Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 1, 15-27.

# Eigenvalues of a Natural Operator of Centro-affine and Graph Hypersurfaces

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**Abstract.** In this article we obtain optimal estimates for the eigenvalues of a natural operator $K_{T^\#}$ for locally strongly convex centro-affine and graph hypersurfaces. Several immediate applications of our eigenvalue estimates are presented. We also provide examples to illustrate that our eigenvalue estimates are optimal.

MSC 2000: 53A15, 53B20, 53B25 (primary); 53C40 (secondary)
Keywords: Centroaffine hypersurface, graph hypersurface, the operator, eigenvalue estimate.

## 1. Introduction

Throughout this article we assume $n \geq 2$. An immersed hypersurface $f : M \to \mathbf{R}^{n+1}$ in an affine $(n+1)$-space $\mathbf{R}^{n+1}$ is called an *affine hypersurface with relative normalization* if there is a transversal vector field $\xi$ such that $D\xi$ has its image in $f_*(T_pM)$, where $D$ is the canonical flat connection on $\mathbf{R}^{n+1}$.

A hypersurface $f\colon M \to \mathbf{R}^{n+1}$ is called *centro-affine* if its position vector field is always transversal to $f_*(TM)$ in $\mathbf{R}^{n+1}$. In this case, for any vector fields $X, Y$ tangent to $M$, one can decompose $D_X f_*(Y)$ into its tangential and transverse components. This is written as

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^f(X,Y)f, \tag{1.1}$$

where $h^f$ is a symmetric tensor of type $(0,2)$ and $\xi = f$.

Throughout this article, we assume that $h^f$ is definite, so $h^f$ defines a semi-Riemannian metric on $M$. In order to consider only a positive definite metric we now make the following changes: if $h^f$ is negative definite, we introduce a transversal vector field $\xi = -f$ and a $(0,2)$-tensor given by $h = -h^f$.

It is well-known that the centro-affine metric $h$ is definite if and only if the hypersurface is locally strongly convex. For this the following terminology is used:

(i) The centro-affine hypersurface $M$ is said to be of *elliptic type* if, for any point $f(p) \in \mathbf{R}^{n+1}$ with $p \in M$, the origin of $\mathbf{R}^{n+1}$ and the hypersurface are on the same side of the tangent hyperplane $f_*(T_pM)$; in this case the centro-affine normal vector field is given by $\xi = -f$.

(ii) The centro-affine hypersurface $M$ is said to be of *hyperbolic type* if, for any point $f(p) \in \mathbf{R}^{n+1}$, the origin of $\mathbf{R}^{n+1}$ and the hypersurface are on the different side of the tangent hyperplane $f_*(T_pM)$; in this case the centro-affine normal vector field is given by $\xi = f$.

An affine hypersurface $f : M \to \mathbf{R}^{n+1}$ is called a *graph hypersurface* if we choose as affine transversal field a constant vector field. For a graph hypersurface we also have the decomposition (1.1) as well. Again in case that $h$ is non-degenerate, it defines a semi-Riemannian metric, called the Calabi metric of the graph hypersurface.

Let $\hat{\nabla}$ denote the Levi-Civita connection of $h$ and let $K$ be the difference tensor $\nabla - \hat{\nabla}$ on $M$. Then, for each $X \in T_pM$, $K_X : Y \mapsto K(X,Y)$ is an endomorphism of $T_pM$. By taking the trace of $K$, one obtains a so-called Tchebychev form

$$T(X) := \frac{1}{n}\,\mathrm{trace}\,\{\, Y \to K(X,Y)\,\}. \tag{1.2}$$

The Tchebychev vector field $T^{\#}$ can then be defined by

$$h(T^{\#}, X) = T(X). \tag{1.3}$$

The Tchebychev form and Tchebychev vector field play an important role in centro-affine differential geometry.

For each integer $k \in [2,n]$, we define an invariant $\hat{\theta}_k$ on the affine hypersurface $M$ in the same way as in [1] (see Section 3 for details).

The main results of this article are the following optimal estimates for the eigenvalues of the operator $K_{T^{\#}}$:

(I) For a locally strongly convex centro-affine hypersurface $M$ in $\mathbf{R}^{n+1}$ we have:

- (I-a) If $\hat{\theta}_k \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^{\#}}$ at $p$ is greater than $(\frac{n-1}{n})(\varepsilon - \hat{\theta}_k(p))$.
- (I-b) If $\hat{\theta}_k = \varepsilon$ at a point $p$, every eigenvalue of $K_{T^{\#}}$ at $p$ is $\geq 0$, where $\varepsilon = 1$ or $-1$ according to $M$ is of elliptic or hyperbolic type.

