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The Apery Algorithm for a Plane Singularity with Two Branches

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Abstract. The equisingularity class of a plane irreducible curve is determined by the semigroup of the curve or, equivalently, by its multiplicity sequence. For a curve with two branches, the semigroup (now a subsemigroup of $\mathbb{N}^2$) still determines the equisingularity class. We introduce the "multiplicity tree" for the curve, which also determines the equisingularity class, and construct an algorithm to go back and forth between the semigroup and the multiplicity tree. Moreover we characterize the multiplicity trees of plane curve singularities with two branches.

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1. Introduction

If $\mathcal{O} = k[[X,Y]]/(F)$, where k is an algebraically closed field of characteristic 0, is an irreducible plane algebroid curve (a branch), then the integral closure $\bar{\mathcal{O}}$ is a DVR, $\bar{\mathcal{O}} \cong k[[t]]$. Hence every nonzero element in $\mathcal{O}$ has a value. The set of values of elements in $\mathcal{O}$ constitutes a numerical semigroup $v(\mathcal{O}) = S$, i.e., a submonoid of $\mathbb{N}$ with finite complement to

$\mathbb{N}$. The smallest nonzero value $e \in S$ is the multiplicity $e(\mathcal{O})$ of $\mathcal{O}$. The semigroup is an important invariant of $\mathcal{O}$ and has been studied e.g. in [2], [3], [6], [5], [8], [9], [10], and [11]. Another important invariant is the sequence of multiplicities of the successive blowups of $\mathcal{O}$, $(e(\mathcal{O}), e(\mathcal{O}'), e(\mathcal{O}^{(2)}), \dots)$. It is well-known that two plane branches have the same semigroup if and only if they have the same multiplicity sequence, cf. [12]. If we instead consider plane analytic branches, these invariants determine the topological class of the branch, cf. [12].

If we have a plane algebroid curve with d branches, $\mathcal{O} = k[[X,Y]]/(F_1 \cdots F_d)$, F_i irreducible, the integral closure is a product of DVR's, $\bar{\mathcal{O}} \cong k[[t_1]] \times \cdots \times k[[t_d]]$, and the set of values $S = v(\mathcal{O})$ is a submonoid of $\mathbb{N}^d$. The projections S_i of S on the coordinate axes are the semigroups of each branch. Two plane algebroid curves $\mathcal{O} = k[[X,Y]]/(F_1 \cdots F_d)$ and $\mathcal{C} = k[[X,Y]]/(G_1 \cdots G_d)$ are said to be formally equivalent if (after a renumbering of the branches) $v(k[[X,Y]]/(F_i)) = v(k[[X,Y]]/(G_i))$ for $i = 1, \dots, d$ and if the intersection multiplicities $l_{\mathcal{O}}(k[[X,Y]]/(F_i, F_j))$ and $l_{\mathcal{C}}(k[[X,Y]]/(G_i, G_j))$ are the same for all pairs (i,j), $i \neq j$. Waldi has shown in [11] that two plane curves are formally equivalent if and only if they have the same semigroup.

In [3], we introduced and studied the multiplicity tree of an algebroid curve. This tree contains exactly the same information as the S_i's together with the pairwise intersection multiplicities in case of plane curves. In fact the i-th branch of the multiplicity tree of $\mathcal{O}$ gives the multiplicity sequence of $k[[X,Y]]/(F_i)$, hence the semigroup of that branch. Furthermore, if the tangents of two branches F_i and F_h are equal in $\mathcal{O}, \mathcal{O}', \dots, \mathcal{O}^{(j-1)}$ but not in $\mathcal{O}^{(j)}$, then, by a theorem of Max Noether, the intersection multiplicity equals $e_0^{(i)} e_0^{(h)} + \cdots + e_j^{(i)} e_j^{(h)}$, where $e_0^{(i)}, e_1^{(i)}, \dots$ is the multiplicity sequence for the branch F_i and $e_0^{(h)}, e_1^{(h)}, \dots$ is the multiplicity sequence for the branch F_h. Thus the multiplicity tree gives the same information as the two semigroups together with the intersection multiplicity between every pair of branches. Thus two plane algebroid curves are formally equivalent if and only if they (after a renumbering of the branches) have the same multiplicity tree.

In [1] Apery showed that, if $\mathcal{O}$ is a plane branch and $\mathcal{O}'$ is its blowup, then the semigroups $v(\mathcal{O})$ and $v(\mathcal{O}')$ are strictly related: there is a formula to get a particular generating set, called the Apery set, for $v(\mathcal{O}')$ from that of $v(\mathcal{O})$ and vice versa (cf. Theorem 2.1). This does not happen in general for non plane branches and it is the reason why for plane branches the semigroup characterizes as well as the multiplicity sequence a class of equivalence. In [4] we used that result of Apery to show how one can get the semigroup from the multiplicity sequence and vice versa. Our aim is now to generalize these results to the case of a plane curve with two branches. We point out that most results in Sections 3 and 4 (included Theorem 4.1) are valid also for more than two branches. However in order to apply a result of [10] in Proposition 4.2 and for the sake of simplicity in notation in the other results, we restrict ourselves to the two branches case.

Since we deal with subsemigroups of $\mathbb{N}^2$, that are not finitely generated, we need first to define a suitable set of generators for S. This "Apery set" is no longer finite, but a finite union of subsets (its components). Each component is the generalization of an element of the Apery set of a numerical semigroup.

The main result of the paper is Theorem 4.1, that shows how also in the two branches case the semigroups $v(\mathcal{O})$ and $v(\mathcal{O}')$ are strictly related. All Section 3 prepares this result. We apply Theorem 4.1 to get the multiplicity tree from the semigroup and vice versa. In the last section a purely numerical characterization of a multiplicity tree of a plane curve with two branches is given (cf. Proposition 5.2). Via the method described in Section 4, this gives also a constructive characterization of the two branches plane curve semigroups.

2. Preliminaries

We start by recalling some results from the one branch case. If $\mathcal{O}$ is a plane branch, we denote the semigroup $v(\mathcal{O})$ by S. The semigroup S is symmetric, i.e., $s \in S$ if and only if $g_S - s \notin S$, where g_S is the largest integer which does not belong to S. For an element $s \in S$, let $\Omega_s = \{\omega_0, \ldots, \omega_{s-1}\}$ consist of the elements $x \in S$ such that $x - s \notin S$, i.e., the smallest elements in S in each congruence class $\pmod{s}$. We suppose that the elements are ordered so that $0 = \omega_0 < \omega_1 < \cdots < \omega_{s-1} = g_S + s$. The set Ω_s is called the Apery set of S with respect to s. In [1] it is shown that there is a duality in Ω_s, namely $\omega_i + \omega_{s-1-i} = \omega_{s-1}$ for $i = 0, \ldots, s-1$. If x is an element of smallest positive value in $\mathcal{O}$, then $\mathcal{O}' = k[[x, y/x]]$ is the blowup of $\mathcal{O}$. The mentioned connection between $v(\mathcal{O})$ and $v(\mathcal{O}')$ is the following, cf. [1]:

Theorem 2.1. (Apery) *If e is the multiplicity of $\mathcal{O}$ and $\Omega_e = \{\omega_0, \ldots, \omega_{e-1}\}$ is the Apery set of $v(\mathcal{O})$ with respect to e, then the Apery set of $v(\mathcal{O}')$ with respect to e is $\{\omega_0, \omega_1 - e, \omega_2 - 2e, \ldots, \omega_{e-1} - (e-1)e\}$.*

Let $\mathcal{O}, \mathcal{O}', \mathcal{O}^{(2)}, \ldots$ be the sequence of consecutive blowups of $\mathcal{O}$ and let $e_0, e_1, e_2, \ldots$ be their respective multiplicities. The sequence $e_0, e_1, e_2, \ldots$ is called the multiplicity sequence of $\mathcal{O}$ and, since $\mathcal{O}^{(i)}$ is regular if $i >> 0$, we have $e_i = 1$ if $i >> 0$, that is $v(\mathcal{O}^{(i)}) = \mathbb{N}$, for $i >> 0$. With the above result of Apery it is possible to determine from $v(\mathcal{O})$ the multiplicity sequence of $\mathcal{O}$ and vice versa. Since the possible multiplicity sequences for plane curves are characterized (cf. e.g. [4, Theorem 3.2]), this gives also a method to characterize the semigroups of plane branches. We now give an example to show how to determine the multiplicity sequence from the semigroup and vice versa.

Example. If the semigroup is $v(\mathcal{O}) = S = \langle 4, 6, 13 \rangle$ (it is in fact the semigroup of $\mathcal{O} = k[[t^4, t^6 + t^7]]$), then the multiplicity $e(\mathcal{O})$ of $\mathcal{O}$ is 4 and the Apery set of S with respect to 4 is $\{0, 6, 13, 19\}$. Hence the Apery set of $S' = v(\mathcal{O}')$ with respect to 4 is $\{0, 2, 5, 7\}$, which gives $S' = \langle 2, 5 \rangle$ and $e(\mathcal{O}') = 2$. The Apery set of $S' = v(\mathcal{O}')$ with respect to 2 is $\{0, 5\}$, so the Apery set of $S'' = v(\mathcal{O}^{(2)})$ with respect to 2 is $\{0, 3\}$, which gives $S'' = \langle 2, 3 \rangle$ and $e(\mathcal{O}^{(2)}) = 2$. The Apery set of $S''' = v(\mathcal{O}^{(3)})$ with respect to 2 is $\{0, 1\}$, so $S''' = \mathbb{N}$ and $e(\mathcal{O}^{(3)}) = 1$. Hence the multiplicity sequence of $\mathcal{O}$ is $4, 2, 2, 1, \ldots$
;. If we start with the multiplicity sequence $4, 2, 2, 1, \ldots$ we can go backwards in the sequence of blowups. We have $e(\mathcal{O}^{(3)}) = 1$, so $S''' = v(\mathcal{O}^{(3)}) = \mathbb{N}$. Since $e(\mathcal{O}^{(2)}) = 2$, we determine the Apery set of $\mathbb{N}$ with respect to 2. This is $\{0, 1\}$, so the Apery set of S'' with respect to 2 is $\{0, 3\}$, so $S'' = \langle 2, 3 \rangle$. Since $v(\mathcal{O}') = 2$ we get that the Apery set of S' with respect to 2 is $\{0, 5\}$, so $S' = \langle 2, 5 \rangle$. Finally $v(\mathcal{O}) = 4$ and the Apery set of S' with respect to 4 is $\{0, 2, 5, 7\}$, so the Apery set of S with respect to 4 is $\{0, 6, 13, 19\}$, and we get $S = \langle 4, 6, 13, 19 \rangle = \langle 4, 6, 13 \rangle$.

