G\rangle \subseteq V(G_{1\,p_i})$. Indeed, $V(S_i)$ contains an element u of exponent $p_i^{k_i}$. Take a left transversal $\{t_1, \dots, t_l\}$ of $\langle u\rangle$ in S_i and set the following elements χ_g in the base subgroup $G_{p_i}^{S_i}$

$$\chi_g(s) = \begin{cases} g, & \text{if } s = 1_{S_i}, \\ 1, & \text{if } s \neq 1_{S_i}. \end{cases}$$

Then $\varphi_g = \prod_{s\in S_i} \chi_g^s = \prod_{j=1}^{l} \left(\prod_{k=0}^{p_i^{k_i}-1} \chi_g^{u^k}\right)^{t_j}$ for any $g \in G$. But, as it is easy to compute, $\prod_{k=0}^{p_i^{k_i}-1} \chi_g^{u^k} = \prod_{k=1}^{p_i^{k_i}-1} [\chi_g, u^k] \in V(G_{1p_i})$ because $g^{-(p_i^{k_i}-1)} = g$ $(i = 1, \dots, n)$, and because the verbal subgroup $V(G_{1p_i})$ is normal in G_{1p_i}. The above part of the proof is a slight variation of the argument of [6]. Therefore, G lies in the verbal subgroup $V(\prod_{i=1}^n G_{1\,p_i})$ of the direct product of groups $G_{1\,p_i}$, $i = 1, \dots, n$. Let now $H = H(G,V) = \overline{W}(G, \prod_{i=1}^n S_i, N)$, where $\overline{W}$ is a new construction that slightly differs from W. Here, too, the group $\prod_{i=1}^n S_i$ functions over the direct product

$$G_{p_1}^{S_1} \times \cdots \times G_{p_n}^{S_n}, \tag{1}$$

but it now functions differently. The group S_i functions on copies on the copies of G_{p_i} as in the previous section, and it functions trivially on the other copies of G_{p_j}, $j \neq i$. The function of the group N on (1) is defined, as above, according to the function of N on G. Define

[3] In fact in the mentioned papers the groups S_i are built for the case when V consists of one word w only, but the proof is very easy to generalize for the case of an arbitrary non-trivial word set V.

further for $g \in G$ the elements $g_{\overline{W}} = (1, f_g)\varphi_{g^{-1}}$. Then the group

$$\begin{aligned}\langle G_{p_1}, \dots, G_{p_n};\ \prod_{i=1}^{n} S_i \rangle &= \langle G_{p_1}, S_1 \rangle \times \cdots \times \langle G_{p_n}, S_n \rangle \\ &= G_{p_1} \operatorname{Wr} S_1 \times \cdots \times G_{p_n} \operatorname{Wr} S_n\end{aligned}$$

is a nilpotent group. Since the extensions of G with $\prod_{i=1}^{n} S_i$ and with N are both nilpotent, the extension H of G by the group $(\prod_{i=1}^{n} S_i) \times N$ also is nilpotent. Further it is clear that the subgroup $G_{\overline{W}} = \langle g_{\overline{W}} |\ g \in G \rangle$ lies in $V(H)$. And since $G_{\overline{W}}$ is isomorphic to G, it remains to show that, $G_{\overline{W}}$ is normal in H. We have to repeat here a part of the computations of the previous section. Set $g_{\overline{W}} \in G_{\overline{W}}$. Firstly, for any $x \in \prod_{i=1}^{n} S_i$ holds

$$(g_{\overline{W}})^{(x,1)} = (1, f_g)^{(x,1)} {\varphi_{g^{-1}}}^{(x,1)} = g_{\overline{W}}.$$

Secondly, for any $f \in N$ holds $(g_{\overline{W}})^{(1,f)} = (1, f_g)^{(1,f)} {\varphi_{g^{-1}}}^{(1,f)} = f(g)_{\overline{W}}$. And finally, for any element $\psi_{g'}$ of type:

$$\psi_{g'}(y) = \begin{cases} g' & \text{for } y = 1_{S_1} \\ 1 & \text{for } y \neq 1_{S_1} \end{cases}$$

for the product (1) holds the following:

$$\begin{aligned}(g_{\overline{W}})^{\psi_{g'}} &= (1, f_g)^{\psi_{g'}} {\varphi_{g^{-1}}}^{\psi_{g'}} = (\psi_{g'})^{-1}(1, f_g)\varphi_{g^{-1}}\psi_{g'} \\ &= (1, f_g)((\psi_{g'})^{-1})^{(1,f_g)}\varphi_{g^{-1}}\psi_{g'} \\ &= (1, f_g)\varphi_{g^{-1}} = g_{\overline{W}},\end{aligned}$$