(II) For a graph hypersurface $M$ in $\mathbf{R}^{n+1}$ we have:

- (II-a) If $\hat{\theta}_k \neq 0$ at a point $p \in M$, every eigenvalue of the operator $K_{T^{\#}}$ at $p$ is greater than $(\frac{1-n}{n})\hat{\theta}_k(p)$.
- (II-b) If $\hat{\theta}_k = 0$ at a point $p \in M$, every eigenvalue of $K_{T^{\#}}$ at $p$ is $\geq 0$.

The proofs of the main results base on the equation of Gauss using the same idea introduced in earlier author's articles [1, 2]. This is done in Section 4. Several immediate applications of our eigenvalue estimates of the operator $K_{T^\#}$ are given in Section 5. In the last two sections, we provide some non-trivial examples to illustrate that our eigenvalue estimates are optimal for both centro-affine and graph hypersurfaces.

## 2. Preliminaries

We recall some basic facts about centro-affine and graph hypersurfaces. For the details, see [3, 4, 5, 6].

Let $f : M \to \mathbf{R}^{n+1}$ be a centro-affine hypersurface with centro-affine normal $\xi$. We assume that the centro-affine hypersurface is definite. As we already mentioned earlier, the centro-affine normal on the hypersurface is chosen in such way that the metric $h$ is positive definite.

The centro-affine structure equations are given by

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X \xi = \mp f_*(X), \tag{2.2}$$

where $D_X\xi = -f_*(X)$ or $D_X\xi = f_*(X)$ according to $\xi = -f$ or $\xi = f$ respectively.

The corresponding equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{2.4}$$

The cubic form is the totally symmetric $(0,3)$-tensor field $C(X,Y,Z) = (\nabla_X h)(Y,Z)$.

Let $\hat{\nabla}$, $\hat{K}$ and $\hat{R}$ denote the Levi-Civita connection, the sectional curvature and the curvature tensor of $h$, respectively. The difference tensor $K$ is then given by

$$K_X Y = K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{2.5}$$

which is a symmetric $(1,2)$-tensor field. The difference tensor $K$ and the cubic form $C$ are related by

$$C(X,Y,Z) = -2h(K_X Y, Z). \tag{2.6}$$

It is well-known that for centro-affine hypersurfaces we have

$$h(K_X Y, Z) = h(Y, K_X Z), \tag{2.7}$$

$$\hat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z + \varepsilon(h(Y,Z)X - h(X,Z)Y), \tag{2.8}$$

$$(\hat{\nabla}K)(X,Y,Z) = (\hat{\nabla}K)(Y,Z,X) = (\hat{\nabla}K)(Z,X,Y), \tag{2.9}$$

where $\varepsilon = 1$ if $M$ is of elliptic type and $\varepsilon = -1$ if $M$ is of hyperbolic type. It follows from (2.7) that the endomorphism $K_X$ is self-adjoint with respect to $h$.

When $f : M \to \mathbf{R}^{n+1}$ is a graph hypersurface, we have (1.1), (2.1), (2.4), (2.5), (2.6), (2.7) and (2.9) as well. However, (2.2), (2.3) and (2.8) shall be replaced by

$$D_X\xi = R(X, Y)Z = 0, \tag{2.10}$$

$$\hat{R}(X, Y)Z = K_Y K_X Z - K_X K_Y Z. \tag{2.11}$$

## 3. Invariant $\hat{\theta}_k$ and relative $K$-null space

Let $M$ be a centro-affine or graph hypersurface with positive definite metric $h$. Denote by $\hat{K}(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM$ relative to $h$. The scalar curvature $\hat{\tau}$ at $p$ is then defined by

$$\hat{\tau}(p) = \sum_{1\le i<j\le n} \hat{K}_{ij}, \tag{3.1}$$

where $\hat{K}_{ij} = \hat{K}(e_i \wedge e_j)$ and $e_1, \dots, e_n$ is an $h$-orthonormal basis of $T_pM$.

