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A Short Note on the Non-negativity of Partial Euler Characteristics

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Abstract. Let $(A, \mathfrak{m})$ be a Noetherian local ring, M a finite A-module and $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\lambda(M/\mathbf{x}M)$ is finite. Serre ([2, Appendix 2]) proved that all partial Euler characteristics of M with respect to $\mathbf{x}$ is non-negative. This fact is easy to show when A contains a field ([1, 4.7.12]). We give an elementary proof of Serre's result when A does not contain a field.

Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finite A-module. Let $x_1, \ldots, x_n \in \mathfrak{m}$ be a multiplicity system of M i.e. $\lambda(M/(\mathbf{x})M)$ is finite. (Here $\lambda(-)$ denotes length.) Let $K(\mathbf{x}, M)$ be the Koszul complex of $\mathbf{x}$ with coefficients in M and let $H_\bullet(\mathbf{x}, M)$ be its homology. Note that $H_\bullet(\mathbf{x}, M)$ has finite length. One defines for all $j \geq 0$ the *partial Euler characteristics*

$$\chi_j(\mathbf{x}, M) = \sum_{i \geq j} (-1)^{i-j} \lambda(H_i(\mathbf{x}, M))$$

of M with respect to $\mathbf{x}$. Serre showed all the partial Euler characteristics are non-negative. It is well known that $\chi_0(\mathbf{x}, M)$ is either zero or the multiplicity of M with respect to the ideal $(x_1, \ldots, x_n)$. It is also easy to see that $\chi_1(\mathbf{x}, M)$ is non-negative, ([1, 4.7.10]). The non-negativity of $\chi_j(\mathbf{x}, M)$ for $j \geq 2$ can be easily proved if A contains a field, ([1, 4.7.12]). In this short note we give an elementary proof of Serre's theorem when A does not contain a field.

Theorem 1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, A not containing a field. Let M be a finite A-module and $x_1, \ldots, x_n \in \mathfrak{m}$ a multiplicity system of M. Then*

$$\chi_j(\mathbf{x}, M) \geq 0 \quad \textit{for each } j \geq 0.$$

Proof. We may assume that A is complete. To prove the theorem we construct a local Noetherian ring $(B, \mathfrak{m})$ with a local homomorphism $\varphi : B \to A$, and $y_1, \ldots, y_n \in \mathfrak{m}$ such that

1. $\varphi(y_i) = x_i$.
2. M becomes a finite B-module (via φ).
3. $y_1, \ldots, y_n$ is a regular sequence and a s.o.p of B.

Since $K(\mathbf{y}, M) \simeq K(\mathbf{x}, M)$ (as B-modules), we have $H_\bullet(\mathbf{y}, M) \cong H_\bullet(\mathbf{x}, M)$ and so $\chi_j(\mathbf{y}, M) = \chi_j(\mathbf{x}, M)$ for each $j \geq 0$.

Suppose we have constructed B as above. The result then follows on similar lines as in [1, 4.7.12]. We give the proof here for the readers convenience. We prove the result by induction on j. For $j = 0, 1$ the result is already known. Let $j > 1$ and consider an exact sequence

$$0 \to U \to F \to M \to 0$$

where F is a finite free B-module. Since $\boldsymbol{y}$ is B-regular we have $H_i(\mathbf{y}, F) = 0$ for $i > 0$. Therefore $H_i(\mathbf{y}, M) \simeq H_{i-1}(\mathbf{y}, U)$ for all $i > 1$. This yields $\chi_j(\mathbf{y}, M) = \chi_{j-1}(\mathbf{y}, U)$ and the proof is complete by induction hypothesis.

Construction of B. Since A is complete there exists a DVR, (R, ρ) and a ring homomorphism $\varphi : R \to A$ which induces an isomorphism $R/\rho R \to A/\mathfrak{m}$. Set $S = R[[X_1, \ldots, X_n]]$ and let $\mathfrak{q}$ be its maximal ideal and consider the natural ring map $\phi : S \to A$, with $\phi(X_i) = x_i$.

We consider M as an S-module via ϕ. Since $M/(\mathbf{X})M = M/(\mathbf{x})M$ is a finite length A-module and so a finite length S-module, since $S/\mathfrak{q}S \cong A/\mathfrak{m}$. So M is a finite S-module. Also note that

$$\mathfrak{q} = \sqrt{\operatorname{ann}_S(M/\mathbf{X}M)} = \sqrt{\operatorname{ann}_S(M) + (\mathbf{X})}.$$

So there exists $\Delta \in \operatorname{ann}_S(M) \setminus (\mathbf{X})$. Observe that $\Delta, X_1, \ldots, X_n$ is an s.o.p. of S. Since S is regular local ring of dimension $n+1$, we have that $\Delta, X_1, \ldots, X_n$ is an S-regular sequence. Set $B = S/\Delta$ and $y_i = \overline{X}_i$ for $i = 1, \ldots, n$. Note that B satisfies our requirements. □

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