because

$$\left[((\psi_{g'})^{-1})^{(1,f_g)}\varphi_{g^{-1}}\psi_{g'}\right](1_{S_1}) = f_g({g'}^{-1})g^{-1}g' = g^{-1}{g'}^{-1}\, g\, g^{-1}g' = g^{-1}$$

and

$$\left[((\psi_{g'})^{-1})^{(1,f_g)}\varphi_{g^{-1}}\psi_{g'}\right](y) = 1 \cdot g^{-1} \cdot 1 = g^{-1} \quad \text{for all } \ y \neq 1_{S_1}.$$

Therefore, $G_{\overline{W}}$ is normal in $H = \overline{W}(G, \prod_{i=1}^{n} S_i, N)$. □

3.3. The cases with various restrictions

The following theorem holds not only for finite nilpotent groups.

Theorem 4. *Let G be an arbitrary nilpotent p-group of finite exponent and $V \subseteq F_\infty$ be a non-trivial word set. Then there exists a nilpotent group $H = H(G, V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if there is a nilpotent subgroup N of* $\operatorname{Aut}(G)$ *such that $V(N) \supseteq \operatorname{Inn}(G)$, and if the split extension of G by N is nilpotent.*

Proof. In the proof of Theorem 3 we needed finiteness of G just to guarantee that G can be presented as direct product of nilpotent p-groups of finite exponent, and that the corresponding wreath product $G_{p_i} \operatorname{Wr} S_i$ is nilpotent. In this theorem, the group G is already a p-group and the wreath product of G and of the corresponding finite p-group S is nilpotent [1]. The rest of the proof remains unchanged. □

Denote, as usual, by $\gamma_c(x_1, \dots, x_c)$ the word $[x_1, \dots, x_c]$. Let $\mathfrak{N}_c = \operatorname{var}(F_\infty/\gamma_c(F_\infty))$ be the variety of nilpotent groups of class at most c. In the following theorem, which is a generalization of Theorem 4 in [7], G can be any nilpotent group.

Theorem 5. *Let G be an arbitrary nilpotent group and $V \subseteq F_\infty$ be a non-trivial word set with a consequence of type $\gamma_c(x_1, \dots, x_c)$. Then there exists a nilpotent group $H = H(G, V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if there is a nilpotent subgroup N of $\operatorname{Aut}(G)$ such that $V(N) \supseteq \operatorname{Inn}(G)$, and if the split extension of G by N is nilpotent.*

Proof. Consider the group $w(G, C, N)$, where $C = \langle c \rangle$ is an infinite cyclic group. The proof of Theorem 3 shows that in this situation it will be sufficient to find in the direct product $\prod_{c^i \in C} G$ such a subgroup L that:

1. L contains all the elements[4] φ_g, $g \in G$ (as above, $\varphi_g(c^i) = g$ for $i \in \mathbb{Z}$),
2. $L^C \subseteq L$ and $L^N \subseteq L$ hold (and, consequently, the subgroup $H = \langle L, C, N \rangle$ contains the isomorphic copy G_W of G),
3. $\{\varphi_g |\ g \in G\} \subseteq V(L, C)$ holds and, thus, $G_W \subseteq V(H)$,
4. the subgroup $\langle L, C \rangle$ is nilpotent (the subgroup $\langle L, N \rangle$ is always nilpotent).

For any family of integers $\{b_i |\ i \in \mathbb{Z}\}$ the system of equations

$$-x_{i-1} + x_i = b_i, \quad i \in \mathbb{Z}$$

always has a solution in $\mathbb{Z}$: one can simply take any $x_0 = a_0 \in \mathbb{Z}$ and then continue:

$$\begin{cases} x_1 = a_1 = b_1 + a_0, \\ x_2 = a_2 = b_2 + a_1, \\ x_3 = a_3 = b_3 + a_2, \\ \dots\dots\dots\dots\dots\dots \end{cases} \quad \text{and} \quad \begin{cases} x_{-1} = a_{-1} = a_0 - b_0, \\ x_{-2} = a_{-2} = a_{-1} - b_{-1}, \\ x_{-3} = a_{-3} = a_2 - b_2, \\ \dots\dots\dots\dots\dots\dots \quad . \end{cases} \tag{2}$$