Assume that $L^k$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L^k$ with respect to $h$. We choose an $h$-orthonormal basis $\{e_1, \dots, e_k\}$ of $L^k$ with $e_1 = X$. Then the $k$-Ricci curvature $\hat{S}_{L^k}(X)$ and the scalar curvature $\hat{\tau}(L^k)$ are defined respectively by

$$\hat{S}_{L^k}(X) = \hat{K}_{12} + \cdots + \hat{K}_{1k}, \tag{3.2}$$

$$\hat{\tau}(L^k) = \sum_{1\le i<j\le k} K_{ij}. \tag{3.3}$$

Obviously, $\hat{S}_{L^2}$ and $\hat{\tau}(L^2)$ are nothing but the sectional curvature $\hat{K}(L^2)$. And $\hat{S}_{L^n}$ and $\hat{\tau}(L^n)$ are the Ricci and scalar curvatures relative to $h$.

For each integer $k \in [2, n]$, we define the invariant $\hat{\theta}_k$ on $M$ by (cf. [1, 2])

$$\hat{\theta}_k(p) = \left(\frac{1}{k-1}\right) \sup_{L^k, X} \hat{S}_{L^k}(X), \quad p \in T_pM, \tag{3.4}$$

where $L^k$ runs over all linear $k$-subspaces in the tangent space $T_pM$ at $p$ and $X$ runs over all $h$-unit vectors in $L^k$.

The *relative $K$-null space* $\mathcal{N}_p^K$ of $M$ in $\mathbf{R}^{n+1}$ is defined by

$$\mathcal{N}_p^K = \{X \in T_pM : K(X, Y) = 0 \text{ for all } Y \in T_pM\}. \tag{3.5}$$

When $\dim \mathcal{N}_p^K$ is constant, $\mathcal{N}^K = U_{p\in M} \mathcal{N}_p^K$ defines a subbundle of the tangent bundle, called the *relative $K$-null subbundle.*

## 4. Optimal estimates for eigenvalues of the operator

For centro-affine hypersurface in $\mathbf{R}^{n+1}$ we have the following result.

**Theorem 4.1.** *Let $f : M \to \mathbf{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbf{R}^{n+1}$. Then, for any integer $k \in [2, n]$, we have:*

(1) *If* $\hat{\theta}_k \neq \varepsilon$ *at a point* $p \in M$, *then every eigenvalue of* $K_{T^\#}$ *at* $p$ *is greater than* $\left(\frac{n-1}{n}\right)(\varepsilon - \hat{\theta}_k(p))$.

(2) *If* $\hat{\theta}_k(p) = \varepsilon$, *every eigenvalue of* $K_{T^\#}$ *at* $p$ *is* $\geq 0$.

(3) *A nonzero vector* $X \in T_pM$ *is an eigenvector of the operator* $K_{T^\#}$ *with eigenvalue* $\left(\frac{n-1}{n}\right)(\varepsilon - \hat{\theta}_k(p))$ *if and only if* $\hat{\theta}_k(p) = \varepsilon$ *and* $X$ *lies in the relative* $K$*-null space* $\mathcal{N}_p^K$ *at* $p$,

*where* $\varepsilon = 1$ *or* $-1$ *according to* $M$ *is of elliptic or hyperbolic type.*

*Proof.* Assume that $f : M \to \mathbf{R}^{n+1}$ is a locally strongly convex centro-affine hypersurface in $\mathbf{R}^{n+1}$. Let $\{e_1, \ldots, e_n\}$ be an arbitrary $h$-orthonormal basis of $T_pM$. From the definition of Tchebychev vector field, (2.8) and (3.1) we have

$$2\hat{\tau} = n(n-1)\varepsilon + h(K,K) - n^2 h(T^\#, T^\#). \tag{4.1}$$

It is well-known that every endomorphism $A$ of $T_pM$ satisfies

$$n\, h(A,A) \geq (\operatorname{trace} A)^2, \tag{4.2}$$

with equality holding if and only if $A$ is proportional to the identity map $I$. By applying (4.1) and (4.2), we obtain

$$2\hat{\tau} \geq n(n-1)\varepsilon - n(n-1)h(T^\#, T^\#) \tag{4.3}$$

with the equality holding at $p$ if and only if we have

(a) $K_{T^\#}$ is proportional to the identity map and

(b) $K_Z = 0$ for $Z$ perpendicular to $T^\#$ at $p$.

