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Completeness Criteria and Invariants for Operation and Transformation Algebras

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Abstract. Operation algebras serve as representations of composition algebras (in the sense of Lausch/Nöbauer). In this paper they are described and characterized by invariant relations as Galois-closed sets w.r.t. a suitable Galois connection. Further, the completeness problem in operation algebras is considered and solved for concrete cases (e.g. for transformation $(\max, \circ)$-semirings).

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Introduction

In this paper *operation algebras* are investigated, i.e. algebras whose elements are operations (of fixed arity k) on a set A and whose fundamental operations are induced by an algebra (from some fixed class (variety) $\mathcal{K}$) on the base set A and also include composition; for unary mappings ($k = 1$) such algebras will be called *transformation algebras*.
One of the motivations to study operation algebras comes from the observation that such algebras are concrete cases of the so-called composition algebras introduced in [4, Ch. 3] (special examples are near-rings, algebras of binary relations and distributive lattices, see 4.2) — the most general approach known to the authors generalizing the classical Cayley theorem

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that every group is isomorphic to a group of permutations where composition plays the role of the group multiplication (see 4.4). It shows that algebras (from some fixed class (variety) $\mathcal{K}$) with an additional $(k+1)$-ary operation $\varkappa$ satisfying "composition-like" properties can be represented as concrete operation algebras where $\varkappa$ is given by composition (see 1.6). The connection between composition algebras (in a setting slightly more general than in [4]) and operation algebras will be reported in Section 1.

In Section 2 we show how operation algebras can be described and characterized via invariant relations; they are the Galois closed elements with respect to a suitable Galois connection (see 2.4).

The completeness problem in operation algebras is considered in Section 3, i.e. we ask for systems of operations generating the full operation $\mathcal{K}$-algebra. For near-rings this was investigated in [1], here we present a much more general approach which can be applied not only to arbitrary operation algebras but also to other structures.

Finally, in Section 4 we shall consider the case where $\varkappa$ is binary (this means $k = 1$, i.e. we deal with representations of algebras by unary operations). Here we discuss some problems connected with the concrete representation of composition algebras (introduced in Section 1) by operations on some base set A, in particular we ask for minimal representations (a representation is called minimal if A has minimal size) and give several examples for transformation algebras. The completeness theorem will be demonstrated for transformation $(\max, \circ)$-semirings over $\{1, \dots, n\}$.

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1. $\mathcal{K}$-composition algebras and their representation by operation $\mathcal{K}$-algebras

Let $\mathrm{Alg}(\tau)$ denote the class of all algebras of a given type τ. Throughout the paper $\mathcal{K}$ denotes a quasivariety of type τ, i.e. a subclass of $\mathrm{Alg}(\tau)$ which is closed with respect to isomorphic copies, subalgebras and direct powers; thus $\mathcal{K}$ may be any variety. Moreover, we assume that $\mathcal{K}$ is nontrivial, i.e. it does not consist of one-element algebras only. The algebras in $\underline{A} \in \mathcal{K}$ must be of type τ. However, often the type is not very essential; then we also use the non-indexed form $\underline{A} = \langle A; U\rangle$; it has to be understood as the algebra $\langle A; (f_i)_{i\in I}\rangle \in \mathcal{K}$ with $U = \{f_i \mid i \in I\}$. Further, k denotes a positive integer.

1.1. $\mathcal{K}$-composition algebras. Let $\langle B; (f_i)_{i\in I}\rangle \in \mathrm{Alg}(\tau)$ and suppose that $\varkappa$ is a $(k+1)$-ary operation on the set B. Then $\langle B; (f_i)_{i\in I}, \varkappa\rangle$ will be called a *(k-dimensional) $\mathcal{K}$-composition algebra* ([4, p. 73]), if

(i) $\langle B; (f_i)_{i\in I}\rangle \in \mathcal{K}$,

(ii) $\varkappa$ is *superassociative*, i.e. we have

$$\varkappa(\varkappa(x_0, x_1, \dots, x_k), y_1, \dots, y_k) = \varkappa(x_0, \varkappa(x_1, y_1, \dots, y_k), \dots, \varkappa(x_k, y_1, \dots, y_k))$$

for all $x_0, \dots, x_k, y_1, \dots, y_k \in B$.

(iii) $\varkappa$ is *right-distributive* with respect to every f_i, i.e.
for every $i \in I$ and all $x_1, \dots, x_n, y_1, \dots, y_k \in B$ (n being the arity of f_i) we have

$$\begin{aligned}
&\varkappa(f_i, y_1, \dots, y_k) = f_i \quad \text{for nullary } f_i,\\
&\varkappa(f_i(x_1, \dots, x_n), y_1, \dots, y_k) =\\
&\qquad\qquad f_i(\varkappa(x_1, y_1, \dots, y_k), \dots, \varkappa(x_n, y_1, \dots, y_k)) \quad \text{for } n\text{-ary } f_i.
\end{aligned}$$

1.2. Selector systems. A *selector system* ([4, p. 73]) for an operation $\varkappa : B^{k+1} \to B$ is a family $s_1, \dots, s_k$ of elements of B such that for all $y_0, y_1, \dots, y_k \in B$ and $i \in \{1, \dots, k\}$

$$\begin{aligned}
&\varkappa(s_i, y_1, \dots, y_k) = y_i \quad \text{and}\\
&\varkappa(y_0, s_1, \dots, s_k) = y_0\,.
\end{aligned}$$

A $\mathcal{K}$-composition algebra $\langle B; (f_i)_{i \in I}, \varkappa\rangle$ is called $\mathcal{K}$-composition algebra with selector system if there exists a selector system for $\varkappa$.

1.3. Remarks. a) The class of $\mathcal{K}$-composition algebras shares many algebraic properties with the underlying class $\mathcal{K}$, e.g. closedness with respect to subalgebras, products or homomorphisms. In particular, if $\mathcal{K}$ is a variety then the class of all $\mathcal{K}$-composition algebras is a variety, too.
b) If there exists a selector system for a $\mathcal{K}$-composition algebra then it is unique ([4, Ch. 3, 1.11]).
c) For $k = 1$ superassociativity of $\varkappa$ reduces to ordinary associativity and a selector system is just an identity for the binary operation $\varkappa$.

1.4. Notation. Let $\underline{A} = \langle A; (f_i)_{i \in I}\rangle \in \mathcal{K}$. For any set B let A^B denote the set of all mappings $f : B \to A$. Let $i \in I$ and let n denote the arity of f_i. Then f_i induces an n-ary operation $\hat{f}_i$ on A^B as follows: For every $h_1, \dots, h_n \in A^B$ define $\hat{f}_i(h_1, \dots, h_n) \in A^B$ by setting for every $b \in B$

$$(\hat{f}_i(h_1, \dots, h_n))(b) := f_i(h_1(b), \dots, h_n(b)) \tag{1.4.1}$$

(note that $\hat{f}_i$ is just the operation f_i in the Cartesian power $\underline{A}^B$ of the algebra $\underline{A}$). If there is no danger of confusion we shall omit the hat and write f instead of $\hat{f}$, in particular if we use special signs for f (like $+$ or $\wedge$).
In case $B = A^k$ the set A^B gives the so-called *(full) Menger algebra* $\langle A^{A^k}; \varkappa_A\rangle$ of k-ary operations, where $\varkappa_A$ is the composition (superposition) defined by

$$\varkappa_A(f_0, f_1, \dots, f_k) := f_0(f_1, \dots, f_k) \tag{1.4.2}$$

(here $f_0 : A^k \to A$ acts as $\hat{f}_0$ according to 1.4.1). For $k = 1$ the operation $\varkappa_A$ is just the usual composition $\circ$ of unary mappings: $(f_0 \circ f_1)(b) = f_0(f_1(b))$.

1.5. Operation $\mathcal{K}$-algebras. Given an algebra $\underline{A} = \langle A; (f_i)_{i \in I}\rangle \in \mathcal{K}$ it is easy to check that $\langle A^{A^k}; (\hat{f}_i)_{i \in I}, \varkappa_A\rangle$ is a (k-dimensional) $\mathcal{K}$-composition algebra; we call this algebra the *full*

(k-dimensional) operation $\mathcal{K}$-algebra over $\underline{A}$. It has a selector system namely the projections $e_1^k, \ldots, e_k^k$ defined by $e_i^k(a_1, \ldots, a_k) = a_i$ $(i \in \{1, \ldots, k\})$.
By a *(k-dimensional) operation $\mathcal{K}$-algebra* we understand any subalgebra of a full (k-dimensional) operation algebra over some $\underline{A} \in \mathcal{K}$ (1-dimensional operation algebras are also called *transformation $\mathcal{K}$-algebras*, see [3]). Since $\mathcal{K}$ is closed under subalgebras, such operation algebras are $\mathcal{K}$-composition algebras in the sense of Definition 1.1. The following theorem shows that the converse is also true up to isomorphism.