Now let $\mathcal{O} = k[[X, Y]]/(F_1 \cdot F_2)$, where F_1, F_2 are irreducible, be an algebroid curve with two branches. The blowing up of the maximal ideal in $\mathcal{O} = k[[x, y]]$ is $\mathcal{O}' = \cup_{n>0}(m^n : m^n)$, where $m = (x, y)$. This is a semilocal ring with at most two maximal ideals. If we continue to blow up the Jacobson radical, we have a sequence of overrings $\mathcal{O} = \mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, \dots$ and eventually get the integral closure $k[[t]] \times k[[u]]$ of $\mathcal{O}$. The integral closure has two maximal ideals, $n_1 = tk[[t]] \times k[[u]]$ and $n_2 = k[[t]] \times uk[[u]]$. The blowing up tree is a tree, where the nodes on level i are the localizations at the maximal ideals $n_j \cap \mathcal{O}^{(i)}$ of $\mathcal{O}^{(i)}$. If we replace the rings with their *fine multiplicities*, we get the multiplicity tree of $\mathcal{O}$. As long as the ring $\mathcal{O}^{(i)}$ is local, the fine multiplicity is $(e(\mathcal{O}^{(i)}/q_1), e(\mathcal{O}^{(i)}/q_2))$, where q_1, q_2 are the minimal primes of $\mathcal{O}^{(i)}$. If $\mathcal{O}^{(i)}$ has two maximal ideals, then $\mathcal{O}^{(i)} = (\mathcal{O}^{(i)})_{n_1\cap\mathcal{O}^{(i)}} \times (\mathcal{O}^{(i)})_{n_2\cap\mathcal{O}^{(i)}}$: the tree has split in two branches, in the first there is a local ring $R_1 = (\mathcal{O}^{(i)})_{n_1\cap\mathcal{O}^{(i)}}$ with integral closure $k[[t]]$, in the second a local ring $R_2 = (\mathcal{O}^{(i)})_{n_2\cap\mathcal{O}^{(i)}}$ with integral closure $k[[u]]$. Their fine multiplicities are $(e(R_1), 0)$ and $(0, e(R_2))$, respectively.

In this case the set of values $v(\mathcal{O})$ of nonzero divisors in $\mathcal{O}$ is a subsemigroup of $\mathbb{N}^2$.

If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, set $\Delta(\boldsymbol{\alpha}) = \{(n, \alpha_2) \mid n > \alpha_1\} \cup \{(\alpha_1, m) \mid m > \alpha_2\}$ and, if S is a subsemigroup of $\mathbb{N}^2$, $\Delta^S(\boldsymbol{\alpha}) = \Delta(\boldsymbol{\alpha}) \cap S$. We call a semigroup *local* if $\Delta^S((0,0)) = \emptyset$. It is immediate that $\mathcal{O}^{(i)}$ is local if and only if $v(\mathcal{O}^{(i)})$ is local. As a matter of fact, $\mathcal{O}^{(i)}$ has two maximal ideals if and only if $n_1 \cap \mathcal{O}^{(i)} \neq n_2 \cap \mathcal{O}^{(i)}$. This happens if and only if there are in $\mathcal{O}^{(i)}$ elements of value $(a, 0)$ and $(0, b)$, for some $a, b \neq 0$, i.e. if and only if $\Delta^{v(\mathcal{O}^{(i)})}((0,0)) \neq \emptyset$.

We refer to [3] for more details on the multiplicity tree and the semigroup of a curve with $d \geq 2$ branches.

We introduce two order relations on $\mathbb{Z}^2$. The first, $\leq$, is defined by $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$ for $i = 1, 2$. We let $\boldsymbol{\alpha} < \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. The second, $\leq\leq$, is defined by $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ if $\alpha_i < \beta_i$ for $i = 1, 2$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$. We write $\boldsymbol{\alpha} << \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$.

Proposition 2.2. *Let S be the value semigroup of an algebroid plane curve $\mathcal{O}$ with two branches. Then:*

a) *There exists a $\boldsymbol{\gamma}_S \in \mathbb{Z}^2$ such that $\Delta^S(\boldsymbol{\gamma}_S) = \emptyset$ and $\boldsymbol{\alpha} \in S$ if $\boldsymbol{\alpha} >> \boldsymbol{\gamma}_S$.*

b) *We have $\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\boldsymbol{\gamma}_S - \boldsymbol{\alpha}) = \emptyset$.*

c) *Let $\mathbf{e}$ be the minimal nonzero element in S, then the blowup $\mathcal{O}'$ of $\mathcal{O}$ is local if and only if $\Delta^S(\mathbf{e}) = \emptyset$, and if and only if the tangents of the two branches are equal.*

Proof. a) is a well-known property of the semigroup of any algebroid curve (cf. e.g. [3, Proposition 2.1]).

Property b) holds, i.e. S is a symmetric semigroup, because $\mathcal{O}$ is a plane curve and thus is a Gorenstein ring (cf. [10] or [9]). If the two branches have the same tangent, they can be parametrized as $(t^k, at^l + \cdots)$, $l > k$, and $(u^m, bu^n + \cdots)$, $n > m$, respectively, hence $\mathcal{O} = k[[(t^k, u^m), (at^l + \cdots, bu^n + \cdots)]] = k[[x, y]]$, $\mathbf{e} = (k, m)$ and $\Delta^S(\mathbf{e}) = \emptyset$. Obviously the blowup $\mathcal{O}' = k[[x, y/x]]$ is local, so $v(\mathcal{O}')$ is local.

If the two branches have distinct tangents, they can be parametrized as $(t^k, at^l + \cdots)$, $l > k$ and $(bu^n + \cdots, u^m), n \geq m$, respectively; it follows that $\mathcal{O} = k[[(t^k, bu^n + \cdots), (at^l + \cdots, u^m)]] = k[[x_1, x_2]]$. Let $x = x_1 + cx_2 = (t^k + act^l + \cdots, cu^m + bu^n + \cdots)$. For a suitable $c \neq 0$ we have $v(x) = (k, m)$, which is the minimal positive value in $v(\mathcal{O})$. Hence $\mathbf{e} = (k, m)$, but $\Delta^S(\mathbf{e}) \neq \emptyset$; moreover we get $v(x_1/x) = (0, n - m)$ and $v(x_2/x) = (l - k, 0)$, so $v(\mathcal{O}')$ is not local. Hence c) is proved.

3. The Apery set

Let $\mathcal{O}$ be a (non necessarily plane) curve with two branches, let $S = v(\mathcal{O})$ and let $\boldsymbol{\alpha} \in S$. We define the *Apery set of* S *with respect to* $\boldsymbol{\alpha}$ to be $\Omega_{\boldsymbol{\alpha}} = \{\boldsymbol{\beta} \in S \mid \boldsymbol{\beta} - \boldsymbol{\alpha} \notin S\}$. The semigroup S, by its good properties (cf. [3, Proposition 2.1]) is entirely described by the elements of S in the rectangle $\{(m,n); 0 \leq m \leq \gamma_1 + 1, 0 \leq n \leq \gamma_2 + 1\}$, where $\boldsymbol{\gamma}_S = (\gamma_1, \gamma_2)$. It follows that, although the Apery set $\Omega_{\boldsymbol{\alpha}}$ of S is not finite, the finite subset $\Omega_{\boldsymbol{\alpha}} \cap \{(m,n); 0 \leq m \leq \gamma_1 + 1 + \alpha_1, 0 \leq n \leq \gamma_2 + 1 + \alpha_2\}$ determines all $\Omega_{\boldsymbol{\alpha}}$.

We are now interested in a finer description of the Apery set in case of a plane curve with two branches. Let $\mathcal{O}$ be a plane curve with two branches, let $S = v(\mathcal{O})$ and let $\boldsymbol{\alpha} \in S$. We set:

Definition. *Let* $\Omega^0_{\boldsymbol{\alpha}} = \{\boldsymbol{\beta} \in \Omega_{\boldsymbol{\alpha}} \mid \boldsymbol{\beta}$ *maximal with respect to* $<<\}$. *Assume that* $\Omega^0_{\boldsymbol{\alpha}}, \dots, \Omega^{i-1}_{\boldsymbol{\alpha}}$ *are defined. Let* $\Omega^i_{\boldsymbol{\alpha}} = \{\boldsymbol{\beta} \in \Omega_{\boldsymbol{\alpha}} \setminus \cup_{j=0}^{i-1} \Omega^j_{\boldsymbol{\alpha}} \mid \boldsymbol{\beta}$ *maximal with respect to* $<<\}$.