Thus, we can inductively define $\mu_g^{(1)} = \varphi_g$, for all $g \in G$. And if $\mu_g^{(j-1)}$ already defined, set $\mu_g^{(j)}(c^i) = g^{a_i}$ for all $g \in G$ and $i \in \mathbb{Z}$, where a_i ($i \in \mathbb{Z}$) is the solution of the system (2) in the case when g^{b_i} is equal to $\mu_g^{(j-1)}(c^i)$. Then it is easy to check that $[c, \mu_g^{(j)}] = \left(\mu_g^{(j)}\right)^{-c} = \mu_g^j = \mu_g^{(j-1)}$. Set

$$L_n = \langle \mu_g^{(n)}, C, N |\ g \in G \rangle \cap \prod_{c^i \in C} G.$$

[4] In order to make our notation more similar to the definition $g_W = (1, f_g)\varphi_{g^{-1}}$, we could consider not the element φ_g but the element $\varphi_{g^{-1}}$. However, to shorten somewhat our notations we will use φ_g. Clearly, if φ_g belongs to a subgroup of W, then $\varphi_{g^{-1}}$ also belongs to that subgroup.

It is clear that $\varphi_g \in L_n$ $(g \in G)$ and $L_n^C \subseteq L_n$, $L_n^N \subseteq L_n$ (for $f \in N$ we have $f(\mu_g^{(j)}) = \mu_{f(g)}^{(j)}$).

Further we show that for our word set V one can choose a number n such that for arbitrary $g \in G$ the element φ_g lies in $V(L_n)$. Since V has a consequence of type $\gamma_k(x_1, \dots, x_k)$, always holds $\gamma_k(L_n) \subseteq V(L_n)$. In order to guarantee that φ_g lies in $\gamma_k(L_n)$, it suffices to take simply $n = k+1$ and to choose $L = L_{k+1}$.

Finally, let us show that the group $\langle L, C\rangle$ is nilpotent. By the criterion of Hall, since the group L is nilpotent, it will be sufficient to show that the factor group $\langle L, C\rangle/L'$ is also nilpotent. Consider the word $\gamma_n(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the generators of $\langle L, C\rangle/L'$. Since L/L' is abelian, $\gamma_n(x_1, \dots, x_n)$ is equal to 1 if in the set $\{x_1, \dots, x_n\}$ there are at least two elements from L. Thus, without loss of generality we can consider the case when x_1 is equal to $\mu_g^{(j)}$ and $x_2, \dots, x_n$ are equal to c. According to our construction of $\mu_g^{(j)}$:

$$\begin{gathered}
[\mu_g^{(j)}, c] \in \langle \mu_g^{(j-1)} \mid g \in G\rangle, \\
[[\mu_g^{(j)}, c], c] \in \langle \mu_g^{(j-2)} \mid g \in G\rangle, \\
\vdots \\
[\dots[\mu_g^{(j)}, c], \dots, c] = t \in \langle \mu_g^{(1)} \mid g \in G\rangle = \langle \varphi_g \mid g \in G\rangle.
\end{gathered}$$

And, evidently, $[t, c] = 1$. Theorem is proved. □

3.4. A generalization of a theorem of Burnside

In [6] Heineken generalized Burnside's theorem [3] that we mentioned at the beginning of this paper. Heineken found the criterion for verbal normal embeddability of finite p-groups in finite p-groups:

Let G be a finite p-group and let L be a Sylow p-subgroup of $\mathrm{Aut}(G)$. *Then for the given word w holds $w(H) \not\supseteq G$ for any nilpotent extension H of G if and only if $w(L) \not\supseteq$* $\mathrm{Inn}(G)$ *for any L.*

Before we generalize this theorem for infinite p-groups, let us notice that here it plays no role whether H is a finite p-group containing G as a normal subgroup, or any nilpotent group with that property. For, the proof of Theorem 3 shows that if the desired normal verbal embedding of a finite p-group into a group H is constructed, then H can also be constructed to be finite. Thus, $H = H_{p_1} \times \dots \times H_{p_n}$, where H_{p_i}, $i = 1, \dots, n$ are the Sylow subgroups of H, and where $p = p_{i_0}$. Then G is normal in $H_{p_{i_0}}$ and lies in $w(H_{p_{i_0}})$.