1.6. Representation Theorem. ([4, Ch. 3, Thm. 1.51]) *Let $\mathcal{K}$ be a quasivariety. Then every (k-dimensional) $\mathcal{K}$-composition algebra is isomorphic to some (k-dimensional) operation $\mathcal{K}$-algebra.*

1.7. Remark. The proof of Theorem 1.6 is constructive and generalizes the proof of the Cayley representation theorem for (semi-)groups. It was given in [4] for varieties but this proof also works for quasivarieties. The proof is particularly easy in case of algebras with a selector system: If $\underline{B} = \langle A; (f_i)_{i \in I}, \varkappa \rangle$ is a k-dimensional $\mathcal{K}$-composition algebra with selector system, then the mapping

$$a \mapsto \varphi_a\,, \quad \text{where } \varphi_a : A^k \to A : (a_1, \ldots, a_k) \mapsto \varkappa(a, a_1 \ldots, a_k),$$

is an embedding of $\underline{B}$ into the full k-dimensional operation $\mathcal{K}$-algebra over $\underline{A} = \langle A; (f_i)_{i \in I} \rangle$.

Examples will be considered in Section 4.

2. Characterization of operation $\mathcal{K}$-algebras by invariant relations

In the previous section we have seen that $\mathcal{K}$-composition algebras can be represented by operation $\mathcal{K}$-algebras. In this section we describe and characterize operation $\mathcal{K}$-algebras by invariant relations. However, now we shall restrict to $\mathcal{K}$-composition algebras with selector system. This means that we shall consider operation $\mathcal{K}$-algebras which always contain the projections $e_1^k, \ldots, e_k^k$. In most cases this is not a real restriction since one can just add a selector system.

2.1. Some notions and notation. For operations and relations on a fixed base set A we introduce the following notation:

$$\begin{aligned}
\mathrm{Op}^{(k)}(A) &:= A^{A^k} = \{f \mid f : A^k \to A\} && (k\text{-ary operations}),\\
\mathrm{Op}(A) &:= \bigcup_{k=1}^{\infty} \mathrm{Op}^{(k)}(A) && (\text{finitary operations}),\\
\mathrm{Rel}^{(m)}(A) &:= \{\varrho \mid \varrho \subseteq A^m\} && (m\text{-ary relations}),\\
\mathrm{Rel}(A) &:= \bigcup_{m=1}^{\infty} \mathrm{Rel}^{(m)}(A) && (\text{finitary relations}).
\end{aligned}$$

An m-tuple $r \in A^m$ may be regarded as a mapping $r : \underline{m} \to A$ (with $\underline{m} := \{1, \ldots, m\}$), and its components are given by $r = (r(1), \ldots, r(m))$.

A relation $\varrho \in \mathrm{Rel}^{(m)}(A)$ is *invariant* for an operation $f \in \mathrm{Op}^{(k)}(A)$ (also f *preserves* ϱ, or f is a *polymorphism* of ϱ) if for all $r_1, \dots, r_k \in \varrho$ we have $f[r_1, \dots, r_k] \in \varrho$. Here the m-tuple $f[r_1, \dots, r_k]$ is defined component-wise by $f[r_1, \dots, r_k](i) := f(r_1(i), \dots, r_k(i))$ $(i \in \underline{m})$.
For $F, S, U \subseteq \mathrm{Op}(A)$ and $Q \subseteq \mathrm{Rel}(A)$ we define

$$\begin{aligned}
\mathrm{Pol}\, Q &:= \{f \in \mathrm{Op}(A) \mid \text{every } \varrho \in Q \text{ is invariant for } f\}\,,\\
{}^{(k)}\mathrm{Pol}\, Q &:= \mathrm{Op}^{(k)}(A) \cap \mathrm{Pol}\, Q\,,\\
\mathrm{End}\, Q &:= {}^{(1)}\mathrm{Pol}\, Q\,,\\
{}^{S}\mathrm{Pol}\, Q &:= S \cap \mathrm{Pol}\, Q\,,\\
\mathrm{Inv}\, F &:= \{\varrho \in \mathrm{Rel}(A) \mid \varrho \text{ is invariant for every } f \in F\}\,,\\
{}^{U}\mathrm{Inv}\, F &:= \mathrm{Inv}\, U \cap \mathrm{Inv}\, F\,.
\end{aligned}$$

Note that the notation Pol stands for polymorphisms not for polynomials!

2.2. Galois connections derived from Pol – Inv. The operators Pol and Inv form a Galois connection between sets of mappings and sets of finitary relations on a base set A. From this one can derive the Galois connections ${}^{(k)}\mathrm{Pol} - \mathrm{Inv}$ and $\mathrm{End} - \mathrm{Inv}$ or, more general, ${}^{S}\mathrm{Pol} - {}^{U}\mathrm{Inv}$.
The Galois closed subsets of $\mathrm{Op}(A)$ and of $\mathrm{Rel}(A)$ are well-known (see e.g. [9, 1.2.1, 1.2.3]). For $F \subseteq \mathrm{Op}(A)$, $H \subseteq \mathrm{Op}^{(1)}(A)$ and $Q \subseteq \mathrm{Rel}(A)$ with finite A we have (for infinite A some modifications are necessary, [6], [7]):

$$\begin{array}{rclcl}
F &=& \mathrm{Pol}\,\mathrm{Inv}\, F & \Longleftrightarrow & F \text{ is a clone,}\\
F &=& {}^{(k)}\mathrm{Pol}\,\mathrm{Inv}\, F & \Longleftrightarrow & F \text{ is a Menger algebra on } A \text{ of order } k,\\
H &=& \mathrm{End}\,\mathrm{Inv}\, H & \Longleftrightarrow & H \text{ is a transformation monoid,}\\
Q &=& \mathrm{Inv}\,\mathrm{Pol}\, Q & \Longleftrightarrow & Q \text{ is a relational algebra,}\\
Q &=& \mathrm{Inv}\,{}^{(k)}\mathrm{Pol}\, Q & \Longleftrightarrow & Q \text{ is a } k\text{-locally closed relational algebra,}\\
Q &=& \mathrm{Inv}\,\mathrm{End}\, Q & \Longleftrightarrow & Q \text{ is a weak Krasner algebra.}
\end{array}$$

All necessary notions will be defined below. In general, for $F, S, U \subseteq \mathrm{Op}(A)$ and finite A we have:

$$ {}^{S}\mathrm{Pol}\,{}^{U}\mathrm{Inv}\, F = S \cap \mathrm{clone}(F \cup U)\,. \tag{2.2.1}$$

In fact, $S \cap \mathrm{clone}(F \cup U) = S \cap \mathrm{Pol}\,\mathrm{Inv}(F \cup U) = S \cap \mathrm{Pol}(\mathrm{Inv}\, F \cap \mathrm{Inv}\, U) = {}^{S}\mathrm{Pol}\,{}^{U}\mathrm{Inv}\, F$.
For an arbitrary S there is no general procedure how to characterize the corresponding Galois closed sets of relations. However, if we assume that S is a clone then there exists a set $Q_0 \subseteq \mathrm{Rel}(A)$ such that $S = \mathrm{Pol}\, Q_0$ (e.g. one can take $Q_0 = \mathrm{Inv}\, S$) and for any such Q_0 we have for finite A (see [8, Thm. 3.2]):

$$ {}^{U}\mathrm{Inv}\,{}^{S}\mathrm{Pol}\, Q = \mathrm{Inv}\, U \cap [Q_0 \cup Q]_{\mathsf{RA}} \tag{2.2.2}$$

(notation below). Recall that a *clone* on A is a composition closed set of operations on A containing all projections $e_i^k \in \mathrm{Op}^{(k)}(A)$, $k \in \mathbb{N}$. The clone generated by a set $F \subseteq \mathrm{Op}(A)$ will be denoted by $\mathrm{clone}(F)$. Note that the k-ary operations $F^{(k)}$ of a clone $F \subseteq \mathrm{Op}(A)$

always form a *(concrete) Menger algebra* $\langle F^{(k)}; \varkappa_A\rangle$, i.e. a subset of $\mathrm{Op}^{(k)}(A)$ which is closed under $\varkappa_A$ (see 1.4).
The notion of relational clone is less common. For finite A it coincides with the following notion of a relational algebra (in the sense of e.g. [9]; however it is different from Tarski's relation algebra of binary relations): A *relational algebra* Q is a set of relations in $\mathrm{Rel}(A)$ which is closed under the following (set-theoretical) operations:

- Δ_A (nullary operation: Q must contain the *diagonal* (or equality) relation $\Delta_A := \{(a,a) \mid a \in A\}$),
- $\cap$ (*intersection* of relations of the same arity),
- $\times$ (*product*: for m-ary ϱ and s-ary σ let

$$\varrho \times \sigma = \{(a_1,\dots,a_m,b_1,\dots,b_s) \in A^{m+s} \mid (a_1,\dots,a_m) \in \varrho,\ (b_1,\dots,b_s) \in \sigma\}\,),$$

- pr_I (*projection* onto a subset I of coordinates: for m-ary ϱ and $I = \{i_1,\dots,i_t\}$ with $1 \le i_1 < i_2 < \dots < i_t \le m$ we define

$$\mathrm{pr}_I(\varrho) := \{(a_{i_1},\dots,a_{i_t}) \mid \exists a_j (j \in \{1,\dots,m\} \setminus I) : (a_1,\dots,a_m) \in \varrho\})\,,$$

- π_α (*permutation* of coordinates: for m-ary ϱ and a permutation α of $\{1,\dots,m\}$ let $\pi_\alpha(\varrho) := \{(a_{\alpha(1)},\dots,a_{\alpha(m)}) \mid (a_1,\dots,a_m) \in \varrho\})$.