Lemma 3.1. a) *We have* $\Omega^0_{\boldsymbol{\alpha}} = \Delta(\boldsymbol{\gamma}_S + \boldsymbol{\alpha})$.

b) *For each* $\boldsymbol{\beta} \in \Omega^i_{\boldsymbol{\alpha}}$ *there exists a* $\boldsymbol{\delta} \in \Omega^{i-1}_{\boldsymbol{\alpha}}$ *such that* $\boldsymbol{\beta} << \boldsymbol{\delta}$.

c) *There is a finite number* m *of non-empty* $\Omega^i_{\boldsymbol{\alpha}}$.

d) $\Omega_{\boldsymbol{\alpha}} = \Omega^0_{\boldsymbol{\alpha}} \cup \cdots \cup \Omega^{m-1}_{\boldsymbol{\alpha}}$.

e) $\Omega^{m-1}_{\boldsymbol{\alpha}} = \{(0,0)\}$.

f) *If* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \Omega^i_{\boldsymbol{\alpha}}$, *then we cannot have* $\boldsymbol{\beta} << \boldsymbol{\delta}$.

Proof. a) By Proposition 2.2 a) we have $\boldsymbol{\beta} \in S$ if $\boldsymbol{\beta} >> \boldsymbol{\gamma}_S$, $\Delta^S(\boldsymbol{\gamma}_S) = \emptyset$, and $\boldsymbol{\gamma}_S \in S$ (since $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\gamma}) = \Delta^S(\mathbf{0}) = \emptyset$). This gives a).
The statement in b) follows from the definitions.
Choose $\boldsymbol{\beta}_i \in \Omega^i_{\boldsymbol{\alpha}}$ with $\boldsymbol{\beta}_0 >> \boldsymbol{\beta}_1 >> \cdots$. Since all $\boldsymbol{\beta}_i$ have positive coordinates, each chain of this type must have length bounded by $\max\{\gamma_1 + \alpha_1, \gamma_2 + \alpha_2\}$; so c) is proved.
The statements in d), e) and f) are trivial.

We have also in this situation a duality. This duality will be made more precise later on.

Proposition 3.2. *Let* $\boldsymbol{\beta} \in S$. *We have* $\boldsymbol{\beta} \in \Omega_{\boldsymbol{\alpha}}$ *if and only if* $\Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta}) \neq \emptyset$. *Furthermore, if* $\boldsymbol{\beta} \in \Omega_{\boldsymbol{\alpha}}$ *then* $\Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta}) \subseteq \Omega_{\boldsymbol{\alpha}}$.

Proof. Suppose $\Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta}) \neq \emptyset$. Then $\boldsymbol{\beta} - \boldsymbol{\alpha} \notin S$, since $\boldsymbol{\beta} - \boldsymbol{\alpha} \in S$ implies $\Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta}) = \emptyset$ according to Proposition 2.2 b). Now suppose $\boldsymbol{\beta} \in \Omega_{\boldsymbol{\alpha}}$. Then $\boldsymbol{\beta} \in S$ and $\boldsymbol{\beta} - \boldsymbol{\alpha} \notin S$ so $\Delta^S(\boldsymbol{\gamma}_S - \boldsymbol{\beta}) = \emptyset$ and $\Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta}) \neq \emptyset$ according to Proposition 2.2 b). If $\boldsymbol{\delta} \in \Delta^S(\boldsymbol{\gamma}_S + \boldsymbol{\alpha} - \boldsymbol{\beta})$, then $\boldsymbol{\delta} - \boldsymbol{\alpha} \in \Delta(\boldsymbol{\gamma}_S - \boldsymbol{\beta})$, but since $\Delta^S(\boldsymbol{\gamma}_S - \boldsymbol{\beta})$ is empty, $\boldsymbol{\delta} - \boldsymbol{\alpha} \notin S$, so $\boldsymbol{\delta} \in \Omega_{\boldsymbol{\alpha}}$.

Discussion. Now suppose that $\mathcal{O} = k[[X,Y]]/I = k[[x,y]]$, where $I = (F_1 \cdot F_2)$ and F_1 and F_2 are irreducible.

If the two branches defined by F_1 and F_2 have the same tangent, we can assume it is $Y = 0$ and, according to Weierstrass' Preparation Theorem, we can assume that $F_1 =$

$Y^{e_1} + \sum_{i=0}^{e_1-1} a_i(X)Y^i, F_2 = Y^{e_2} + \sum_{i=0}^{e_2-1} b_i(X)Y^i$, where e_1 and e_2 are the minimal powers such that F_1 (F_2 respectively) contains a term Y^{e_1} (a term Y^{e_2} respectively). Thus $F_1F_2 = Y^E + \sum_{i=0}^{E-1} c_i(X)Y^i$, where $E = e_1 + e_2$ is the multiplicity of the curve.

If the tangents of the two branches are distinct, we can assume that one is $Y = 0$, so $F_1 = Y^{e_1} + \sum_{i=0}^{e_1-1} a_i(X)Y^i$; as for F_2, if we write it as $F_2(X+Y, Y)$ we get a term Y^{e_2}, where e_2 is the minimal degree of the nonzero terms of F_2. Hence after the substitution $X = X+Y$ and applying Weierstrass' Preparation Theorem we get again $F_1F_2 = Y^E + \sum_{i=0}^{E-1} c_i(X)Y^i$, where $E = e_1 + e_2$ is the multiplicity of the curve.

It is clear that, in both cases, we can express $\mathcal{O}$ as a $k[[x]]$-module minimally generated by $1, y, y^2, \dots, y^{E-1}$, with $v(x) = (e_1, e_2)$ and $e_1 + e_2 = E$.

Let now $u, z \in \mathcal{O}$ be two elements such that $\mathcal{O}$ is a $k[[u]]$-module minimally generated by $1, z, z^2, \dots, z^{N-1}$, where $N = n_1 + n_2$ and $v(u) = (n_1, n_2) = \mathbf{n}$. Hence $\mathcal{O} \cong k[[U, Z]]/(F)$, where $F(U, Z) = Z^N + \sum_{i=0}^{N-1} b_i(U)Z^i$. Indeed there is the natural surjective homomorphism $\phi : k[[U, Z]] \longrightarrow \mathcal{O}$, whose kernel contains (F); now $\ker\phi$ has to be an intersection of two prime ideals P_1 and P_2 of height 1 (hence $P_1 = (G)$, $P_2 = (H)$ and $\ker\phi = (GH)$, since $k[[U, Z]]$ is a 2-dimensional UFD); moreover if GH divides F it has to be of the form $Z^j + \Psi(U, Z)$ with $j \leq N$ and, since $\mathcal{O}$ is minimally generated by $1, z, z^2, \dots, z^{N-1}$ as $k[[u]]$-module, then $j = N$ and $(GH) = (F)$.

Notice that, by the first part of this discussion, the classes $x = X + I, y = Y + I \in \mathcal{O}$ always satisfy the condition requested for u and z. Hence we can always assume that $\mathcal{O} = k[[x]] + k[[x]]y + k[[x]]y^2 + \ldots + k[[x]]y^{E-1}$, where $v(x) = (e_1, e_2) = \min(v(\mathcal{O}) \setminus \{(0, 0)\})$ and $v(y) = (r, s)$, with $r > e_1$ and $s \geq e_2$.

Definitions. Assume that up to the end of the section $u, z \in \mathcal{O}$ are fixed and let $\mathcal{O}_i$ be the $k[[u]]$-submodule of $\mathcal{O}$ generated by $1, z, \dots, z^i$, i.e., $\mathcal{O}_i = k[[u]] + k[[u]]z + \cdots + k[[u]]z^i$.

Moreover we set $Y_0 = \{1\}$, $Y_i = \{z^i + \phi_{i-1}(u, z) \mid \phi_{i-1}(u, z) \in \mathcal{O}_{i-1}, v(z^i + \phi_{i-1}(u, z)) \notin v(\mathcal{O}_{i-1})\}$.

Lemma 3.3. *For each $i < N - 1$ and each $\boldsymbol{\alpha} \in v(Y_i)$ there is a $\boldsymbol{\beta} \in v(Y_{i+1})$ such that $\boldsymbol{\alpha} << \boldsymbol{\beta}$.*

Proof. Let $\boldsymbol{\alpha} = v(z^i + \phi_{i-1}(u, z))$, where $z^i + \phi_{i-1}(u, z) \in Y_i$. Consider $z^{i+1} + z\phi_{i-1} \in \mathcal{O}_{i+1}$. If $v(z^{i+1} + z\phi_{i-1}) \in v(\mathcal{O}_{i+1}) \setminus v(\mathcal{O}_i)$, we are finished. Otherwise there is an $f_1 \in \mathcal{O}_i$ such that $v(f_1) = v(z^{i+1} + z\phi_{i-1})$. Hence there is a $c_1 \in k$ such that $v(z^{i+1} + z\phi_{i-1} - c_1f_1) > v(z^{i+1} + z\phi_{i-1})$. If $v(z^{i+1} + z\phi_{i-1} - c_1f_1) \in v(\mathcal{O}_{i+1}) \setminus v(\mathcal{O}_i)$ we are finished, otherwise we go on. If at some point $v(z^{i+1} + z\phi_{i-1} - c_1f_1 - \cdots - c_nf_n) \notin v(\mathcal{O}_i)$ we are finished. Otherwise we get two power series, $z^{i+1} + z\phi_{i-1}$ and $c_1f_1 + \cdots$ such that their difference belongs to any power of the maximal ideal (u, z) in $k[[u, z]]$, hence they coincide, since $k[[u, z]]$ is complete. But this is a contradiction since $z^{i+1} + z\phi_{i-1} \in \mathcal{O}_{i+1} \setminus \mathcal{O}_i$ and $c_1f_1 + \cdots \in \mathcal{O}_i$.