Theorem 6. *Let G be a p-group of finite exponent and let L be a Sylow p-subgroup of* $\mathrm{Aut}(G)$. *Then for an arbitrary non-trivial word set V holds $V(H) \not\supseteq G$ for any extension H of G (such that H is a p-group) if and only if $V(L) \not\supseteq$* $\mathrm{Inn}(G)$ *for any L.*

Proof. The necessity of this condition follows from the proof of Lemma 4: If the p-group H exists, then in $\mathrm{Aut}(G)$ there is a (to $H/C_H(\hat{G})$ isomorphic) p-subgroup P. The subgroup P lies in a Sylow p-subgroup L of the finite or infinite group [9] $\mathrm{Aut}(G)$. Thus: $V(L) \supseteq \mathrm{Inn}(G)$. Now assume that the p-subgroup L exists. The proof of Theorem 5 shows that it will be sufficient to consider as H the group $W(G, D, L)$. □

4. The normal verbal embeddings of SN^*-groups

4.1. SN^*-groups

A modification of our construction allows us to consider the normal verbal embeddings of SN^*-groups. This consideration seems to be natural because the generalized soluble and generalized nilpotent groups are natural generalizations of soluble and nilpotent groups. For background information on SN^*-groups we refer to [15, 9, 16]. A group G is an SN^*-group if for an ordinal number κ it has a system $\{G_\delta |\ \delta \leq \kappa\}$ of subgroups such that:

(1) $\{1\}, G \in \{G_\delta |\ \delta \leq \kappa\}$,

(2) $\bigcup_{\delta<\nu} G_\delta = G_\nu$ for limit ordinal numbers $\nu \leq \kappa$,

(3) $G_\beta \triangleleft G_{\beta+1}$ for non-limit ordinal numbers $\beta + 1 \leq \kappa$,

(4) the factor $G_{\beta+1}/G_\beta$ for any $\beta + 1 \leq \kappa$.

4.2. The criterion for the normal verbal embeddability for SN^*-groups

Theorem 7. *Let G be an arbitrary SN^*-group and $V \subseteq F_\infty$ be a non-trivial word set. Then there exists an SN^*-group $H = H(G, V)$ with a normal subgroup $\hat{G} \subseteq V(H)$ isomorphic to G if and only if there is an SN^*-subgroup B of* $\mathrm{Aut}(G)$ *such that $V(B) \supseteq \mathrm{Inn}(G)$.*

Remark 4. Of course, here we need not to consider the cases when the groups G and H are finite because the finite SN^*-groups simply are soluble. But here, too, we can prove that *if G is finitely generated, then G can also be chosen to be finitely generated; and if G is infinite, then G can be chosen to be of the same cardinality as G.*

Proof of Theorem 7. In the proof of Theorem 2, in order to construct a soluble group $W(G, D, B)$, we used a soluble subgroup $B \subseteq \mathrm{Aut}(G)$. The case of SN^*-groups is a little more complicated: We have to consider not the entire group $W(G, D, S)$, but a certain part of the latter. The problem is that if $\{G_\delta |\ \delta \leq \kappa\}$ is a subnormal soluble ascending series for G, then the system of subgroups $\left\{\prod_{d\in D} G_\delta |\ \delta \leq \kappa\right\}$ of the Cartesian product $\prod_{d\in D} G$ must not necessarily be a soluble ascending series for $\prod_{d\in D} G$ (the condition (2) may fail).

As in the proof of Lemma 2 let D be the group that we found for the variety $\mathfrak{V}$, let $a \in V(D)$ be an element of infinite order, and let τ_g, $g \in G$ be the elements defined in that proof. Denote $W^* = \langle \tau_g, D, S |\ g \in G\rangle$. It is clear that:

a. G_W lies in W^* because $\varphi_g \in W^*$ and $S \subseteq W^*$;

b. G_W is normal in W^* because G_W is normal even in $W = W(G, D, S)$;

c. G_W lies in $V(W^*)$ (see the proof of Lemma 2).

It remains to show that W^* is an SN^*-group provided that S and G are SN^*-groups. Denote

$$G^*_\delta = W^* \cap \prod_{d\in D} G_\delta, \quad \delta \leq \kappa.$$

It is easy to calculate that the conditions (1), (3) and (4) hold for G^*_δ, $\delta \leq \kappa$. Let us prove that the condition (2) also holds. Let θ be any element of G^*_ν for the limit ordinal ν. Since