If, in addition, Q is also closed with respect to

- $\cup$ (*union* of relations of the same arity),

then Q is called a *weak Krasner algebra.*

For a positive integer k a relational algebra Q is called *k-locally closed* if $k\text{-LOC}\,Q = Q$ where

$$k\text{-LOC}\,Q := \{\varrho \in \mathrm{Rel}(A) \mid \forall r_1,\dots,r_k \in \varrho\, \exists \sigma \in Q : \{r_1,\dots,r_k\} \subseteq \sigma \subseteq \varrho\}\,.$$

The relational algebra (weak Krasner algebra) generated by a set $Q \subseteq \mathrm{Rel}(A)$ will be denoted by $[Q]_{\mathsf{RA}}$ ($[Q]_{\mathsf{WKA}}$, resp.).

2.3. Lemma. *$F \subseteq A^{A^k}$ is an operation $\mathcal{K}$-algebra with selector system over $\underline{A} = \langle A; U\rangle$ if and only if $F = \mathrm{Op}^{(k)}(A) \cap \mathrm{clone}(F \cup U)$.*

Proof. Let $\langle F; (\hat{f}_i)_{i\in I}, \varkappa_A\rangle$ be an operation $\mathcal{K}$-algebra over $\underline{A} = \langle A; (f_i)_{i\in I}\rangle$, $U := \{f_i \mid i \in I\}$. By Definition 1.5 it can be characterized as a subalgebra $\langle F; \varkappa_A\rangle$ of the full Menger algebra $\langle A^{A^k}; \varkappa_A\rangle$ which is closed with respect to each $\hat{f}_i$, i.e. $\hat{f}_i(g_1,\dots,g_{n_i}) \in F$ for $g_1,\dots,g_{n_i} \in F$, $i \in I$.
But these properties can be reformulated: $\hat{f}_i(g_1,\dots,g_{n_i})$ is just a composition of the operations $f_i, g_1,\dots,g_{n_i}$ (see 1.4) and thus belongs to the clone generated by F and f_i, and so also to $\mathrm{clone}(F \cup U)$. Conversely, every k-ary operation in $\mathrm{clone}(F \cup U)$ must belong to F.
To conlude the proof we mention that the projections $e_1^k,\dots,e_k^k$ belong to every clone and therefore only algebras with selector system are characterized by the condition of the lemma. □

Note that by 2.2 every Menger algebra $F \subseteq A^{A^k}$ can be characterized by invariant relations: $F = {}^{(k)}\mathrm{Pol}\,\mathrm{Inv}\,F$. Therefore it makes sense to ask how to characterize those Menger algebras which are at the same time operation $\mathcal{K}$-algebras with respect to a given algebra $\underline{A} = \langle A; U\rangle \in \mathcal{K}$. Following a general approach described already in [7, 15.1, page 84] we get those Menger algebras as Galois closed elements of the restricted Galois connection ${}^{(k)}\mathrm{Pol} - {}^{U}\mathrm{Inv}$.

2.4. Theorem. *Let $\underline{A} = \langle A; U\rangle$ be some finite algebra in $\mathcal{K}$. Then $F \subseteq A^{A^k}$ is an operation $\mathcal{K}$-algebra over $\underline{A}$ if and only if*

$$F = {}^{(k)}\mathrm{Pol}\,{}^{U}\mathrm{Inv}\,F\,.$$

In particular (for $k = 1$), a set $H \subseteq A^A$ is a transformation $\mathcal{K}$-algebra over $\underline{A}$ if and only if

$$H = \mathrm{End}\,{}^{U}\mathrm{Inv}\,H\,.$$

Proof. By Lemma 2.3, F is an operation $\mathcal{K}$-algebra if and only if $F = A^{A^k} \cap \mathrm{clone}(F \cup U)$. Now take into account that $A^{A^k} \cap \mathrm{clone}(F \cup U) = {}^{(k)}\mathrm{Pol}\,{}^{U}\mathrm{Inv}\,F$ (by 2.2.1). □

2.5. Remark. Let us consider the case of transformation $\mathcal{K}$-algebras ($k = 1$). According to 2.2 we have $[Q]_{\mathsf{WKA}} = \mathrm{Inv}\,\mathrm{End}\,Q$ (for finite A). Consequently, from 2.4 we get

$$\mathrm{Inv}\,H = [{}^{U}\mathrm{Inv}\,H]_{\mathsf{WKA}}$$

for a transformation $\mathcal{K}$-algebra H over $\langle A; U\rangle$. The U-invariants ${}^{U}\mathrm{Inv}\,H$ always form a relational algebra (since it is the intersection of relational algebras, see 2.1 and 2.2). It follows that every invariant relation $\varrho \in \mathrm{Inv}\,H$ is a union of U-invariant relations.

3. Completeness

In this section we want to characterize generating sets F of the full operation $\mathcal{K}$-algebras over some given finite algebra $\underline{A} = \langle A; U\rangle \in \mathcal{K}$. It follows from Theorem 2.4 that the subalgebra of $\langle A^{A^k}; \hat{U}, \varkappa_A\rangle$ generated by F equals $A^{A^k} \cap \mathrm{clone}(F \cup U) = {}^{(k)}\mathrm{Pol}\,{}^{U}\mathrm{Inv}\,F$. We slightly generalize this problem using an arbitrary set S of operations instead of A^{A^k}. Thus we are faced with a typical completeness problem "does $S \cap \mathrm{clone}(F \cup U) = S$ hold?" and we shall connect it with methods known from clone theory.

3.1. Notions and Notation. Let $S, U \subseteq \mathrm{Op}(A)$ be arbitrary fixed sets of operations. For $F \subseteq S$ set

$$\overline{F} := S \cap \mathrm{clone}(F \cup U) \tag{3.1.1}$$

(see Fig. 1). We shall say that $F \subseteq S$ is a *U-set* (with respect to S) if $\overline{F} = F$. A U-set $N \neq S$ is said to be *maximal* if for every U-set P, from $N \subset P \subseteq S$ it follows that $P = S$. A set $F \subseteq S$ is called *U-complete* (or F is a *generating system*) if $\overline{F} = S$, i.e. together with U it generates all elements of S.

3.2. Lemma. 1. *The operator $F \mapsto \overline{F}$ is an algebraic closure operator on $\mathcal{P}(S)$.*

2. *If C is a clone containing U, then $S \cap C$ is a U-set.*

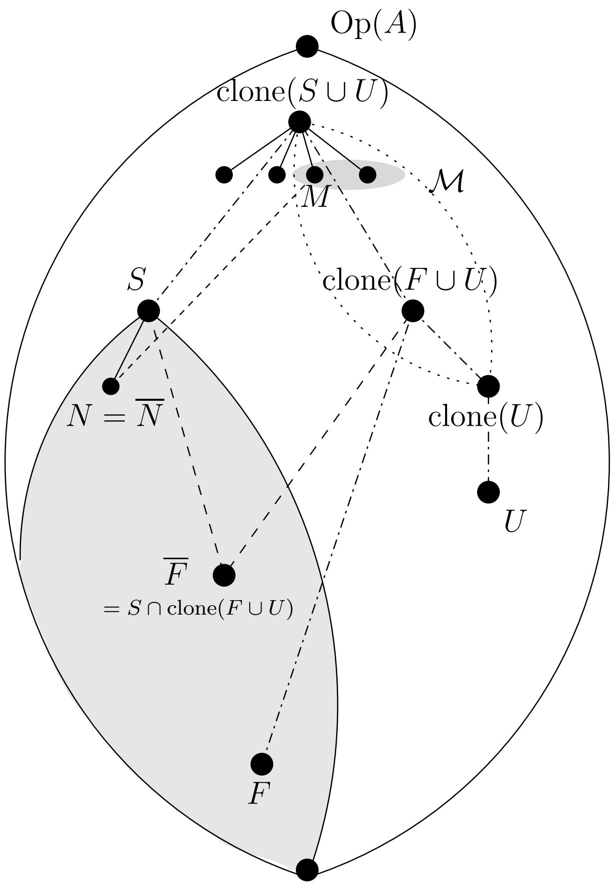

Figure 1. The closure operator $F \mapsto \overline{F}$ and a U-maximal set N

3. *$\overline{F}$ is the least U-set containing F (for $F \subseteq S$).*

Proof. 1. $\overline{F}$ is a Galois closure for the Galois connection ${}^S\mathrm{Pol} - {}^U\mathrm{Inv}$ (see 2.2.1), thus $F \mapsto \overline{F}$ is a closure operator. It is also algebraic since it is an intersection (of S) with an algebraic closure operator (namely $F \mapsto \text{clone}(F \cup U)$).