Let $L_{i_1\cdots i_k}$ be the $k$-plane section spanned by the orthonormal vectors $e_{i_1}, \ldots, e_{i_k}$. It follows from (3.2) and (3.3) that

$$\hat{\tau}(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \hat{S}_{L_{i_1\cdots i_k}}(e_i), \tag{4.4}$$

$$\hat{\tau}(p) = \frac{(k-2)!(n-k)!}{(n-2)!} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \hat{\tau}(L_{i_1\cdots i_k}). \tag{4.5}$$

By combining (3.4), (4.4) and (4.5) we find

$$\hat{\tau} \leq \frac{n(n-1)}{2} \hat{\theta}_k. \tag{4.6}$$

Thus (4.3) and (4.6) ensure that

$$h(T^\#, T^\#) \geq \varepsilon - \hat{\theta}_k. \tag{4.7}$$

Hence the Tchebychev vector field $T^\#$ vanishes at a point $p$ only when $\hat{\theta}_k(p) \geq \epsilon$. Therefore, if $T^\#(p) = 0$, statements (1) and (2) of Theorem 4.1 hold automatically.

Next, let us assume that $T^{\#}(p) \neq 0$. Since $K_{T^{\#}}$ is self-adjoint with respect to $h$, we may choose an $h$-orthonormal basis $e_1, \dots, e_n$ of $T_pM$ which diagonalizes the operator $K_{T^{\#}}$. Let $e_1^*$ be the $h$-unit vector at $p$ in the direction of $T^{\#}$ and let us choose $h$-orthonormal vectors $e_2^*, \dots, e_n^*$ at $p$ perpendicular to $T^{\#}$. Then we have

$$K_{e_1^*} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \tag{4.8}$$

and trace $(K_{e_r^*}) = 0$ for $r = 2, \dots, n$.

Let us put $K_{ij}^{r^*} = h(K(e_i, e_j), e_r^*)$. Then (2.8) implies that

$$K_{ij} = \varepsilon - a_i a_j + \sum_{r=2}^{n} \big(K_{ij}^{r^*}\big)^2 - \sum_{r=2}^{n} K_{ii}^{r^*} K_{jj}^{r^*}, \quad 1 \le i \neq j \le n. \tag{4.9}$$

Now, by applying the same argument as the proof of Theorem 1 of [1], we obtain

$$a_1(a_1 + \cdots + a_n) \ge (n-1)\big(\varepsilon - \hat{\theta}_k(p)\big) + a_1^2 \ge (n-1)\big(\varepsilon - \hat{\theta}_k(p)\big), \tag{4.10}$$

with both equality holding if and only if we have $\hat{S}_L(e_1) = \hat{\theta}_k(p)$ and $a_1 = K_{1j}^{r^*} = 0$ for $r = 2, \dots, n$; $j = 2, \dots, n$. The same inequality holds if the lower index 1 in (4.10) were replaced by any $j \in \{2, \dots, n\}$. Hence, we have

$$K_{T^{\#}} \ge \frac{n-1}{n}\big(\varepsilon - \hat{\theta}_k(p)\big) I. \tag{4.11}$$

If $K_{T^{\#}} X = \frac{n-1}{n}\big(\varepsilon - \hat{\theta}_k(p)\big) X$ holds for some nonzero vector $X \in T_pM$, then $X$ is an eigenvector of $K_{T^{\#}}$ with eigenvalue $(n-1)\big(\varepsilon - \hat{\theta}_k(p)\big)/n$. Without loss of generality, we may choose $e_1 = X/\sqrt{h(X,X)}$. In this case we get

$$a_1(a_1 + \cdots + a_n) = (n-1)\big(\varepsilon - \hat{\theta}_k(p)\big). \tag{4.12}$$

On the other hand, from (4.10) and (4.12), we find $a_1 = 0$ and $\hat{\theta}_k(p) = \varepsilon$. Moreover, we know from (4.10) that $e_1$ lies in the relative $K$-null space $\mathcal{N}_p^K$. Consequently, we obtain statements (1) and (2) of Theorem 4.1 and also one part of statement (3). The remaining part of statement (3) is obvious. □

For graph hypersurfaces we have the following.