$\theta \in W^*$, there is a finite set of elements

$$\begin{aligned} &\tau_{g_1}, \dots, \tau_{g_p} \text{ in } \prod_{d\in D} G, \\ &d_1, \dots, d_q \text{ in } D, \\ &s_1, \dots, s_r \text{ in } S, \end{aligned}$$

such that $\theta \in \langle \tau_{g_1}, \dots, \tau_{g_p};\ d_1, \dots, d_q;\ s_1, \dots, s_r \rangle$. Let $G_{\delta(\theta)}$ be the element of $\{G_\delta |\ \delta \le \kappa\}$ that contains the finite set of the elements

$$\tau_{g_1}(1), \dots, \tau_{g_p}(1) \text{ and } \big(s_j(\tau_{g_1})\big)(1), \dots, \big(s_j(\tau_{g_p})\big)(1);\ \text{where } j = 1, \dots, r.$$

The analysis of the operation of $d_1, \dots, d_q$ and of $s_1, \dots, s_r$ on $\prod_{d\in D} G$ shows that $\theta \in \prod_{d\in D} G_{\delta(\theta)}$. Thus $\theta \in G^*_{\delta(\theta)}$ holds, and $\{G^*_\delta |\ \delta \le \kappa\}$ is a subnormal soluble ascending series for $W^* \cap \prod_{d\in D} G$. In order to continue this series for the entire group W^* it remains to notice that the factor group $W^*/\big(W^* \cap \prod_{d\in D} G\big) = D \times S$ also contains subnormal soluble ascending series $\{Y_\rho |\ \rho \le \sigma\}$ (of length σ). Thus, we can define $G^*_{\kappa+\rho} = Y_\rho \cdot \big(W^* \cap \prod_{d\in D} G\big)$, $\rho \le \sigma$.

The necessity of the condition of the theorem is proved in the same manner as it was done in Lemma 4 and Lemma 5. We use the fact that the factor groups of SN^*-groups are also SN^*-groups. □

Proof of Remark 4. Here the same condition works as in the proof of the Remark in Section 2, together with the fact that the subgroups of SN^*-groups also are SN^*-groups. □

5. Economical embeddings

5.1. Embeddings into smaller varieties, the case of the word γ_n

Theorem 1 finds a general criterion for the normal verbal embeddability without taking into account the form of the words in V. In [6] and [7] a few more economical constructions are built for a few "well-known" words (x^m, $[x_1, x_2]$, etc.). This means: The analogs of the criterion of Theorem 1 can be proved for some words in such a way that the constructed group H is much smaller than what we get in the proof of Theorem 1.

The proof of Theorem 1 guarantees the embeddability of the group G into a group H from the variety $\operatorname{var}(G) \cdot (\mathfrak{N}_c \cup \operatorname{var}(A))$, where c is the smallest number such that for some rank n holds:

$$F_n(\mathfrak{N}_c) \notin \mathfrak{V} = \operatorname{var}\left(F_\infty / V(F_\infty)\right). \tag{3}$$

Let $\mathfrak{A}$ be the variety of all abelian groups and let $\mathfrak{A}_m$ be the variety of abelian groups of exponents dividing m.

Theorem 8. *Let G be any group. Only then there exists a group $H = H(G, \gamma_c)$ with a to G isomorphic normal subgroup $\hat{G} \subseteq \gamma_c(H)$, if $\gamma_c(\operatorname{Aut}(G)) \supseteq \operatorname{Inn}(G)$ holds. And if this group H exists, it can be chosen to belong to the variety* $\operatorname{var}(G) \cdot (\mathfrak{A} \cup \operatorname{var}(A))$.

Notice that the construction of Theorem 1 in this situation would only guarantee that H lies in $\mathrm{var}(G) \cdot (\mathfrak{N}_{c+1} \cup \mathrm{var}(A))$.

Proof. Apply the construction of the proof of Theorem 5. Here the group G is not necessarily nilpotent. But regardless of that fact the construction embeds G in a group H, where H is a subgroup of $W(G, C, \mathrm{Aut}(G))$. Evidently $W(G, C, \mathrm{Aut}(G)) \in \mathrm{var}(G) \cdot (\mathfrak{A} \cup \mathrm{var}(A))$ holds. □

5.2. The case with the word x^m

Theorem 9. *Let G be any group. Only then there exists a group $H = H(G, x^m)$ with a to G isomorphic normal subgroup $\hat{G} \subseteq (H)^m$, if $(\mathrm{Aut}(G))^m \supseteq \mathrm{Inn}(G)$ holds. And if this group H exists, it can be chosen to belong to the variety $\mathrm{var}(G) \cdot (\mathfrak{A}_m \cup \mathrm{var}(A))$.*

The proof of Theorem 1 guarantees the embeddability of the group G into a group H from the variety $\mathrm{var}(G) \cdot (\mathfrak{A} \cup \mathrm{var}(A))$ only.