Lemma 3.4. a) *If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in v(Y_i)$ we cannot have $\boldsymbol{\alpha} << \boldsymbol{\beta}$.*

b) *For every $\boldsymbol{\alpha} \in v(Y_{i-1})$ and for every $\boldsymbol{\beta} \in v(Y_i)$, we cannot have $\boldsymbol{\beta} \leq\leq \boldsymbol{\alpha}$.*

Proof. a) Let $\boldsymbol{\alpha} = v(z_i)$ and $\boldsymbol{\beta} = v(z_i')$, where $z_i = z^i + \phi_{i-1}(u, z)$ and $z_i' = z^i + \phi_{i-1}'(u, z)$. If $\boldsymbol{\alpha} << \boldsymbol{\beta}$, then $v(z_i - z_i') = \boldsymbol{\alpha}$. But $z_i - z_i' = \phi_{i-1} - \phi_{i-1}' \in \mathcal{O}_{i-1}$. This contradicts $\boldsymbol{\alpha} \notin v(\mathcal{O}_{i-1})$.

b) If $\boldsymbol{\beta} \leq\leq \boldsymbol{\alpha}$, since, by Lemma 3.3, there exists $\boldsymbol{\beta}' \in v(Y_i)$ such that $\boldsymbol{\alpha} << \boldsymbol{\beta}'$, we get a contradiction with a).

Proposition 3.5. *Let* $\mathbf{n} = (n_1, n_2)$ *and* $N = n_1 + n_2$. *For every* $i = 1, \ldots, N-1$, *the set* $v(\mathcal{O}_i)$ *is a free* $\mathbb{N}\mathbf{n}$*-submodule of* $\mathbb{N}^2$ *generated by* $\mathbf{0}, v(Y_1), \ldots, v(Y_i)$, *i.e.,* $v(\mathcal{O}_i) = \mathbb{N}\mathbf{n} \cup (v(Y_1) + \mathbb{N}\mathbf{n}) \cup \cdots \cup (v(Y_i) + \mathbb{N}\mathbf{n})$, *where the unions are pairwise disjoint.*

Proof. Let $g(u,z) = g_0(u) + g_1(u)z + \cdots + g_i(u)z^i \in \mathcal{O}_i$ and let $g(U,Z) = g_0(U) + g_1(U)Z + \cdots + g_i(U)Z^i$. If all $g_i(U)$ contain a factor U^k, we get $v(g(u,z)) = k\mathbf{n} + v(g'(u,z))$, hence we may assume that there exists a minimal $j \leq i$ such that $g_j(U)$ is a unit in $k[[U]]$. Multiplying with its inverse, we can assume that $g(U,Z) = g_0(U) + \cdots + Z^j + g_{j+1}(U)Z^{j+1} + \cdots + g_i(U)Z^i$. Weierstrass' Preparation Theorem implies that there is a unit $u(U,Z) \in k[[U,Z]]$ such that $g(U,Z) = u(U,Z)(Z^j + h_{j-1}(U)Z^{j-1} + \cdots + h_0(U)) = u(U,Z)h(U,Z)$. Thus $v(g(u,z)) = v(h(u,z))$. If $j < i$, then $v(h(u,z)) \in v(\mathcal{O}_j)$, and the result follows by induction. If $j = i$ either $h(u,z) \notin Y_i$, and the result follows by induction, or $h(u,z) \in Y_i$ so $v(h(u,z)) \in v(Y_i)$.

By Lemma 3.4 b), it follows that, for every $\boldsymbol{\beta} \in v(Y_i)$ and for every $\boldsymbol{\alpha} \in v(\mathcal{O}_{i-1})$, $\boldsymbol{\beta} \not\equiv \boldsymbol{\alpha} \pmod{\mathbf{n}}$; hence $\boldsymbol{\beta} = v(z^i + \phi_{i-1}(u,z)) \notin v(\mathcal{O}_{i-1})$ if and only if $\boldsymbol{\beta} \not\equiv \boldsymbol{\alpha} \pmod{\mathbf{n}}$. It follows that the unions are pairwise disjoint.

Lemma 3.6. *If* $g(u,z) \in Y_i$, *then* $v(g(u,z)) \in \Omega_{\mathbf{n}}$.

Proof. Let $g(u,z) = z^i + \phi_{i-1}(u,z) \in Y_i$, where $\phi_{i-1} \in \mathcal{O}_{i-1}$. If $v(g) \notin \Omega_{\mathbf{n}}$, then $v(g) - \mathbf{n} \in v(\mathcal{O}) = S$. Hence $v(g) \in v(\mathcal{O}) + \mathbf{n}$, so $v(g) = v(h) + \mathbf{n}$ for some $h \in \mathcal{O}$. If $v(h) \in v(\mathcal{O}_{i-1})$, then $v(g) = v(h) + \mathbf{n} \in v(\mathcal{O}_{i-1})$ which is a contradiction. Thus we can suppose that $h \in \mathcal{O}_l \setminus \mathcal{O}_{l-1}$ for some $l \geq i$. By Proposition 3.5 $v(h) = m\mathbf{n} + v(z^l + \psi_{l-1}(u,z))$ for some $z^l + \psi_{l-1} \in Y_l$. Now $v(z^l + \psi_{l-1} - z^{l-i}g) = v(z^l + \psi_{l-1} - z^l - z^{l-i}\phi_{i-1}) = v(\psi_{l-1} - z^{l-i}\phi_{i-1}) \in v(\mathcal{O}_{l-1})$. But $v(z^l + \psi_{l-1}) = v(h) - m\mathbf{n} \leq\leq v(h) << v(g) \leq\leq v(z^{l-i}g)$ so $v(z^l + \psi_{l-1} - z^{l-i}g) = v(z^l + \psi_{l-1}) \notin v(\mathcal{O}_{l-1})$, which is a contradiction.

Lemma 3.7. *We have* $\Omega_{\mathbf{n}} = \cup_{i=0}^{N-1} v(Y_i)$.

Proof. By Lemma 3.6 we have $v(Y_i) \subseteq \Omega_{\mathbf{n}}$, in particular we see that all elements in $\cup v(Y_i)$ are in different congruence classes $\pmod{\mathbf{n}}$. Since, by Proposition 3.5, $S = v(\mathcal{O}) = \mathbb{N}\mathbf{n} \cup \cdots \cup (v(Y_{N-1}) + \mathbb{N}\mathbf{n})$, we get the other inclusion.

Proposition 3.8. *We have* $v(Y_{N-1-i}) = \Omega_{\mathbf{n}}^i$.

Proof. We show first that $\Omega_{\mathbf{n}}^0 = v(Y_{N-1})$. Let $\boldsymbol{\alpha} \in \Omega_{\mathbf{n}}^0$. If $\boldsymbol{\alpha} \in v(Y_i)$ for some $i < N-1$, then $\boldsymbol{\alpha}$ is not maximal according to Lemma 3.3, hence $\boldsymbol{\alpha} \notin \Omega_{\mathbf{n}}^0$, a contradiction. Suppose now $\boldsymbol{\alpha} \in v(Y_{E-1})$ and not in $\Omega_{\mathbf{n}}^0$. Then $\boldsymbol{\alpha}$ is not maximal in $\Omega_{\mathbf{n}} = \cup_{i=0}^{N-1} v(Y_i)$. Thus there exists a $\boldsymbol{\beta} \in v(Y_j)$ for some j such that $\boldsymbol{\alpha} << \boldsymbol{\beta}$. Hence, if $j = N-1$, we contradict Lemma 3.4. If $j < N-1$ we get, by Lemma 3.3, that there exists a $\boldsymbol{\delta} \in v(Y_{N-1})$ such that $\boldsymbol{\alpha} << \boldsymbol{\beta} << \boldsymbol{\delta}$, which again contradicts Lemma 3.4. Assume now that the statement is proved for $j = 0, 1, \ldots, i-1$. Then $\Omega_{\mathbf{n}} \setminus \cup_{j=0}^{i-1} \Omega_{\mathbf{n}}^j = \Omega_{\mathbf{n}} \setminus \cup_{j=0}^{i-1} v(Y_{N-1-j})$. The statement $\Omega_{\mathbf{n}}^i = v(Y_{N-1-i})$ has now the same proof as above.

Corollary 3.9. *There are the same number of $\Omega^i_{\mathbf{n}}$'s as Y_i's, namely N.*

Notation. We will from now on denote $v(Y_i)$ by $\Omega^{\mathbf{n}}_i$. Thus $\Omega^{\mathbf{n}}_i = \Omega^{N-1-i}_{\mathbf{n}}$. We call it the *i-th component* of the Apery set of S with respect to $\mathbf{n}$.

Example. Consider the semigroup $v(\mathcal{O})$, where $\mathcal{O} = k[[X,Y]]/(Y^4 - 2X^3Y^2 - 4X^5Y + X^6 - X^7)(Y^2 - X^3)$ and choose $\mathbf{n} = \mathbf{e} = (4,2)$ (cf. Fig. 1, where the elements of $\Omega_{\mathbf{e}}$ are marked with circles). We have $\Omega_{\mathbf{e}} = \cup_{i=0}^{5} \Omega^{\mathbf{e}}_i$, where $\Omega^{(4,2)}_0 = \{(0,0)\}$, $\Omega^{(4,2)}_1 = \{(6,3)\}$, $\Omega^{(4,2)}_2 = \{(13,n) \mid n \geq 7\}$, $\Omega^{(4,2)}_3 = \{(19,n) \mid n \geq 10\}$, $\Omega^{(4,2)}_4 = \{(n,13) \mid n \geq 27\} \cup \{(26,n) \mid n \geq 14\}$, and $\Omega^{(4,2)}_5 = \{(n,16) \mid n \geq 33\} \cup \{(32,n) \mid n \geq 17\}$.