**Theorem 4.2.** *Let $f : M \to \mathbf{R}^{n+1}$ be a graph hypersurface in $\mathbf{R}^{n+1}$ with positive definite Calabi metric. Then, for any integer $k \in [2, n]$, we have:*

(1) *If $\hat{\theta}_k \neq 0$ at a point $p \in M$, then every eigenvalue of $K_{T^{\#}}$ at $p$ is greater than $\left(\frac{1-n}{n}\right) \hat{\theta}_k(p)$.*

(2) *If $\hat{\theta}_k = 0$ at $p$, then every eigenvalue of $K_{T^{\#}}$ at $p$ is $\ge 0$.*

(3) *A nonzero vector $X \in T_pM$ is an eigenvector of the operator $K_{T^\#}$ with eigenvalue $\left(\frac{1-n}{n}\right)\hat{\theta}_k(p)$ if and only if we have $\hat{\theta}_k(p) = 0$ and $X \in \mathcal{N}_p^K$.*

*Proof.* For graph hypersurfaces in $\mathbf{R}^{n+1}$ we have

$$\hat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z. \tag{4.13}$$

Thus, by applying the same argument given in Theorem 4.1, we obtain Theorem 4.2. □

## 5. Some applications

When $k = 2$, statement (1) of Theorem 4.1 implies immediately the following.

**Corollary 5.1.** *Let $f : M \to \mathbf{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbf{R}^{n+1}$. If $\sup \hat{K} \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $\left(\frac{n-1}{n}\right)(\epsilon - \sup \hat{K}(p))$.*

Similarly, if we denote by $\sup \hat{S}(p)$ the supremum of the Ricci curvature of $(M, h)$ at a point $p \in M$, then statement (1) of Theorem 4.1 with $k = n$ implies immediately the following.

**Corollary 5.2.** *Let $f : M \to \mathbf{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbf{R}^{n+1}$. If $\sup \hat{S} \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $\left(\frac{n-1}{n}\right)(\epsilon - \sup \hat{S}(p))$.*

From Theorem 4.1 we also obtain the following.

**Corollary 5.3.** *Let $f : M \to \mathbf{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbf{R}^{n+1}$. If we have $\hat{\theta}_k < \varepsilon$ on $M$ for some integer $k \in [2, n]$, then every eigenvalue of $K_{T^\#}$ is positive.*

Theorem 4.1 also gives rise to the following simple geometric characterization of hyper-ellipsoids and two-sheeted hyperboloids.

**Corollary 5.4.** *An elliptic centro-affine hypersurface $M$ in $\mathbf{R}^{n+1}$ is centroaffinely equivalent to an open portion of a hyperellipsoid if and only if we have $nK_{T^\#} = (n-1)(1-\hat{\theta}_k)I$ on $M$ for some integer $k \in [2, n]$.*

*Proof.* Let $f : M \to \mathbf{R}^{n+1}$ be an elliptic centro-affine hypersurface in $\mathbf{R}^{n+1}$. If $M$ is an open portion of a hyperellipsoid, then $K$ vanishes identically which implies that $K_{T^\#} = 0$. Hence, according to (2.10), $(M, h)$ is of constant curvature one. Therefore we obtain $\hat{\theta}_2 = \cdots = \hat{\theta}_n = 1$. Consequently, we have $nK_{T^\#} = (n-1)(1-\hat{\theta}_k)I$ identically.

Conversely, let us assume that $nK_{T^\#} = (n-1)(1-\hat{\theta}_k)I$ holds identically for some integer $k \in [2, n]$, then statement (3) of Theorem 4.1 implies that every tangent vector of $M$ lies in the relative $K$-null subbundle. In this case $K$ vanishes identically on $M$. Consequently, by applying a theorem of Berwald [6, Section 7.1.1], we conclude that $M$ is centroaffinely equivalent to an open portion of a hyper-ellipsoid centered at the origin. □

**Corollary 5.5.** *A hyperbolic centro-affine hypersurface $M$ in $\mathbf{R}^{n+1}$ is centroaffinely equivalent to an open portion of a two-sheeted hyperboloid if and only if, for some integer $k \in [2, n]$, we have $nK_{T\#} = (1-n)(1+\hat{\theta}_k)I$ identically on $M$.*

*Proof.* This can be done in the same way as Corollary 5.4. □

Similarly Theorem 4.2 implies the following.