2. Assume $U \subseteq C$. From $(S \cap C) \cup U \subseteq C$ it follows immediately that $\text{clone}((S \cap C) \cup U) \subseteq C$. Now, taking intersection of both sides with S gives $\overline{S \cap C} \subseteq S \cap C$.

3. From 2. it follows that $\overline{F}$ is a U-set. Let H be a U-set such that $F \subseteq H$. Then $\overline{F} \subseteq \overline{H} = H$. □

In clone theory there are several completeness criteria via maximal clones which can easily be adapted to U-completeness (here the maximal subclones of a clone B will be called *B-maximal*):

3.3. *U*-completeness criterion. *Suppose $B := \text{clone}(S \cup U)$ is a finitely generated clone. Then we have:*

(1) *A set $F \subseteq S$ is U-complete if and only if for every B-maximal clone M with $U \subseteq M$ we have $F \not\subseteq M$.*

(1′) *A set $F \subseteq S$ is U-complete if and only if for every maximal U-set N we have $F \not\subseteq N$.*

Thus, for the completeness problem it becomes essential to know the maximal U-sets. Therefore, before giving the proof of 3.3, we shall relate maximal U-sets to maximal subclones of the clone $B = \mathrm{clone}(S \cup U)$.

Recall that for a finitely generated clone B there are only finitely many B-maximal clones and every proper subclone of B is contained in a B-maximal clone (see e.g. [9, 4.1.2, 4.1.1]).

3.4. Proposition. *Suppose $B := \mathrm{clone}(S \cup U)$ is a finitely generated clone and let $\mathcal{M} = \{M_1, M_2, \dots, M_s\}$ be the set of all B-maximal clones that contain U. Then we have*

(2) *S is a finitely generated U-set and every U-set not equal to S is contained in some maximal U-set.*

(3) *Let N be a maximal U-set. Then there exists an $M \in \mathcal{M}$ such that $N = S \cap M$ (see Fig. 1).*

(4) *Let N be a maximal U-set and let $P \supset N$ be a subset of S properly containing N. Then* $\mathrm{clone}(P \cup U) = B$.

(5) *Let $\mathcal{N}$ be the partially ordered set formed by $S \cap M_1$, $S \cap M_2$, ... , $S \cap M_s$ under inclusion. A U-set N is maximal if and only if it is a maximal element of $\mathcal{N}$. In particular, there are only finitely many maximal U-sets.*

Proof of 3.3 and 3.4. (1) "$\Leftarrow$": Let $C := \mathrm{clone}(F \cup U)$. Clearly, C is contained in no B-maximal clone (for every B-maximal clone M we have either $F \not\subseteq M$, or $U \not\subseteq M$). Since B is finitely generated, it follows that $C = B$ whence $\overline{F} = S \cap C = S \cap B = S$.

"$\Rightarrow$": Suppose $\overline{F} = S$ but $F \subseteq M$ for some B-maximal clone $M \supseteq U$. Then $\mathrm{clone}(F \cup U) \subseteq M$ since $U \subseteq M$, so $S = \overline{F} = S \cap \mathrm{clone}(F \cup U) \subseteq M$. Therefore, $S \cup U \subseteq M$ whence $B \subseteq M$ which contradicts the fact that M is B-maximal.

(2): It is easy to see that $S \not\subseteq M$ for all $M \in \mathcal{M}$ (if $S \subseteq M$ then $U \cup S \subseteq M$ whence $B = M$, which contradicts the fact that M is a maximal B-clone). Take arbitrary $f_1 \in S \setminus M_1$, ... , $f_s \in S \setminus M_s$. Then by 3.3(1) we have $\overline{\{f_1, \dots, f_s\}} = S$. By Lemma 3.2(1), the lattice of U-sets is algebraic. Since S is finitely generated, it follows now that maximal U-sets exist and every U-set $\neq S$ is contained in a maximal one.

(3): Let $C := \mathrm{clone}(N \cup U)$. Since N is a maximal U-set, $S \cap C = N \subset S$, and thus $C \neq B$. Since B is finitely generated, there exists a B-maximal clone M such that $C \subseteq M$. Since $U \subseteq C \subseteq M$ we have $M \in \mathcal{M}$. We show that M is the B-maximal clone we are looking for, i.e., $S \cap M = N$. Suppose $S \cap M \neq N$. Then $S \cap M \supset N$, so take any $f \in (S \cap M) \setminus N$ and let $P := \overline{N \cup \{f\}}$. Since $N \subset P$ and the U-set N is maximal, it follows that $P = S$. Now, $S = P = S \cap \mathrm{clone}(N \cup \{f\} \cup U) \subseteq S \cap M$. This shows $S \subseteq M$. On the other hand, we already know that $U \subseteq M$, whence $\mathrm{clone}(S \cup U) = B \subseteq M$. This contradicts the fact that M is a maximal B-clone.

(4): Let $C := \mathrm{clone}(P \cup U)$ and suppose $C \neq B$. Since B is finitely generated, C is contained in some B-maximal clone M. From $U \subseteq C \subseteq M$ and Lemma 3.2(2), it follows that $S \cap M$ is a U-set. Clearly, $S \cap M \neq S$, so by (3) there is a maximal U-set N' such that $S \cap M \subseteq N'$. We now have $N \subset P \subseteq S \cap C \subseteq S \cap M \subseteq N'$, that is $N \subset N'$, which contradicts the fact that both N and N' are maximal U-sets.

(5) "$\Rightarrow$": Suppose that N is a maximal U-set. Then by (3) there is a $j \in \{1, \dots, s\}$ such that $N = S \cap M_j$. If $S \cap M_j$ is not a maximal element of $\mathcal{M}$, there is a $k \in \{1, \dots, s\}$ such that $S \cap M_j \subset S \cap M_k$. Since the U-set N is maximal, by (4), we obtain $\mathrm{clone}((S \cap M_k) \cup U) = B$. Now $S \cap M_k = \overline{S \cap M_k} = S \cap \mathrm{clone}((S \cap M_k) \cup U) = S \cap B = S$. Therefore, $S \cup U \subseteq M_k$ which implies $M_k = B$ – contradiction with the fact that M_k is B-maximal.

"$\Leftarrow$": Take $j \in \{1, \dots, s\}$ such that $S \cap M_j$ is a maximal element of $\mathcal{N}$ and let $N = S \cap M_j$. Suppose further that the U-set N is not maximal. Then there is a maximal U-set P such that $P \supset N$. According to (3) there is a k such that $P = S \cap M_k$. But then $S \cap M_k \supset S \cap M_j$, which is impossible due to the choice of j.

(1′) is a direct consequence of (1) and (5), but it follows also from (2). $\square$

The maximal clones in the lattice of all clones (i.e. the $\mathrm{Op}(A)$-maximal clones) are well known. They can be described as $\mathrm{Pol}\,\varrho$ where ϱ is a relation from one of six finite classes described by I. Rosenberg ([12], see also [9, Ch. 4.3]) – we shall call these relations *Rosenberg relations*. Thus combining 3.3(1′) with 3.4(5) we immediately get the following theorem.

3.5. Theorem. *Suppose* $\mathrm{clone}(S \cup U) = \mathrm{Op}(A)$. *Let* Q_0 *denote the set of all Rosenberg relations which are contained in* $\mathrm{Inv}\,U$, *and let* Q_1 *be the set of all* $\varrho \in Q_0$ *such that* $S \cap \mathrm{Pol}\,\varrho$ *is maximal in the partially ordered set* $\mathcal{N} := \langle \{S \cap \mathrm{Pol}\{\varrho\} \mid \varrho \in Q_0\}; \subseteq \rangle$. *Then:*

(1) *A U-set N is maximal if and only if there exists a* $\varrho \in Q_1$ *such that* $N = S \cap \mathrm{Pol}\{\varrho\}$.