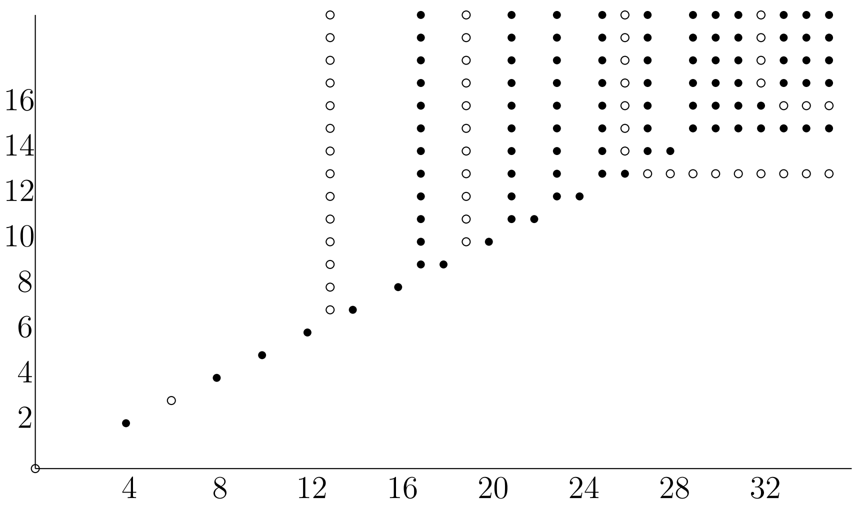

Fig. 1. The semigroup of $\mathcal{O}$ $=k[[X,Y]]/(Y^4 - 2X^3Y^2 - 4X^5Y + X^6 - X^7)(Y^2 - X^3)$

Proposition 3.10. *We have $\boldsymbol{\alpha} \in \Omega^{\mathbf{n}}_i$ if and only if $\Delta^S(\boldsymbol{\gamma}_S + \mathbf{n} - \boldsymbol{\alpha}) \subseteq \Omega^{\mathbf{n}}_{N-1-i}$.*

Proof. Let $\boldsymbol{\alpha} = v(z^i + \phi_{i-1}(u,z)) \in \Omega^{\mathbf{n}}_i$, with $\phi_{i-1} \in \mathcal{O}_{i-1}$. We know that $\Delta^S(\boldsymbol{\gamma}_S + \mathbf{n} - \boldsymbol{\alpha}) \subseteq \Omega_{\mathbf{n}}$ by Proposition 3.2. Pick a $\boldsymbol{\beta} \in \Delta^S(\boldsymbol{\gamma}_S + \mathbf{n} - \boldsymbol{\alpha})$. Suppose $\boldsymbol{\beta} \in \Omega^{\mathbf{n}}_j$, so $\boldsymbol{\beta} = v(y^j + \psi_{j-1}(x,y))$ for some $\psi_{j-1} \in \mathcal{O}_{j-1}$. Now $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \Delta^S(\boldsymbol{\gamma}_S + \mathbf{n}) = v(Y_{N-1})$. But $\boldsymbol{\alpha} + \boldsymbol{\beta} = v((z^i + \phi_{i-1})(z^j + \psi_{j-1})) = v(z^{i+j} + \Phi(u,z))$. We want to show that $j = N - i - 1$, i.e., that $i+j = N-1$. If $i+j < N-1$, then $y^{i+j} + \Phi \in \mathcal{O}_{i+j}$, so $v(y^{i+j} + \Phi) \in v(\mathcal{O}_{i+j})$ which contradicts $\boldsymbol{\alpha} + \boldsymbol{\beta} \in v(Y_{N-1})$, since, for every $f(u,z) \in Y_{N-1}$, $f(u,z) \notin \mathcal{O}_{i+j}$. If $i+j > N-1$, then $z^{i+j} = z^{i+j-N} f(u,z)$, where $Z^N + f_{N-1}(U)Y^{N-1} + \cdots + f_0(U) = Y^N - f(U,Z)$ is the equation of our curve in $k[[U,Z]]$. Since none of the f_i's is a unit, we can factor out an u from $f(u,z)$. Now neither ϕ_{i-1} nor ψ_{j-1} contains a pure power of z, since if there were a pure power of z in e.g. ϕ_{i-1}, then $z^i + \phi_{i-1}$ would lie in some $\mathcal{O}_k$ with $k < i$ by Weierstrass' Preparation Theorem. The same applies for ψ_{j-1}. Thus we can factor out an u from $\Phi = z^j \phi_{i-1} + z^i \psi_{j-1} + \phi_{i-1}\psi_{j-1}$, and thus from $z^{i+j} + \Phi$. Thus $v(z^{i+j} + \Phi) - \mathbf{n} \in S$, so $v(y^{i+j} + \Phi) \notin \Omega_{\mathbf{n}}$, a contradiction.

4. Blowing up

Let $\mathcal{O} = k[[x,y]]$ be a plane curve with two branches, where $v(x) = \mathbf{e} = (e_1, e_2)$ is the smallest value in $S \setminus \{\mathbf{0}\}$. The blowup of $\mathcal{O}$ is $\mathcal{O}' = k[[x, y/x]]$. We will now derive the connection between the Apery sets of $v(\mathcal{O})$ and $v(\mathcal{O}')$ with respect to $\mathbf{e}$, in case also $\mathcal{O}'$ is local.

Theorem 4.1. *Assume that $\mathcal{O}$ and $\mathcal{O}'$ are both local rings. Let $\Omega_i^{\mathbf{e}}$ and $(\Omega_i')^{\mathbf{e}}$ denote the i-th component of the Apery set with respect to $\mathbf{e}$ of $v(\mathcal{O})$ and $v(\mathcal{O}')$, respectively. Then $\Omega_i^{\mathbf{e}} = (\Omega_i')^{\mathbf{e}} + i\mathbf{e}$.*

Proof. For $0 \leq i \leq E-1$, denote by Y_i the subsets of $\mathcal{O}$ defined as in Section 3.

Consider the elements in $\mathcal{O}'$ as power series in x and $z = y/x$. Then $\mathcal{O}'$ is a $k[[x]]$-module minimally generated by $1, z, \dots, z^{E-1}$, where $E = e_1 + e_2$. Denote by $\mathcal{O}_i'$ the $k[[x]]$-module generated by $1, z, \dots, z^i$ and let $Y_i' = \{z^i + \phi_{i-1}(x,z) \mid \phi_{i-1}(x,z) \in \mathcal{O}_{i-1}', v(z^i + \phi_{i-1}(x,z)) \notin v(\mathcal{O}_{i-1}')\}$, $(\Omega_i')^{\mathbf{e}} = v(Y_i')$. We know, by the results of previous section, that $v(Y_i) = \Omega_i^{\mathbf{e}}$ (respectively $v(Y_i') = (\Omega_i')^{\mathbf{e}}$) is the i-th component of the Apery set of $v(\mathcal{O})$ (respectively $v(\mathcal{O}')$) with respect to $\mathbf{e}$.

Notice that $v(\mathcal{O}_i')$ and $v(\mathcal{O}_i)$ contain exactly the same congruence classes (mod $\mathbf{e}$), that is, if $\boldsymbol{\alpha} \in \mathbb{N}^2$, then there exists $\boldsymbol{\beta} \in v(\mathcal{O}_i)$ such that $\boldsymbol{\beta} \equiv \boldsymbol{\alpha}$ (mod $\mathbf{e}$) if and only if there exists $\boldsymbol{\beta}' \in v(\mathcal{O}_i')$ such that $\boldsymbol{\beta}' \equiv \boldsymbol{\alpha}$ (mod $\mathbf{e}$). In fact if $\boldsymbol{\alpha} \equiv \boldsymbol{\beta} = v(f_0(x) + f_1(x)y + \ldots + f_i(x)y^i) \in v(\mathcal{O}_i)$, then $\boldsymbol{\beta} = v(f_0(x) + f_1(x)xz + \ldots + f_i(x)x^iz^i) \in v(\mathcal{O}_i')$ and, conversely, if $\boldsymbol{\alpha} \equiv \boldsymbol{\beta}' = v(g_0(x) + g_1(x)z + \ldots + g_i(x)z^i) \in v(\mathcal{O}_i')$, then $\boldsymbol{\beta} = \boldsymbol{\beta}' + i\mathbf{e} = v((g_0(x) + g_1(x)y/x + \ldots + g_i(x)y^i/x^i)x^i) \in v(\mathcal{O}_i)$.

We claim that $Y_i'x^i \subseteq Y_i$. If $f(x,z) \in Y_i'$, then $f(x, y/x)x^i$ is an element of the form requested in Y_i and, if $\boldsymbol{\alpha} = v(f(x,z))$, then $v(f(x,x/y)x^i)) = \boldsymbol{\alpha} + i\mathbf{e} \notin v(\mathcal{O}_{i-1})$, since $\boldsymbol{\alpha} \notin v(\mathcal{O}_{i-1}')$.

By the inclusion $Y_i'x^i \subseteq Y_i'$ we get, passing to the values, that $(\Omega_i')^{\mathbf{e}} + i\mathbf{e} \subseteq \Omega_i^{\mathbf{e}}$. To prove the equality, assume that i is the smallest index such that $(\Omega_i')^{\mathbf{e}} + i\mathbf{e} \subsetneq \Omega_i^{\mathbf{e}}$. Then in $v(\mathcal{O}_i) = \mathbb{N}\mathbf{e} \cup (\Omega_0^{\mathbf{e}} + \mathbb{N}\mathbf{e}) \cup \cdots \cup (\Omega_i^{\mathbf{e}} + \mathbb{N}\mathbf{e})$ (cf. Proposition 3.5) there is a congruence class not appearing in $v(\mathcal{O}_i') = \mathbb{N}\mathbf{e} \cup ((\Omega_0')^{\mathbf{e}} + \mathbb{N}\mathbf{e}) \cup \cdots \cup ((\Omega_i')^{\mathbf{e}} + \mathbb{N}\mathbf{e})$, which is in contradiction with what we have showed above.

Assume now that $\mathcal{O}$ is local and $\mathcal{O}'$ is not local; this happens when the two branches of $\mathcal{O}$ have distinct tangents (cf. Proposition 2.2c)). In this case $\mathcal{O}'$ is the direct product of localizations at its maximal ideals (cf. [3, Proposition 3.1]), hence $S' = v(\mathcal{O}') = S_1' \times S_2'$, where S_1' and S_2' are the two projections of S'.