**Corollary 5.6.** *Let $f : M \to \mathbf{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If we have either $\sup \hat{K} \neq 0$ or $\sup \hat{S} \neq 0$ at a point $p \in M$, then every eigenvalue of the operator $K_{T\#}$ is greater than $\left(\frac{1-n}{n}\right) \sup \hat{K}$ at $p$.*

**Corollary 5.7.** *Let $f : M \to \mathbf{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k < 0$ holds on $M$, then every eigenvalue of $K_{T\#}$ is positive.*

From Corollaries 5.3 and 5.7 we obtain the following.

**Corollary 5.8.** *Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 1$ at some point $p \in M$, then $M$ cannot be realized as an elliptic proper affine hypersphere in $\mathbf{R}^{n+1}$.*

**Corollary 5.9.** *Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < -1$ at some point $p \in M$, then $M$ cannot be realized as a hyperbolic proper affine hypersphere in $\mathbf{R}^{n+1}$.*

**Corollary 5.10.** *Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 0$ at some point $p \in M$, then $M$ cannot be realized as an improper affine hypersphere in $\mathbf{R}^{n+1}$.*

## 6. Some examples of centro-affine hypersurfaces

In this section we provide some examples of locally strongly convex centro-affine hypersurfaces. From these examples we know that the eigenvalue estimates given in Theorem 4.1 are best possible.

**Example 6.1.** Let $M$ be the elliptic locally strongly convex centro-affine hypersurface defined by:

$$e^{bs}\left(e^{(b^{-1}-b)s}, \sin(ax_2), \dots, \sin(ax_n)\prod_{j=2}^{n-1}\cos(ax_j), \prod_{j=2}^{n}\cos(ax_j)\right), \tag{6.1}$$

with $a = \sqrt{1-b^2}$, $b \in (0, 1)$. Then the affine metric $h$ on $M$ is

$$h = ds^2 + dx_2^2 + \cos^2(ax_2)dx_3^2 + \cdots + \prod_{j=2}^{n-1}\cos^2(ax_j)dx_n^2. \tag{6.2}$$

The Levi-Civita connection of $h$ satisfies

$$\begin{aligned}
&\hat{\nabla}_{\partial\ /\partial s}\frac{\partial}{\partial s}=\hat{\nabla}_{\partial/\partial s}\frac{\partial}{\partial x_k}=\hat{\nabla}_{\partial/\partial x_2}\frac{\partial}{\partial x_2}=0,\\
&\hat{\nabla}_{\partial/\partial x_i}\frac{\partial}{\partial x_j}=-a\tan(ax_i)\,\frac{\partial}{\partial x_j},\quad 2\le i<j,\\
&\hat{\nabla}_{\partial/\partial x_j}\frac{\partial}{\partial x_j}=a\sum_{k=2}^{j-1}\left(\frac{\sin(2ax_k)}{2}\prod_{l=k+1}^{j-1}\cos^2(ax_l)\right)\frac{\partial}{\partial x_k},\ j=3,\dots,n.
\end{aligned}\tag{6.3}$$

It follows from (6.1) and (6.2) that $\hat{K}_{1j}=0$ and $\hat{K}_{jk}=a^2$ for $2\le j\ne k\le n$. Hence we have

$$\hat{\theta}_n=\left(\frac{n-2}{n-1}\right)(1-b^2).\tag{6.4}$$

On the other hand, from (6.1) and a straight-forward computation, we find

$$\begin{aligned}
&\nabla_{\partial/\partial s}\frac{\partial}{\partial s}=\left(b+\frac{1}{b}\right)\frac{\partial}{\partial s},\ \nabla_{\partial/\partial s}\frac{\partial}{\partial x_j}=b\frac{\partial}{\partial x_j},\\
&\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j}=-a\tan(ax_i)\,\frac{\partial}{\partial x_j},\quad 2\le i<j\le n,\\
&\nabla_{\partial/\partial x_j}\frac{\partial}{\partial x_j}=b\prod_{i=2}^{j-1}\cos^2(ax_i)\frac{\partial}{\partial s}+a\sum_{k=2}^{j-1}\left(\frac{\sin(2ax_k)}{2}\prod_{l=k+1}^{j-1}\cos^2(ax_l)\right)\frac{\partial}{\partial x_k}
\end{aligned}\tag{6.5}$$