(2) *A set* $F \subseteq S$ *is U-complete if and only if for every* $\varrho \in Q_1$ *there exists an* $f \in F$ *such that* $f \notin \mathrm{Pol}\{\varrho\}$.

(3) *Suppose that all U-sets* $S \cap \mathrm{Pol}\{\varrho\}$, $\varrho \in Q_0$, *are mutually incomparable. Then a U-set N is maximal if and only if* $N = S \cap \mathrm{Pol}\{\varrho\}$ *for some* $\varrho \in Q_0$.

Examples how to apply the completeness criteria will be given in the next section.

4. 1-dimensional composition algebras: Examples and problems

In this section we present examples for the main results of the paper and discuss some problems. We restrict mainly to 1-dimensional composition algebras and corresponding transformation algebras (see 1.5), i.e. we deal with the representation by unary mappings (although the problems can be formulated for arbitrary k-dimensional composition algebras as well).

4.1. Specializing 1.1 to the case $k = 1$ we get that a 1-dimensional $\mathcal{K}$-composition algebra is an algebra $\underline{B} = \langle B; (f_i)_{i \in I}, \cdot \rangle$ satisfying

(i) $\langle B; (f_i)_{i \in I} \rangle \in \mathcal{K}$,

(ii) $\langle B; \cdot \rangle$ is a semigroup,

(iii) The operation $\cdot$ is *right distributive* over every f_i $(i \in I)$,
i.e., we have for all $b, c_1, c_2, \dots, c_n \in B$:

$$f_i \cdot b = f_i \quad \text{for every nullary } f_i \text{, and}$$
$$f_i(c_1, c_2, \dots, c_n) \cdot b = f_i(c_1 \cdot b, c_2 \cdot b, \dots, c_n \cdot b) \quad \text{for every } n\text{-ary } f_i.$$

Further, $\underline{B}$ has a selector system (see 1.3c) iff $\langle B; \cdot \rangle$ is a monoid.

4.2. Examples of 1-dimensional $\mathcal{K}$-composition algebras.

(A) Near-rings. An algebra $\langle N; +, \cdot\rangle$ is called a *near-ring* (see, e.g., [5]), if $\langle N; +\rangle$ is a group, $\langle N; \cdot\rangle$ is a semigroup and $\cdot$ is right distributive over $+$. Obviously, near-rings $\langle N; +, \cdot\rangle$ are 1-dimensional $\mathcal{K}$-composition algebras for $\mathcal{K}$ being the class of all groups.

(B) Binary relations. Let $B := \mathrm{Rel}^{(2)}(Y)$ be the set of all binary relations on Y. It is easy to check that $\langle B; \cup, \circ\rangle$ is a 1-dimensional $\mathcal{K}$-composition algebra for the variety $\mathcal{K}$ of semilattices. Here $\circ$ denotes the relational product

$$\varrho \circ \sigma = \{(x, y) \in Y^2 \mid \exists z \in Y : (x, z) \in \varrho \,\&\, (z, y) \in \sigma\}$$

for $\varrho, \sigma \in B$, and $\cup$ is the set-theoretical union.

(C) Distributive lattices. Obviously, any distributive lattice $\underline{L} = \langle L; \wedge, \vee\rangle$ can be considered as a 1-dimensional $\mathcal{K}$-composition algebra over the variety $\mathcal{K}$ of semilattices, where $\vee$ plays the role of $\cdot$.

Note that e.g. for Boolean lattices we cannot add complementation to the signature, because $\vee$ is not right distributive over complementation. Note further that, according to 4.1(iii), the largest element 1 is right distributive with respect to $\vee$, but the least element 0 is not.

(D) Semirings. Semirings $\underline{S} = \langle S; +, \cdot\rangle$ are rings with the usual axioms except that $\langle S; +\rangle$ is a commutative semigroup and not necessarily a group (see, e.g., [2]). Therefore semirings are 1-dimensional $\mathcal{K}$-composition algebras for the variety of commutative semigroups.

4.3. The Representation Theorem 1.6 provides us with a representation of composition algebras $\underline{B}$ by transformation $\mathcal{K}$-algebras, i.e. subalgebras of $\langle A^A; (\hat{f}_i)_{i\in I}, \circ\rangle$ for some $\underline{A} = \langle A; (f_i)_{i\in I}\rangle \in \mathcal{K}$, where $\circ$ is the composition of (unary) mappings. It can be checked easily (see also [3]) that the representation sketched in 1.7 works here not only for $\mathcal{K}$-composition algebras $\underline{B}$ with selector system but for any $\mathcal{K}$-composition algebra satisfying one of the following conditions:

(ii_0) all left translations are distinct, i.e. $\big([\forall c \in B : a \cdot c = b \cdot c] \implies a = b\big)$ for all $a, b \in B$,

(ii_1) $\langle B; \cdot\rangle$ is a monoid,

(ii_2) $\langle B; \cdot\rangle$ is a semigroup with a right unit,

(ii_3) $\langle B; \cdot\rangle$ is a right cancellative semigroup (i.e. $x \cdot y = z \cdot y$ implies $x = z$).

Note (ii_2) $\implies$ (ii_1) $\implies$ (ii_0) $\impliedby$ (ii_3), in particular (ii_0) is satisfied if any of the other three conditions holds.

Consequently, given a $\mathcal{K}$-composition algebra $\underline{B} = \langle B; (f_i)_{i\in I}, \cdot\rangle$, under any of these conditions, the mapping

$$b \mapsto \varphi_b \quad \text{where } \varphi_b : B \to B : x \mapsto b \cdot x \tag{4.3.1}$$

is an embedding of $\underline{B}$ into the full transformation algebra $\langle B^B; (\hat{f}_i)_{i\in I}, \circ\rangle$ over $\langle B; (f_i)_{i\in I}\rangle$.

4.4. Remark. The representation theorem for 1-dimensional $\mathcal{K}$-composition algebras (1.6, $k = 1$) generalizes various Cayley-type theorems for special structures; we list some of them:

- For the class $\mathcal{K}$ of all sets (without any algebraic structure) and for a group operation $\cdot$ we obtain the classical Cayley theorem: *every group is isomorphic to some permutation group.*
- Likewise, if $\mathcal{K}$ is the class of all sets and $\cdot$ is a semigroup operation, then we obtain that *every semigroup is isomorphic to some transformation semigroup.*
- If we consider ordinary rings as 1-dimensional $\mathcal{K}$-composition algebras (whereby $\mathcal{K}$ is the class of commutative groups), then 1.6 shows that every ring is isomorphic to a ring of transformations. Moreover, every ring R can be considered as an R-module (over itself, just by left multiplication). Therefore we have more precisely (due to the property 4.1(iii)) the known result (see e.g. [11, §38, Satz 66]): *every ring R with unit element is isomorphic to a subring of the full endomorphism ring of the module R.* The unit element of R hereby ensures property 4.3(ii$_1$). The weaker property 4.3(ii$_0$) says in the case of rings that 0 is the only left annullator in R. In [11, Satz 66] it was proved that this is a necessary and sufficient condition that the Cayley representation $\Phi : R \to R^R : b \mapsto \varphi_b$ (as given in (4.3.1) with $B := R$) is an isomorphism.

4.5. The minimal representation problem. With Theorem 1.6 in hand we are faced with the *minimal representation problem*: Find a minimal representation of a given 1-dimensional $\mathcal{K}$-composition algebra $\underline{B}$, i.e., find a representation of $\underline{B}$ as a transformation $\mathcal{K}$-algebra over some $\underline{Y} = \langle Y; (f_i)_{i\in I}\rangle \in \mathcal{K}$ of minimal cardinality $|Y|$. From the construction (4.3.1) above we conclude that such a minimal Y satisfies $|Y| \le |B|$ if $\underline{B}$ satisfies 4.3(ii$_0$); otherwise it is known that e.g. $|Y| \le |B|^2$ (which can be further improved, see [3]).

The following three examples illustrate the representation theorem and the minimization problem.