If $S = v(\mathcal{O})$, the projections S_1 and S_2 of S are the (numerical) value semigroups of the two branches of $\mathcal{O}$, so the connection between S_1 and S_1' (and between S_2 and S_2', respectively) is described in 2.1. Thus, if we know S and consequently its projections S_1 and S_2, we get easily $S' = S_1' \times S_2'$.

On the other hand, if we know S' and consequently its projections S_1' and S_2', we get the projections S_1 and S_2 of S, but they are not enough to determine S, that is not a direct product. To get a precise description of S we can use some results from [10]. Let $0 = a_0 < \cdots < a_{e_1-1}$ and $0 = b_0 < \cdots < b_{e_2-1}$ be the Apery sets of S_1 and S_2, respectively, with respect to their smallest positive elements e_1 and e_2, then we have the following result.

Proposition 4.2. [10, Theorems 6 and 18] *Suppose $\mathcal{O}$ has distinct tangents at the origin. Let* $P = \{(a_j+ie_1, b_i+je_2) \mid 0 \le i \le e_1-1, 0 \le j \le e_2-1\}$. *Then* $v(\mathcal{O}) = S_1 \times S_2 \setminus \cup_{p \in P} \Delta^S(p)$.

Finally, if $\mathcal{O}$ is not local (and thus also $\mathcal{O}'$ is not local), both $S = v(\mathcal{O})$ and $S' = v(\mathcal{O}')$ are direct products of their projections: $S = S_1 \times S_2$, $S' = S'_1 \times S'_2$; it follows that the result for the one branch case (2.1) is enough to describe the connection between S and S'.

4.1. From the semigroup to the multiplicity tree

We will use our results to determine the multiplicity tree of a plane curve with two branches $\mathcal{O}$ from its semigroup and vice versa. The fine multiplicty of $\mathcal{O}$ is the smallest nonzero value in $v(\mathcal{O})$. By Theorem 4.1, we get the semigroup of the blowup $\mathcal{O}'$ from the semigroup of $\mathcal{O}$. If $\mathcal{O}'$ is local, we get the fine multiplicity as the smallest nonzero value in $v(\mathcal{O}')$. We continue like this as long as the blowup is local. At some point a blowup will have two maximal ideals. Then the semigroup is the product of the two projections, $S = S_1 \times S_2$, and the multiplicity tree splits in two branches with fine multiplicity $(e_1, 0)$ and $(0, e_2)$, where e_i is the smallest positive value in S_i, $i = 1, 2$. After that point, the blowup is the product of the blowups of the two branches, so we can use the corresponding results for one branch to continue. In this way the multiplicity tree is determined.

We illustrate our result with an example.

Example. Let us start with a semigroup for a plane curve as in Fig. 1. This semigroup is local, and by Theorem 4.1 we get for the blowup $\Omega_0^{(4,2)} = \{(0,0)\}$, $\Omega_1^{(4,2)} = \{(2,1)\}$, $\Omega_2^{(4,2)} = \{(5,n) \mid n \ge 3\}$, $\Omega_3^{(4,2)} = \{(7,n) \mid n \ge 4\}$, $\Omega_4^{(4,2)} = \{(n,5) \mid n \ge 11\} \cup \{(10,n) \mid n \ge 6\}$, and $\Omega_5^{(4,2)} = \{(n,6) \mid n \ge 13\} \cup \{(12,n) \mid n \ge 7\}$. This gives the whole semigroup of the blowup, since the semigroup is the free $(4,2)$-module on this set, cf. Proposition 3.5. The semigroup is shown in Fig. 2. This semigroup is still local with fine multiplicity $(2,1)$. We show $\Omega^{(2,1)}$ in Fig. 3. In fact $\Omega_0^{(2,1)} = \{(0,0)\}$, $\Omega_1^{(2,1)} = \{(5,n) \mid n \ge 3\}$, $\Omega_2^{(2,1)} = \{(n,5) \mid n \ge 11\} \cup \{(10,n) \mid n \ge 6\}$. Applying again Theorem 4.1 and Proposition 3.5, we get the semigroup shown in Fig. 4. This semigroup is still local with fine multiplicity $(2,1)$ and Apery sets $\Omega_0^{(2,1)} = \{(0,0)\}$, $\Omega_1^{(2,1)} = \{(3,n) \mid n \ge 2\}$, $\Omega_2^{(2,1)} = \{(n,3) \mid n \ge 7\} \cup \{(6,n) \mid n \ge 4\}$. In the next blowup we get the semigroup in Fig. 5 which is still local with fine multiplicity $(1,1)$. The branches now have different tangents, since $\Delta^S((1,1)) \neq \emptyset$, cf. Proposition 2.2 c). Both projections of the semigroup are $\mathbb{N}$, hence the next blowup has semigroup $\mathbb{N} \times \mathbb{N}$. Thus we get the multiplicity tree in Fig. 6.

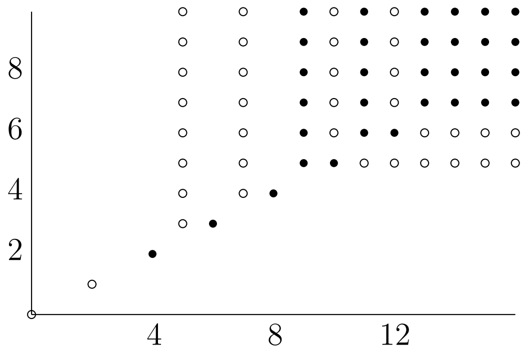

Fig. 2. The semigroup of the blowup of $\mathcal{O}$

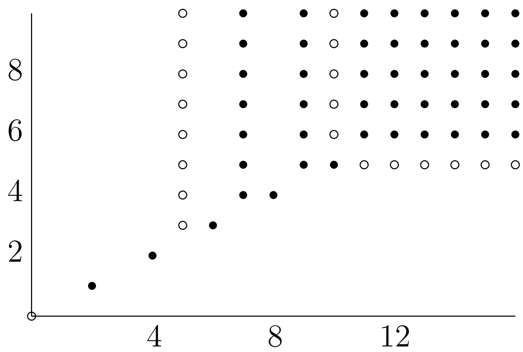

Fig. 3. The semigroup of the blowup of $\mathcal{O}$

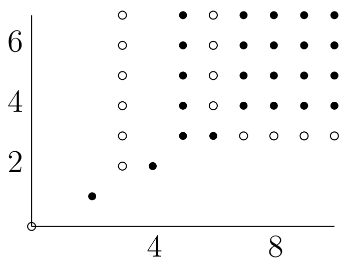

Fig. 4. The semigroup of the second blowup of $\mathcal{O}$

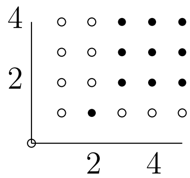

Fig. 5. The semigroup of the third blowup of $\mathcal{O}$

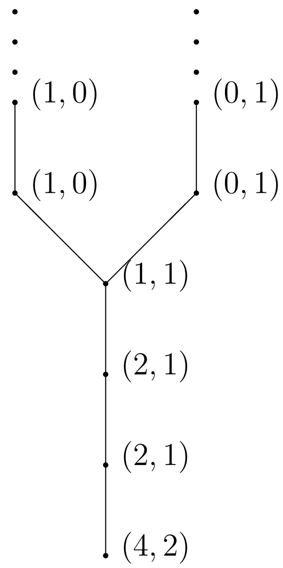

Fig. 6

4.2. From the multiplicity tree to the semigroup

Now suppose that we have a multiplicity tree for a plane curve with two branches. At some level we have the fine multiplicities $(1, 0)$ and $(0, 1)$ on the two branches. If on the previous level we still have two maximal ideals, we can blow down each branch and the semigroup will be the product of the two semigroups. We continue like this to a level j, where on level $j-1$ we have a local ring. We can, in the same way as above, get the projections S_1 and S_2 of $S = v(\mathcal{O}^{(j-1)})$ on the coordinate axes and apply Proposition 4.2. After that we can continue with our Theorem 4.1. Finally we get the semigroup of $\mathcal{O}$.

Also here we give an example.

Example Now we start with the multiplicity tree in Fig. 7. The semigroup on level 3 is $\mathbb{N} \times \mathbb{N}$, since the fine multiplicities are $(1, 0)$ and $(0, 1)$, respectively. On the previous level the ring still has two maximal ideals, both with multiplicity 2. The Apery set of $\mathbb{N}$ with respect to 2 is $\{0, 1\}$, so on the previous level the Apery set with respect to 2 is $\{0, 3\}$, which gives the semigroup $\langle 2, 3\rangle$ on level 2. Thus the semigroup of the curve is $\langle 2, 3\rangle \times \langle 2, 3\rangle$. Proposition 4.2 gives the local semigroup in Fig. 8 on level 1, with fine multiplicity $(2, 2)$, and Apery sets with respect to $(2, 2)$ equal to: $\Omega_0^{(2,2)} = \{(0, 0)\}$, $\Omega_1^{(2,2)} = \{(4, 2), (5, 2), (2, 4), (2, 5)\}$, $\Omega_2^{(2,2)} = \{(n, 4) \mid n \geq 8\} \cup \{(4, n) \mid n \geq 8\} \cup \{(6, 7), (7, 7), (7, 6)\}$, and $\Omega_3^{(2,2)} = \{(n, 9) \mid n \geq 10\} \cup \{(9, n) \mid n \geq 10\}$. Then we use Theorem 4.1 to get the semigroup at level 0. We have the fine multiplicity $(2, 2)$, and for the Apery sets we get $\Omega_0^{(2,2)} = \{(0, 0)\}$, $\Omega_1^{(2,2)} = \{(6, 4), (7, 4), (4, 6), (4, 7)\}$, $\Omega_2^{(2,2)} = \{(n, 8) \mid n \geq 12\} \cup \{(8, n) \mid n \geq 12\} \cup \{(10, 11), (11, 11), (11, 10)\}$, and $\Omega_3^{(2,2)} = \{(n, 15) \mid n \geq 16\} \cup \{(15, n) \mid n \geq 16\}$. This gives the semigroup in Fig. 9.