for $j=2,\dots,n$. By applying (2.5), (6.3) and (6.5) we find

$$\begin{aligned}
&K\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right)=\left(b+\frac{1}{b}\right)\frac{\partial}{\partial s},\ \ K\left(\frac{\partial}{\partial s},\frac{\partial}{\partial x_j}\right)=b\frac{\partial}{\partial x_j},\\
&K\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_j}\right)=b\prod_{i=2}^{j-1}\cos^2(ax_i)\frac{\partial}{\partial s},\ \ K\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=0
\end{aligned}\tag{6.6}$$

for $2\le i\ne j\le n$. Therefore we obtain from (1.3), (6.2) and (6.6) that

$$T^{\#}=\left(b+\frac{1}{nb}\right)\frac{\partial}{\partial s},\ \ K_{T^{\#}}\left(\frac{\partial}{\partial x_j}\right)=\lambda_j\frac{\partial}{\partial x_j},\quad \lambda_j=b^2+\frac{1}{n}\tag{6.7}$$

for $j=2,\dots,n$. Consequently, we conclude that the eigenvalue $\lambda_j$ of the operator $K_{T^{\#}}$ associated with eigenvector $\partial/\partial x_j$ satisfies

$$\lambda_j-\frac{n-1}{n}\big(1-\hat{\theta}_n\big)=\frac{2b^2}{n}\longrightarrow 0\quad\text{as } b\to 0.$$

**Example 6.2.** Consider the hyperbolic locally strongly convex centro-affine hypersurface defined by:

$$e^{bs}\left(e^{-(b^{-1}+b)s},\sinh(ax_2),\dots,\sinh(ax_n)\prod_{j=2}^{n-1}\cosh(ax_j),\prod_{j=2}^{n}\cosh(ax_j)\right)\tag{6.8}$$

with $a = \sqrt{1+b^2}$, $b \in (0,\infty)$. The induced affine metric $h$ of this hypersurface is given by

$$h = ds^2 + dx_2^2 + \cosh^2(ax_2)dx_3^2 + \cdots + \prod_{j=2}^{n-1}\cosh^2(ax_j)dx_n^2, \tag{6.9}$$

which implies that $\hat{K}_{1j} = 0, \hat{K}_{jk} = -a^2$ for $2 \le j \ne k \le n$. Hence we have

$$\hat{\theta}_n = \left(\frac{2-n}{n-1}\right)(1+b^2). \tag{6.10}$$

From (2.1), (2.5), (6.8) and a straight-forward computation we find

$$\begin{aligned} &K\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) = \left(b-\frac{1}{b}\right)\frac{\partial}{\partial s},\quad K\left(\frac{\partial}{\partial s},\frac{\partial}{\partial x_j}\right) = b\frac{\partial}{\partial x_j},\\ &K\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_j}\right) = b\prod_{i=2}^{j-1}\cosh^2(ax_i)\frac{\partial}{\partial s},\quad K\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) = 0 \end{aligned} \tag{6.11}$$

for $2 \le i \ne j \le n$. Therefore we have

$$T^{\#} = \left(b - \frac{1}{nb}\right)\frac{\partial}{\partial s},\quad K_{T^{\#}}\left(\frac{\partial}{\partial x_j}\right) = \left(b^2 - \frac{1}{n}\right)\frac{\partial}{\partial x_j},\quad j = 2,\ldots,n. \tag{6.12}$$

Consequently, the eigenvalue $\lambda_j$ of the operator $K_{T^{\#}}$ associated with eigenvector $\partial/\partial x_j$ satisfies

$$\lambda_j + \frac{n-1}{n}\big(1+\hat{\theta}_n\big) = \frac{2b^2}{n} \longrightarrow 0 \quad \text{as } b \to 0.$$

Examples 6.1 and 6.2 show that the eigenvalue estimate given in statement (1) of Theorem 4.1 is optimal for locally strongly convex centro-affine hypersurfaces of both elliptic and hyperbolic types.

**Example 6.3.** Consider the following elliptic centro-affine locally strongly convex hypersurface:

$$\begin{aligned} 5\Bigg(&\sin x_1, \sin x_2\cos x_1, \ldots, \sin x_{n-1}\prod_{j=1}^{n-2}\cos x_j,\\ &e^{\frac{1}{2}(b+\sqrt{b^2-4})x_n}\prod_{j=1}^{n-1}\cos x_j, e^{\frac{1}{2}(b+\sqrt{b^2-4})x_n}\prod_{j=1}^{n-1}\cos x_j\Bigg) \end{aligned} \tag{6.13}$$

with $b > 2$. The affine metric of this hypersurface is given by

$$h = dx_1^2 + \cos^2 x_1 dx_2^2 + \cdots + \prod_{j=1}^{n-1}\cos^2 x_j dx_n^2. \tag{6.14}$$

It follows from (6.14) that $\hat{\theta}_k = 1$ for $k = 2, \dots, n$.