4.6. Example (Semirings). Let $\underline{S} = \langle \{s_1, s_2, s_3\}; +, \cdot\rangle$ be the algebra with the following operation tables:

$+$	s_1	s_2	s_3
s_1	s_1	s_2	s_3
s_2	s_2	s_2	s_3
s_3	s_3	s_3	s_3

$\cdot$	s_1	s_2	s_3
s_1	s_1	s_1	s_3
s_2	s_2	s_2	s_3
s_3	s_3	s_3	s_3

This is a semiring (it can be found in [2, page 24], however here we used the transposed multiplication table in order to ensure condition 4.3(ii$_2$) so that we can use (4.3.1)). Thus $\underline{S}$ is a 1-dimensional $\mathcal{K}$-composition algebra for the variety $\mathcal{K}$ of commutative semigroups (see 4.2(D)). Using the notation

$$\varphi = \begin{pmatrix}\varphi(s_1) & \varphi(s_2) & \varphi(s_3)\end{pmatrix}$$

for mappings $\varphi : S \to S$ we have

$$\varphi_{s_1} = \begin{pmatrix}s_1 & s_1 & s_3\end{pmatrix}, \varphi_{s_2} = \begin{pmatrix}s_2 & s_2 & s_3\end{pmatrix}, \varphi_{s_3} = \begin{pmatrix}s_3 & s_3 & s_3\end{pmatrix}.$$

Put $C = \{\varphi_{s_1}, \varphi_{s_2}, \varphi_{s_3}\}$. By (4.3.1), the algebra $\langle C; +, \circ\rangle$ is isomorphic to $\underline{S}$ (according to 1.4 we write $+$ instead of $\hat{+}$). E.g., we have

$$\varphi_{s_1} + \varphi_{s_2} = \begin{pmatrix}s_1+s_2 & s_1+s_2 & s_3+s_3\end{pmatrix} = \begin{pmatrix}s_2 & s_2 & s_3\end{pmatrix} = \varphi_{s_2} = \varphi_{s_1+s_2},$$
$$\varphi_{s_1} \circ \varphi_{s_2} = \begin{pmatrix}s_1 & s_1 & s_3\end{pmatrix} = \varphi_{s_1} = \varphi_{s_1\cdot s_2}.$$

It is easy to check that this representation is minimal.

4.7. Example (Lattices). Let $\underline{L} = \langle\{0, u, v, 1\}; \wedge, \vee\rangle$ be the 4-element free boolean lattice (see Fig. 2(i)). Thus $\underline{L}$ is a 1-dimensional $\mathcal{K}$-composition algebra for the variety $\mathcal{K}$ of semilattices ($\vee$ plays the role of multiplication, see 4.2(C)). Moreover, it satisfies 4.3(ii$_1$). According to 4.3.1 we obtain $\varphi_a(x) = a \vee x$ for $a, x \in L$, and the transformation $\mathcal{K}$-algebra $\underline{D_0} := \langle\{\varphi_0, \varphi_1, \varphi_u, \varphi_v\}; \wedge, \circ\rangle$ (over $\underline{Y} = \langle\{0, 1, u, v\}; \wedge\rangle$) is isomorphic to $\underline{L}$.
Let us ask for a minimal representation of $\underline{L}$ (in the sense of the minimization problem, see 4.5). Does there exist a transformation algebra $\underline{D}$ over some semilattice $\underline{Y} = \langle Y; \wedge\rangle$ such that $\underline{L} \cong \underline{D}$ but Y has less elements as in the above representation (where $Y = L$ has 4 elements)?
Case $|Y| = 2$*:* There is only one semilattice $\underline{Y}$ on a 2-element set, namely the chain $Y = \{0, 1\}$ with $0 \wedge 1 = 0$. The corresponding full transformation algebra $\langle Y^Y; \wedge, \circ\rangle$ is not isomorphic to $\underline{L}$ since $\circ$ is not commutative but $\vee$ is. Thus there does not exist a representation of $\underline{L}$ over a 2-element $\underline{Y}$.
Case $|Y| = 3$*:* In fact there is a representation with a 3-element semilattice $\underline{Y}$.
Let $Y := \{a, b, c\}$ and let $\underline{Y} = \langle Y; \wedge\rangle$ be the semilattice corresponding to the chain $\langle Y; \leq\rangle$ (as shown in Fig. 2(ii); as usual, $\leq$ is defined by $x \leq y :\iff x \wedge y = x$).

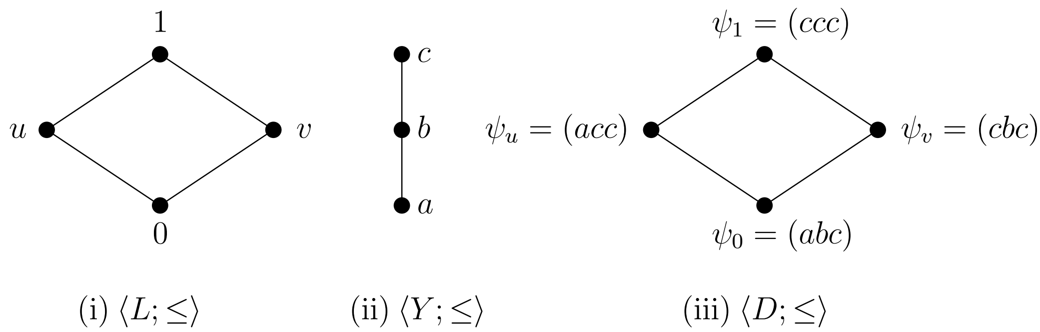

Figure 2. The lattice $\underline{L}$ and the isomorphic transformation algebra $\underline{D}$ over the semilattice $\underline{Y}$

Let $D := \{\psi_0, \psi_u, \psi_v, \psi_1\} \subseteq Y^Y$ with

$$\psi_0 = (abc),\ \psi_u = (acc),\ \psi_v = (cbc),\ \psi_1 = (ccc)$$

(recall that we represent $\psi \in Y^Y$ as the triple $(\psi(a)\psi(b)\psi(c))$). Then $\underline{D} := \langle D; \wedge, \circ\rangle$ is a transformation algebra over $\underline{Y}$, where the poset $\langle D; \leq\rangle$ corresponding to $\langle D; \wedge\rangle$ (induced componentwise by $\langle Y; \leq\rangle$) is given in Fig. 2(iii). Thus

$$\Psi : L \to D : x \mapsto \psi_x$$

is a semilattice isomorphism from $\langle L; \wedge\rangle$ onto $\langle D; \wedge\rangle$. But we also have

$$\Psi(x \vee y) = \Psi(x) \circ \Psi(y)$$

since $\psi_0 \circ \psi_x = \psi_x \circ \psi_0 = \psi_x$ (ψ_0 is the identity map), $\psi_x \circ \psi_1 = \psi_1 \circ \psi_x = \psi_1$ and $\psi_u \circ \psi_v = \psi_v \circ \psi_u = \psi_1$. Therefore Ψ is also an isomorphism from $\underline{L}$ onto $\underline{D}$, and $\underline{D}$ is a minimal representation of $\underline{L}$.

4.8. Example (Lattices continued). Let $\mathcal{K}$ be the variety of all semilattices enriched with an additional unary operation f which is a semilattice endomorphism, i.e. satisfies $f(x \wedge y) = f(x) \wedge f(y)$. Thus, e.g., $\underline{Y_0} := \langle L; \wedge, f\rangle$ with $L = \{0, u, v, 1\}$ from the previous example together with $f : L \to L$ given by $f(0) = f(v) = v$, $f(u) = f(1) = 1$, belongs to $\mathcal{K}$.
Since this f also satisfies $f(x) \vee y = f(x \vee y)$, i.e., $\vee$ is right distributive over f, the algebra

$$\underline{M} := \langle\{0, u, v, 1\}; \wedge, f, \vee\rangle$$

is a 1-dimensional $\mathcal{K}$-composition algebra. To get a representation as transformation $\mathcal{K}$-algebra we have to extend the representation $\underline{D_0}$ in 4.7 by the operation $\hat{f} : D_0 \to D_0 : \varphi_x \mapsto \varphi_{f(x)}$.
Now, contrary to Example 4.7, this representation is minimal.
To see this, assume that there is a representation of $\underline{M}$ as a transformation $\mathcal{K}$-algebra $\underline{D} = \langle D; \wedge, \hat{g}, \circ\rangle$ over a 3-element semilattice $\underline{Y} = \langle Y; \wedge, g\rangle$ with a semilattice endomorphism g. Consequently, $D \subseteq Y^Y$ and there would exist an isomorphism, say $\Phi : x \mapsto \psi_x$, from $\underline{M}$ onto $\underline{D}$ (the mappings ψ_x are still not known and have nothing to do with those from Example 4.7). Assume that $Y = \{a, b, c\}$ where a denotes the least element (w.r.t. the corresponding poset $\langle Y; \leq\rangle$ induced by $\langle Y; \wedge\rangle$).
Since Φ is an isomorphism, we get for any $x \in L$

$$\psi_x = \Phi(x) = \Phi(x \vee x) = \Phi(x) \circ \Phi(x) = \psi_x \circ \psi_x\,.$$

Therefore every ψ_x ($x \in L$) belongs to the following set $S \subset Y^Y$ of idempotent mappings (where ψ_x is again written as the triple $\psi_x = (\psi_x(a)\psi_x(b)\psi_x(c))$):

$$S := \{(aaa), (bbb), (ccc), (aac), (bbc), (aba), (cbc), (abb), (acc), (abc)\}.$$

Up to isomorphism there are only two 3-element semilattices $\underline{Y}$, which we shall examine separately.
Case 1: $\underline{Y}$ is the semilattice with $b \wedge c = a$ (see Fig. 3).