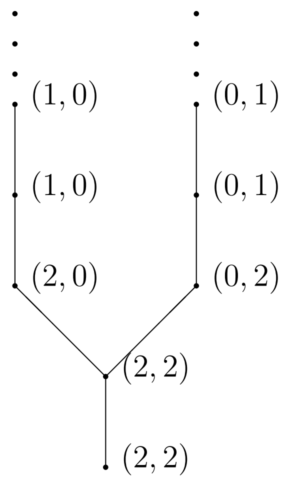

Fig. 7

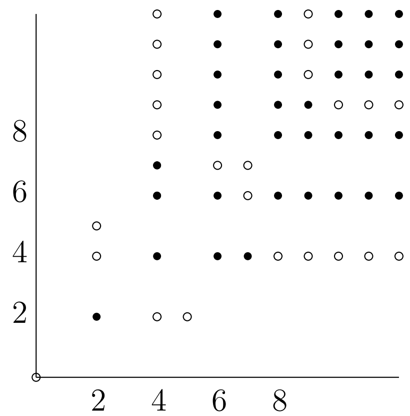

Fig. 8

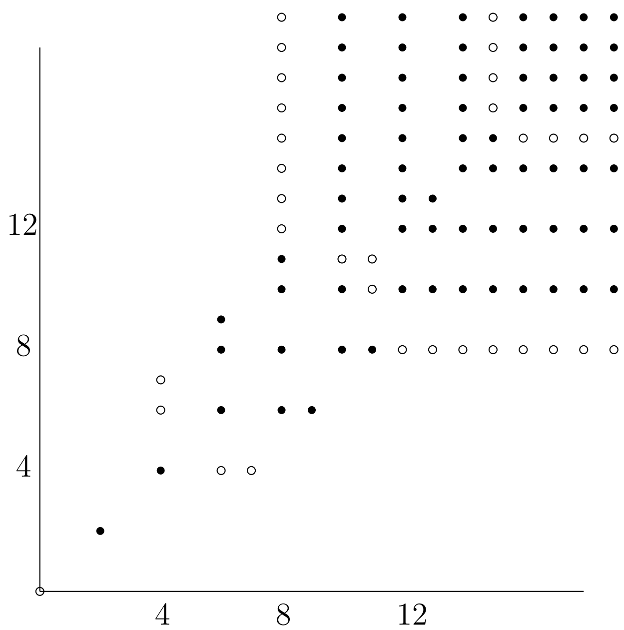

Fig. 9

5. The multiplicity tree of a plane curve with two branches

The possible multiplicity sequences of a plane branch, and thus also the semigroups are characterized (cf. e.g. [4, Theorem 3.2]). In order to extend also these results to curves with two branches, we will now give an explicit characterization of the multiplicity tree of a plane curve with two branches. With the results of previous sections this is also a characterization of the semigroup of a plane curve with two branches.

Suppose we have a plane branch with multiplicity sequence $e_0, e_1, \ldots$. It is well-known that, for each $i \geq 0$, $e_i = \sum_{h=1}^{k} e_{i+k}$, for some $k \geq 1$ (cf. [3, Theorem 5.11]). In particular the sequence is not increasing. The *restriction number* $r(e_j)$ of e_j is defined to be the number of sums $e_i = \sum_{h=1}^{k} e_{i+k}$ where e_j appears as a summand. It is also well-known that, if R is a plane branch, $r(e_j) = 1$ or 2, for $j \geq 1$ (cf. [7, Corollary 3.5.7]). The ring $R^{(j)}$ obtained by blowing up j times R is classically called a *free point* if $r(e_j) = 1$, where $e_j = e(R^{(j)})$ and a *satellite point* if $r(e_j) = 2$. When, in a multiplicity sequence of a plane branch, we have $e_i > e_{i+1}$, then, if the Euclidean division gives

$$e_i = e_{i+1} q_i + r_i \quad (0 \leq r_i < e_{i+1})$$

we will have in the sequence q_i times e_{i+1} and then, if $r_i \neq 0$, r_i:

$$e_i > e_{i+1} = e_{i+2} = \cdots = e_{i+q_i} > r_i = e_{i+q_i+1}.$$

It is clear that the elements $e_{i+2}, \ldots, e_{i+q_i}, e_{i+q_i+1}$ have restriction number 2, because any of them is a summand of the previous one and of e_i. Since $e_{i+q_i} > e_{i+q_i+1}$, our multiplicity sequence is uniquely determined with elements of restriction number 2 (corresponding to satellite points), up to the $\gcd(e_i, e_{i+1}) = e_N$. After e_N, we will have any number q $(0 \leq q)$ of elements equal to e_N, before an element $e_{N+q+1} < e_{N+q}$, that starts another Euclidean algorithm. The elements $e_{N+1}, \ldots, e_{N+q+1}$ have restriction number 1, because any of them is a summand only for the previous one and corresponds to free points.

What is recalled above (and it is well-known) is got from the fact that, if $R^{(i)}$ has a parametrization

$$(t^{e_i}, ct^{n_i} + \cdots)$$

with $e_i < n_i$, so that $e(R^{(i)}) = e_i$, and $c \neq 0$ (always possible by Puiseux Theorem), then the blowup $R^{(i+1)}$ has a parametrization

$$(t^{e_i}, ct^{n_i - e_i} + \cdots)$$

so that $e(R^{(i+1)}) = e_{i+1} = \min(e_i, n_i - e_i)$.

If $e_i > e_{i+1}$, i.e. $e_{i+1} = n_i - e_i$, the exponent n_i is uniquely determined. In this case the couple e_i, e_{i+1} (or e_i, n_i) is enough to determine the multiplicity sequence as long as the restriction number is 2. If we include the coefficient c of the parametrization of $R^{(i)}$ as information, then also the tangents of $R^{(i)}$, $R^{(i+1)}$, ... are determined as long as we have satellite points.

Example. Consider the multiplicity sequence $e_0 = 7, e_1 = 2, e_2 = 2, e_3 = 2, e_4 = 1, e_5 = 1, e_6 = 1, \ldots$, that is the multiplicity sequence of a plane branch (cf. [4, Theorem 3.2]). For the restriction numbers we get: $r(e_0) = 0, r(e_1) = 1, r(e_2) = 2, r(e_3) = 2, r(e_4) = 2, r(e_5) = 2, r(e_6) = 1, \ldots$. Notice that, after 7,2 (since $7 > 2$) all the numbers in the multiplicity sequence are determined as long as the restriction number is 2. We can suppose that a plane branch $R = R^{(0)}$ with multiplicity sequence $7, 2, 2, 2, 1, 1, \ldots$ has parametrization $(t^7, ct^9+\cdots)$ with $c \neq 0$. Hence the successive blowups are $R^{(1)} = (t^7, ct^2 + \cdots)$, $R^{(2)} = (c^{-1}t^5 + \cdots, ct^2 + \cdots)$, $R^{(3)} = (c^{-2}t^3+\cdots, ct^2+\cdots)$, $R^{(4)} = (c^{-3}t+\cdots, ct^2+\cdots)$, $R^{(5)} = (c^{-3}t+\cdots, c^4t+\cdots)$, where $R^{(2)}, \ldots, R^{(5)}$ are satellite points.

Another example. Consider the multiplicity sequence 2,2,2,1,1, ... The restriction numbers are 0,1,1,1,2,1,1, ... In this case the first two elements of the multiplicity sequence do not determine the following ones. We can suppose that a plane branch $R = R^{(0)}$ with multiplicity sequence $2, 2, 2, 1, 1, \ldots$ has parametrization $(t^2, ct^n + \cdots)$ with $2 < n$ and $c \neq 0$, but the exponent n is not uniquely determined in this case. The value semigroup is $\langle 2, 7\rangle$, so the characteristic exponents are 2 and 7, so R has a parametrization $(t^2, at^4 + bt^6 + ct^7 + \cdots)$, with a and b arbitrary but with $c \neq 0$.

We denote the tangent vector of R at the origin by $\operatorname{tg} R$.

Lemma 5.1. *Let R be a plane branch with multiplicity sequence $e_0, e_1, \ldots$. Suppose that $R^{(i)}$ has a parametrization*

$$(t^{e_i}, ct^{n_i} + \cdots)$$

with $e_i < n_i$, and $c \neq 0$, so that $e(R^{(i)}) = e_i$ and $\operatorname{tg} R^{(i)} = (1, 0)$, then:

if $n_i < 2e_i$ (equivalently $e_i > e_{i+1}$), we have $\operatorname{tg} R^{(i+1)} = (0, 1)$;

if $n_i \geq 2e_i$ (equivalently $e_i = e_{i+1}$), we have to distinguish two cases:

1) *In case $r(e_{i+1}) = 2$:*
   if $r(e_{i+2}) = 2$, then $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, 0)$;
   if $r(e_{i+2}) = 1$, then $\operatorname{tg} R^{(i+1)} = (1, c)$, with $c \neq 0$.
   Moreover if we have a plane branch multiplicity sequence $e_0, e_1, \ldots,$ with $r(e_{i+1}) = 2$ and $r(e_{i+2}) = 1$, then, for each $c \neq 0$, there exists a plane branch R such that $\operatorname{tg} R^{(i)} = (1, 0)$ and $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, \mathrm{c})$.
2) *In case $r(e_{i+1}) = 1$, $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, \mathrm{c})$.*
   Moreover if we have a plane branch multiplicity sequence $e_0, e_1, \ldots,$ with $r(e_{i+1}) = 1$, then, for each c, there exists a plane branch R such that $\operatorname{tg} \mathrm{R}^{(i)} = (1, 0)$ and $\operatorname{tg} R^{(i+1)} = (1, c)$.