Proof. We choose

$$h = W(G, C_m, A),$$

where $C_m = \langle c \rangle$ is a cyclic group of order m. Since the subgroup $G_W = \{(1, f_g)\varphi_{g^{-1}} |\ g \in G\}$ is normal in H, and since $\mathrm{Inn}(G)$ lies in $(\mathrm{Aut}(G))^m$, it remains to show that $\varphi_g \in H^m$, for all $g \in G$. Take $x = (c, 1)\pi$, where

$$\pi_g(c^i) = \begin{cases} g, & \text{if } i = 0, \\ 1, & \text{if } i \neq 0. \end{cases}$$

Then it is easy to compute that $x^m = \varphi_g$ and, thus, $\varphi_g \in H^m$. □

5.3. The case with the word $\delta_n(x_1, \ldots, x_{2^n})$

Let $\delta_n(x_1, \ldots, x_{2^n})$ be the word "of solubility": $\delta_0 = x$, and

$$\delta_{n+1}(x_1, \ldots, x_{2^{n+1}}) = [\delta_n(x_1, \ldots, x_{2^n}), \delta_n(x_{2^n+1}, \ldots, x_{2^{n+1}})],$$

$(n \in \mathbb{N})$. Let $\mathfrak{S}_n$ be the variety of the soluble groups of maximal solubility length n.

Theorem 10. *Let G be any group. Only then there exists a group $H = H(G, \delta_n)$ with a to G isomorphic normal subgroup $\hat{G} \subseteq \delta_n(H)$, when $\delta_n(\mathrm{Aut}(G)) \supseteq \mathrm{Inn}(G)$ holds. And if this group H exists, it can be chosen to belong to the variety $\mathrm{var}(G) \cdot (\mathfrak{S}_n \cup \mathrm{var}(A))$.*

The proof of Theorem 1 guarantees the embeddability of the group G into a group H from the variety $\mathrm{var}(G) \cdot (\mathfrak{N}_c \cup \mathrm{var}(A))$ only (the class c must be so large that one of the free groups of $\mathfrak{N}_c$ does not belong to $\mathfrak{V}$).

Proof. The group $W(G, C, A)$ (with an abelian C) cannot be used here because the extension E of G with an abelian group C must be soluble and it cannot contain a non-trivial verbal subgroup $\delta_n(E)$ (for the number n big enough). We embed G in the Cartesian wreath product $G \,\mathrm{Wr}\, C$ (where $C = \langle c \rangle$ is in infinite cyclic group): $g \mapsto \varphi_g$, $\quad g \in G$. Let $\mu_g = \mu_g^{(2)}$ be the element that we defined in the proof of Theorem 5. Then

1. the isomorphic copy $\{\varphi_g |\ g \in G\}$ of G lies in the commutator subgroup G_1' of the group $G_1 = \langle \mu_g, c |\ g \in G\rangle$,
2. for an arbitrary automorphism $f \in \mathrm{Aut}(G)$ there is an $f_1 \in \mathrm{Aut}(G_1)$ whose restriction on the copy of G coincides with f.

We can continue this process and embed G_1 into a corresponding subgroup G_2 of $G_1 \,\mathrm{Wr}\, C$), etc.. Finally, we embed G into a soluble group G_n so that

1. the isomorphic copy $\{\varphi_g |\ g \in G\}$ of G lies in $\delta_n(G_n)$,
2. for an arbitrary automorphism $f \in \mathrm{Aut}(G)$ there is an $f_1 \in \mathrm{Aut}(G_n)$ whose restriction on the copy of G coincides with f.

Let $\varphi_g^{(n)}$ be the image of g in G_n. The condition 2 allows us to build an extension of G_n with $\mathrm{Aut}(G)$. This extension E contains in its verbal subgroup $\delta_n(E)$ the normal subgroup $\{f_g \varphi_{g^{-1}}^{(n)} |\ g \in G\}$. □

We could continue the research of the current paper and prove, for example, the analogs of theorems of this section for normal verbal embeddings of nilpotent and soluble groups into nilpotent and soluble groups, respectively. Also, the results of the previous section about SN^*-groups have analogs for a few other classes of generalized soluble and generalized nilpotent groups. However, we do not include those results in the current paper because here our main aim is to present a construction for normal verbal embeddings for groups.

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