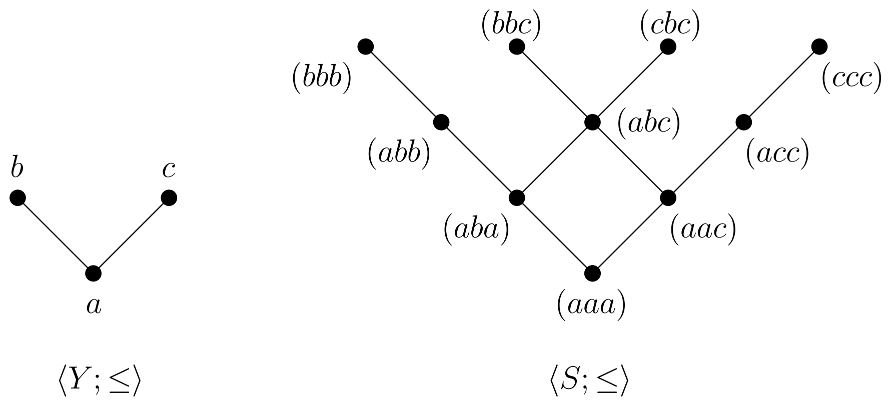

Figure 3. The semilattice $\underline{Y}$ (case 1) and the induced poset on S

Then the induced poset $\langle S; \leq\rangle$ is as in Fig. 3. Since ψ_u and ψ_v must be incomparable but must have a least upper bound, namely ψ_1 (see Fig. 2(i)), we get $\{\psi_u, \psi_v\} = \{(aba), (aac)\}$.

But $\psi_1 = \Phi(1) = \Phi(u \vee v) = \psi_u \circ \psi_v = (aaa)$. Hence $\psi_1 \leq \psi_u$, which is a contradiction to $\underline{D} \cong \underline{M}$.

Case 2: $\underline{Y}$ is the semilattice with $b \wedge c = b$ (see Fig. 4).

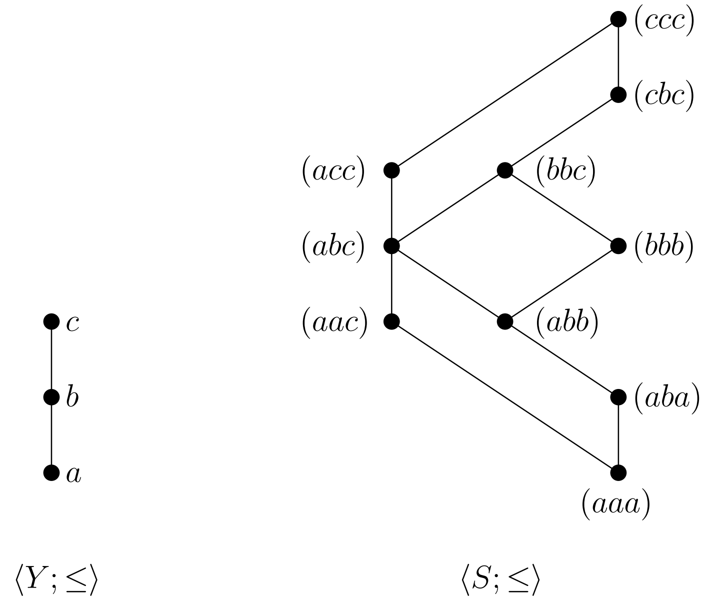

Figure 4. The semilattice $\underline{Y}$ (case 2) and the induced poset on S

Then the induced poset $\langle S; \leq\rangle$ is as in Fig. 4. The elements ψ_u, ψ_v must be incomparable, thus one of them, say ψ_u, must belong to $\{(aac), (abc), (acc)\}$ (see Fig. 4). We are going to exclude each case by contradiction.

(1) $\psi_u = (aac)$. Then $\psi_v \in \{(aba), (abb), (bbb)\}$ (by incomparability, see Fig. 4), but this gives $\Phi(1) = \Phi(u \vee v) = \psi_u \circ \psi_v = (aaa)$ which is not a least upper bound of ψ_u, ψ_v in $\langle S; \leq\rangle$.

(2) $\psi_u = (abc)$. Then $\psi_v = (bbb)$ and $\psi_u \circ \psi_v = \psi_v$ gives the contradiction (analogously to case (1)).

(3) $\psi_u = (acc)$. Then $\psi_v \in \{(bbb), (bbc), (cbc)\}$ (since ψ_u, ψ_v must be incomparable). But $(bbb) \circ \psi_u = (bbb) \neq (ccc) = \psi_u \circ (bbb)$ and $(bbc) \circ \psi_u = (bcc) \notin S$, thus $\psi_v = (cbc)$. Consequently the unique remaining possibility is $D = \{\psi_0, \psi_u, \psi_v, \psi_1\}$ with $\psi_0 = (abc)$, $\psi_1 = (ccc)$ as in 4.7. Contrary to 4.7 now we have to take into account the additional unary operation. From the isomorphism $\Phi : \underline{M} \to \underline{D}$ we get $\hat{g}(\Phi(0)) = \Phi(f(0)) = \Phi(v)$, i.e., $\hat{g}(\psi_0) = \psi_v$. Thus $(g(a)g(b)g(c)) = \hat{g}((abc)) = (cbc)$, but this g is not a semilattice endomorphism since $g(a \wedge b) = g(a) = c \neq b = c \wedge b = g(a) \wedge g(b)$, a contradiction.

Therefore no $\underline{Y}$ with 3 elements exists.

Finally, we shall demonstrate how the Completeness Theorem 3.5(2) applies to produce completness criteria for concrete structures. A completeness theorem for near-rings (see 4.2(A)) can be found in [1]. Here we consider semirings (see 4.2(D)).

4.9. (max, ∘)-semirings. Let $A = \{1, 2, \dots, n\}$ be a finite at least two element set and let $\underline{T} := \langle A^A; \max, \circ\rangle$ be the full concrete $(\max, \circ)$-semiring. Here max denotes the maximum

(acting as $\widehat{\max}$ pointwise on A^A, see 1.4.1). Note that $\underline{T}$ is the full transformation algebra over $\underline{A} := \langle A; \max\rangle$ with selector ($= \mathsf{id}_A$). For the purposes of this example we shall consider only subsemirings of $\underline{T}$ containing the identity map.
In accordance with notation from Theorem 3.5, put $S := A^A$, $U := \{\max\}$ and for $F \subseteq A^A$ let

$$\mathrm{Srg}(F) := \overline{F} = A^A \cap \mathrm{clone}(F \cup \{\max\})$$

be the corresponding closure operator (see 3.1.1). It follows from 2.3 that $\mathrm{Srg}(F)$ is the least $(\max, \circ)$-semiring containing $F \cup \{\mathsf{id}_A\}$.
We shall say that F is *semiring-complete* if $\mathrm{Srg}(F) = \underline{T}$.
Therefore, U-sets in the terminology of Theorem 3.5 are just subsemirings of $\underline{T}$ containing the identity map (or 1-dimensional operation algebras as defined in 1.5). Since max is a so-called Słupecki operation (i.e. surjective and essentially binary), $\mathrm{clone}(A^A \cup \{\max\})$ is the clone of all operations on A, so by Theorem 3.5 we now know that in order to find maximal U-sets it suffices to find those Rosenberg relations that are invariant under max. In the next lemma we single out possible candidates for these relations.

4.10. Lemma. *If a Rosenberg relation on $A = \{1, \dots, n\}$ is invariant under* max*, then it belongs to one of the following classes of relations:*

- *bounded partial orders on A where n is either the least or the greatest element;*
- *equivalence relations whose blocks are intervals in the usual linear order $\langle A; \leq\rangle$;*
- *proper subsets of A;*
- *binary central relations where* 1 *is not a central element.*

Proof. We examine the six classes of Rosenberg relations (assuming the reader is familiar with them, otherwise see e.g. [10] or [9, 4.3.21]) and eliminate those not invariant under max.

Bounded partial orders. Let $\preccurlyeq$ be a bounded partial order on A with the least element a and the greatest element b and suppose $\preccurlyeq \in \mathrm{Inv}\{\max\}$. Then clearly $a \preccurlyeq n \preccurlyeq b$. If $a < b$ then from $a \preccurlyeq n$ and $b \preccurlyeq b$ we infer $b = \max(a, b) \preccurlyeq \max(n, b) = n$ so $b = n$, i.e. n is the greatest element of $\preccurlyeq$. Analogously, if $a > b$ then n is the least element of $\preccurlyeq$.