Proof. If $n_i < 2e_i$, the claim follows immediately from the parametrization of the blowup.

Suppose $n_i \geq 2e_i$. In case 1), in the parametrization $(t^{e_i}, ct^{n_i}+\cdots)$ of $R^{(i)}$, the exponent n_i is uniquely determined by the multiplicity sequence, thus, if $r(e_{i+2}) = 2$, we have necessarily $n_i > 2e_i$ and $(t^{e_i}, ct^{n_i-e_i}+\cdots)$, with $e_i = e_{i+1} < n_i-e_i = n_{i+1}$, is a parametrization for $R^{(i+1)}$. Hence $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, 0)$. If, on the other hand, $r(e_{i+2}) = 1$, we have necessarily $n_i = 2e_i$ and $R^{(i+1)} = (t^{e_i}, ct^{n_i-e_i} + \cdots)$, with $e_i = e_{i+1} = n_i - e_i = n_{i+1}$, has $\operatorname{tg}(1, \mathrm{c})$, with $c \neq 0$.

In case 2), in the parametrization $(t^{e_i}, ct^{n_i} + \cdots)$ of $R^{(i)}$, the exponent n_i is not uniquely determined by the multiplicity sequence (although the value semigroup of $R^{(i)}$ and the characteristic exponents are). Also in this case of course, if $n_i > 2e_i$, $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, 0)$ and, if $n_i = 2e_i$, $\operatorname{tg} \mathrm{R}^{(i+1)} = (1, \mathrm{c})$, but the two possibilities are not forced by any other condition.

Recall that if $\mathcal{O}$ is a plane curve with two branches R and T, then $\mathcal{O}$ is local, so the tree has not split, as long as the branches have the same tangents, cf. Proposition 2.2 c). If i is the first index such that $R^{(i)}$ and $T^{(i)}$ have different tangents, then i is the splitting level of (the multiplicity tree of) the curve. With this in mind, we can now give the following characterization for a multiplicity tree of a plane curve with two branches.

Proposition 5.2. *Let $e_0, e_1, \dots$ and $f_0, f_1, \dots$ be two plane branch multiplicity sequences. They give a multiplicity tree of a plane curve with two branches with splitting node at level k if and only if the following conditions are satisfied:*

1) *$e_{i-1} = e_i$ if and only if $f_{i-1} = f_i$, for $i = 1, \dots, k-1$.*
2) *$r(e_i) = r(f_i)$, for $i = 0, \dots, k$.*
3) *If $e_{k-1} > e_k$, then $f_{k-1} = f_k$.*
4) *If $r(e_k) = r(f_k) = r(e_{k+1}) = r(f_{k+1}) = 2$ and if $e_{k-1} = e_k$, then $f_{k-1} > f_k$.*

Proof. Conditions 1) and 2) are necessary and sufficient in order that the two multiplicity sequences are "very similar" up to level k. Conditions 3) and 4) are necessary and sufficient in order that the two multiplicity sequences are not "too similar" around level k.

Denote by R and T the two branches and by $R = R^{(0)} \subset R^{(1)} \subset \cdots$ and $T = T^{(0)} \subset T^{(1)} \subset \cdots$ the respective blowing ups.

Proof of necessity:

1) If for example $e_{i-1} > e_i$ and $f_{i-1} = f_i$, supposing that $\operatorname{tg} R^{(i-1)} = \operatorname{tg} T^{(i-1)} = (1,0)$, we have by Lemma 5.1 $\operatorname{tg} R^{(i)} = (0,1)$ and $\operatorname{tg} T^{(i)} = (1,c)$, for some constant c, so level i ($i \leq k-1$) is necessarily a splitting level, a contradiction.

2) Suppose $r(e_i) = 1$ and $r(f_i) = 2$, for some $i \leq k$. It means that e_i is a summand only for e_{i-1} and f_i is a summand for f_{i-1} and for f_{i-j}, for some $j > 1$. This is in contradiction with the fact that the node (e_{i-j}, f_{i-j}) is the sum of the nodes of a subtree rooted in it (and we are supposing that the tree has splitting level $k \geq i$) (cf. [3, Theorem 5.11, c)]).

3) If $e_{k-1} > e_k$ and $f_{k-1} > f_k$, supposing that $\operatorname{tg} R^{(k-1)} = \operatorname{tg} T^{(k-1)} = (1,0)$, we get $\operatorname{tg} R^{(k)} = \operatorname{tg} T^{(k)} = (0,1)$ and level k is not a splitting level, a contradiction.

4) If $r(e_k) = r(f_k) = r(e_{k+1}) = r(f_{k+1}) = 2$, $e_{k-1} = e_k$ and $f_{k-1} = f_k$, then supposing that $\operatorname{tg} R^{(k-1)} = \operatorname{tg} T^{(k-1)} = (1,0)$, we get by Lemma 5.1 $\operatorname{tg} R^{(k)} = \operatorname{tg} T^{(k)} = (1,0)$, impossible for a splitting level.

We have to prove now that the conditions are sufficient. Suppose we have two plane branch multiplicity sequences $e_0, e_1, \dots$ and $f_0, f_1, \dots$ fulfilling conditions 1), 2), 3) and 4). We want to show that there exists a curve $\mathcal{O}$ with the following multiplicity tree

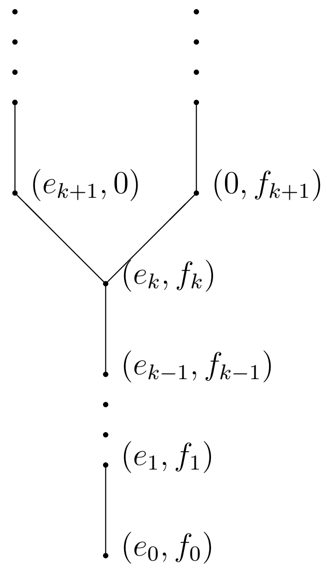

Fig. 10

Suppose we have a plane curve $\mathcal{O}$ with two branches R and T such that $R^{(j)}$ has the same tangent as $T^{(j)}$, for $j = 0, \dots, i$ and suppose $\operatorname{tg} R^{(i)} = \operatorname{tg} T^{(i)} = (1,0)$, where $i \leq k-2$. We want to show that there exists a plane curve such that $\operatorname{tg} R^{(i+1)} = \operatorname{tg} T^{(i+1)}$.

If $e_i > e_{i+1}$, then by 1) also $f_i > f_{i+1}$ and so $R^{(i+1)}$ and $T^{(i+1)}$ have same tangent ($= (0,1)$). If $e_i = e_{i+1}$, then by 1) also $f_i = f_{i+1}$. Now, if $r(e_{i+1}) = r(f_{i+1}) = 2$ and $r(e_{i+2}) = r(f_{i+2}) = 2$ (cf. condition 2)), then $\operatorname{tg} R^{(i+1)} = \operatorname{tg} T^{(i+1)} = (1,0)$, if $r(e_{i+1}) = r(f_{i+1}) = 2$ and $r(e_{i+2}) = r(f_{i+2}) = 1$ (cf. condition 2)), by Lemma 5.1 we can choose the two branches with the same coefficient c in the parametrizations of $R^{(i)}$ and $T^{(i)}$, so that $\operatorname{tg} R^{(i+1)} = \operatorname{tg} T^{(i+1)} = (1,c)$. Finally, if $r(e_{i+1}) = r(f_{i+1}) = 1$, then, again by Lemma 5.1, we can choose a curve such that $\operatorname{tg} R^{(i+1)} = \operatorname{tg} T^{(i+1)} = (1,c)$.

Now, if we have $e_{k-1} > e_k$ and $f_{k-1} = f_k$ (cf. condition 3)), then, supposing $\operatorname{tg} R^{(k-1)} = \operatorname{tg} T^{(k-1)} = (1,0)$, we get $\operatorname{tg} R^{(k)} = (0,1)$ and $\operatorname{tg} T^{(k)} = (1,c)$, so the curve must split at level k. On the other hand, if we have $e_{k-1} = e_k$ and $f_{k-1} = f_k$, then by condition 2) $r(e_k) = r(f_k)$. If $r(e_k) = r(f_k) = 1$, then, supposing $\operatorname{tg} R^{(k-1)} = \operatorname{tg} T^{(k-1)} = (1,0)$, by Lemma 5.1, case 2), we get $\operatorname{tg} R^{(k)} = (1,c)$ and $\operatorname{tg} T^{(k)} = (1,d)$ and we can choose the branches such that $c \neq d$. If $r(e_k) = r(f_k) = 2$, then by condition 4), $r(e_{k+1}) \neq 2$ or $r(f_{k+1}) \neq 2$, so that again by Lemma 5.1 we can always realize different tangents for $R^{(k)}$ and $T^{(k)}$.

Example. Consider the two plane branch multiplicity sequences $62, 62, 18, 18, 18, 8, 8, 2, 2, 2, 2, 1, \dots$ and $30, 30, 10, 10, 10, 4, 4, 2, 2, 2, 2, 1, \dots$. According to Proposition 5.2, the possible splitting levels for a multiplicity tree (of a plane curve with two branches with those multiplicity sequences for the branches) are only $k = 0, 1, 4$. In fact $k = 2$ is not possible for condition 3), $k = 3$ is not possible for condition 4) and $k \geq 5$ is not possible for condition 2). Examples of curves that realize these three different multiplicity trees are $k[[(t^{62}, u^{70} + u^{74} + u^{79}), (t^{142} + t^{143}, u^{30})]]$, that splits at level 0, $k[[(t^{62}, u^{30}), (t^{124} + t^{142} + t^{143}, u^{70} + u^{74} + u^{79})]]$ that splits at level 1 and $k[[(t^{62}, u^{30}), (t^{142} + t^{143}, u^{70} + u^{74} + u^{79})]]$ that splits at level 4.

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