Nontrivial equivalence relations. Consider an equivalence relation $\varrho \in \mathrm{Inv}\{\max\}$ and let B be one of its blocks. Take any $b, b' \in B$ and let $b \leq x \leq b'$. From $(b, b') \in \varrho$ and $(x, x) \in \varrho$ we get $(\max(b, x), \max(b', x)) = (x, b') \in \varrho$ whence $x \in B$. Consequently B is an interval w.r.t. the order $\leq$.

Permutational relations. Let $\varrho = \{(x, \alpha(x)) \mid x \in A\} \in \mathrm{Inv}\{\max\}$ be a permutational relation given by some permutation α of A where all cycles of α have the same prime length p. Thus, in particular, $n \neq \alpha(n)$. On the other hand from $(\alpha^{-1}(n), n) \in \varrho$ and $(n, \alpha(n)) \in \varrho$ it follows that $(\max(\alpha^{-1}(n), n), \max(n, \alpha(n))) = (n, n) \in \varrho$, i.e. $\alpha(n) = n$. Contradiction. Therefore, no permutational relation is invariant under max.

Affine relations. Let $\langle A; +, -, 0\rangle$ be an elementary abelian p-group on A and suppose that $\lambda_+ := \{(x, y, u, v) \mid x + y = u + v\} \in \mathrm{Inv}\{\max\}$. Let $a \in A \setminus \{n\}$. Then $(n, a, n, a) \in \lambda_+$ and $(n, a, a, n) \in \lambda_+$. Consequently

$$(\max(n, n), \max(a, a), \max(n, a), \max(a, n)) = (n, a, n, n) \in \lambda_+$$

whence $n = a$. Contradiction. Therefore, no affine Rosenberg relation λ_+ is invariant under max.

Regular relations and central relations of arity at least three. These relations are totally reflexive and of arity at least three. Suppose that $\varrho \in \mathrm{Inv}\{\max\}$ is a nontrivial h-ary ($h \geq 3$) totally reflexive relation. Take any $(a_1, \dots , a_h) \notin \varrho$. We have $(a_1, 1, \dots , 1)$, $(1, a_2, 1, \dots , 1)$, $\dots$, $(1, \dots , 1, a_h) \in \varrho$. Applying max to these tuples component-wise and recalling the fact that ϱ is invariant under max yields $(a_1, \dots , a_h) \in \varrho$. Contradiction. Thus no such ϱ is invariant under max.

Unary and binary relations. Note that unary central relations are just proper subsets of A and that every subset of A is invariant under max. Finally, let $\varrho \in \mathrm{Inv}\{\max\}$ be a binary central relation. Suppose that 1 is a central element of ϱ. Then for every $x, y \in A$ we have $(x, 1), (1, y) \in \varrho$ whence $(x, y) \in \varrho$ by applying max component-wise. Thus $\varrho = A^2$ is trivial which contradicts the requirement that central relations are nontrivial. □

Some of the relations ϱ listed in Lemma 4.10 need not produce maximal semirings $\mathrm{End}\{\varrho\}$ and a careful examination would improve (i.e. shorten) the list. However, all maximal semirings are among the semirings $\mathrm{End}\{\varrho\}$ produced from the above relations. This already provides us the following:

4.11. Proposition (Completeness criterion). *A set $F \subseteq A^A$ is semiring-complete if and only if for every relation ϱ listed in Lemma 4.10 there is an $f \in F$ such that $f \notin \mathrm{End}\{\varrho\}$.* □

We specialise this in two ways. First, we shall describe 1-element generating sets of $\underline{T}$. We shall say that an equivalence relation ϱ on A is *$\leq$-regular* if it is nontrivial and all its blocks are intervals in $\langle A; \leq\rangle$ of the same length.

4.12. Proposition (1-Generators of $\underline{T}$). *Let $f \in A^A$. Then $\mathrm{Srg}(f) = A^A$ if and only if f is a cyclic permutation of A such that $f \notin \mathrm{End}\{\varrho\}$ for all $\leq$-regular equivalence relations on A. In particular, if $|A|$ is prime then $\mathrm{Srg}(f) = A^A$ if and only if f is a cyclic permutation of A.*

Proof. “$\Rightarrow$:” Suppose $\mathrm{Srg}(f) = A^A$. Then f preserves no relation listed in Lemma 4.10, in particular no $\leq$-regular equivalence relation and no proper subset of A. The latter implies that f has to be a cyclic permutation.

“$\Leftarrow$:” Suppose f is a cyclic permutation of A with $f \notin \mathrm{End}\{\varrho\}$ for all $\leq$-regular equivalence relations on A. By 4.11 it suffices to show that f preserves no relation mentioned in Lemma 4.10.

A cyclic permutation preserves no bounded partial order and no proper subset of A. If f preserves an equivalence relation ϱ, then it acts on the set of blocks of ϱ as a cyclic permutation as well, whence follows that all the blocks of ϱ have the same length. Therefore, ϱ is $\leq$-regular, and f preserves no $\leq$-regular equivalence relation by the assumption. Finally, suppose $f \in \mathrm{End}\{\varrho\}$ for some binary central relation. Take any $(x, y) \notin \varrho$ and let s be an integer such that $f^{-s}(x)$ is a central element of ϱ. Then $(f^{-s}(x), f^{-s}(y)) \in \varrho$ and from the fact that f preserves ϱ we get $(x, y) \in \varrho$. Contradiction. Thus f preserves no binary central relation.

For the second part of the statement it suffices to observe that if $|A|$ is prime, no nontrivial equivalence relation on A is $\leq$-regular. $\square$

4.13. As a further illustration of 4.11 we shall describe all maximal semirings on $A = \{1, 2, 3\}$ in order to get a completeness criterion for the full transformation $(\max, \circ)$-semiring on A. The relations listed in Lemma 4.10 are the following:

Bounded partial orders: $\tau_{123} := 1 \prec 2 \prec 3$ and $\tau_{213} := 2 \prec 1 \prec 3$ (as well as their duals which we do not have to consider since $\mathrm{End}\{\tau\} = \mathrm{End}\{\tau^{-1}\}$);

Equivalence relations: $\varepsilon_{12|3} := \Delta_A \cup \{(1,2),(2,1)\}$ and $\varepsilon_{1|23} := \Delta_A \cup \{(2,3),(3,2)\}$;

Unary relations: $\sigma_1 := \{1\}$, $\sigma_2 := \{2\}$, $\sigma_3 := \{3\}$, $\sigma_{12} := \{1,2\}$, $\sigma_{13} := \{1,3\}$, $\sigma_{23} := \{2,3\}$; and

Binary central relations: $\zeta_2 := \Delta_A \cup \{(1,2),(2,1),(2,3),(3,2)\}$ whose only central element is 2, and $\zeta_3 := \Delta_A \cup \{(1,3),(3,1),(2,3),(3,2)\}$ whose only central element is 3.

We know (see 3.5(1)) that the maximal subsemirings of the full $(\max, \circ)$-semiring A^A are among $\mathrm{End}\{\varrho\}$ where ϱ is one of the above relations. It is straightforward to determine the partially ordered set $\mathcal{N}$ (ordered by inclusion) of all these $\mathrm{End}\{\varrho\}$, see Figure 5.

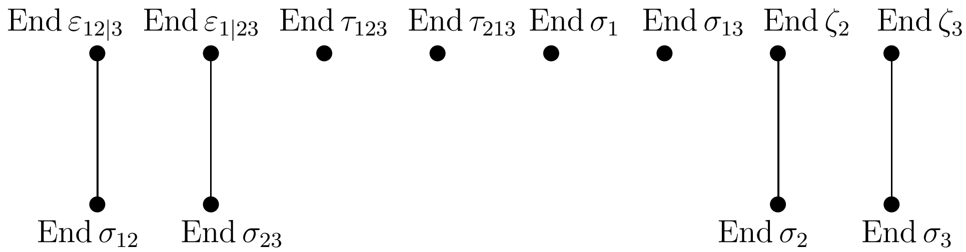

Figure 5. The partially ordered set $\mathcal{N}$

Thus, by 3.5(1), there are precisely 8 maximal subsemirings of $\underline{T}$ containing the identity map, namely

$$\mathrm{End}\{\varrho\} \text{ for } \varrho \in \{\varepsilon_{12|3}, \varepsilon_{1|23}, \tau_{123}, \tau_{213}, \sigma_1, \sigma_{13}, \zeta_2, \zeta_3\}\,.$$

Therefore, we can now infer the following completeness criterion:

4.14. Proposition. *Let $A = \{1, 2, 3\}$. A set $F \subseteq A^A$ is semiring complete if and only if for every $\varrho \in \{\varepsilon_{12|3}, \varepsilon_{1|23}, \tau_{123}, \tau_{213}, \sigma_1, \sigma_{13}, \zeta_2, \zeta_3\}$ there is an $f \in F$ such that $f \notin \mathrm{End}\{\varrho\}$.* $\square$

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