On the other hand, from (6.13) and a direct computation, we have

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = K\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_n}\right) = 0, \quad K\left(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n}\right) = b\frac{\partial}{\partial x_n} \tag{6.15}$$

for $1 \leq i, j \leq n-1$, which ensures that

$$T^{\#} = \left(\frac{b}{n}\prod_{j=1}^{n-1}\sec^2 x_j\right)\frac{\partial}{\partial x_n}, \quad K_{T^{\#}}\left(\frac{\partial}{\partial x_j}\right) = 0, \quad j = 1, \dots, n-1. \tag{6.16}$$

**Example 6.4.** Let $M$ be the hyperbolic locally strongly convex centro-affine hypersurface defined by

$$\begin{aligned}\Bigg(&\sinh x_1, \sinh x_2 \cosh x_1, \dots, \sinh x_{n-1}\prod_{j=1}^{n-2}\cos x_j, \\ &e^{\frac{1}{2}(b+\sqrt{b^2+4})x_n}\prod_{j=1}^{n-1}\cosh x_j, e^{\frac{1}{2}(b-\sqrt{b^2+4})x_n}\prod_{j=1}^{n-1}\cosh x_j\Bigg),\end{aligned} \tag{6.17}$$

with nonzero $b$. Since the induced affine metric is given by

$$h = dx_1^2 + \cosh^2 x_1 dx_2^2 + \cdots + \prod_{j=1}^{n-1}\cosh^2 x_j dx_n^2, \tag{6.18}$$

thus we have $\hat{\theta}_2 = \cdots = \hat{\theta}_n = -1$.

On the other hand, by (6.17) and a straight-forward computation, we find

$$K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = K\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_n}\right) = 0, \quad K\left(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n}\right) = b\frac{\partial}{\partial x_n} \tag{6.19}$$

for $1 \leq i, j \leq n-1$. Hence we obtain

$$T^{\#} = \frac{b}{n}\prod_{j=1}^{n-1}\operatorname{sech}^2 x_j\frac{\partial}{\partial x_n}, \quad K_{T^{\#}}\left(\frac{\partial}{\partial x_j}\right) = 0, \quad j = 1, \dots, n-1. \tag{6.20}$$

Clearly, Examples 6.3 and 6.4 illustrate that the estimate given in statement (2) of Theorem 4.1 is optimal for locally strongly convex centro-affine hypersurfaces of both elliptic and hyperbolic types.

## 7. Examples of graph hypersurfaces

**Example 7.1.** Consider the graph hypersurface $M$ in $\mathbf{R}^{n+1}$:

$$\left(u_2, \dots, u_n, \frac{s^4}{4} + \sum_{j=2}^{n} u_j^2, \frac{s^2}{4}\right) \tag{7.1}$$

with constant affine normal $\xi$ given by $(0,\dots,0,-1)$ and Calabi metric given by $h = ds^2 + s^{-2}\left(du_2^2 + \cdots + du_n^2\right).$

A direct computation shows that $\hat{K}_{1j} = -s^{-2}$ and $\hat{K}_{ij} = -1$ for $2 \le i \ne j \le n$. Thus we get

$$\hat{\theta}_2 = \cdots = \hat{\theta}_n = \begin{cases} -\dfrac{1}{s^2} & \text{if } s^2 \ge 1; \\ -1 & \text{if } s^2 < 1. \end{cases} \tag{7.2}$$

From (7.1) and a straight-forward computation, we find

$$\begin{aligned} &K\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = \frac{3}{s}\frac{\partial}{\partial s}, \quad K\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u_j}\right) = \frac{1}{s}\frac{\partial}{\partial u_j}, \\ &K\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = 0, \quad K\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_j}\right) = \frac{1}{s^3}\frac{\partial}{\partial s}, \quad 2 \le i \ne j \le n, \end{aligned} \tag{7.3}$$

which yields

$$T^{\#} = \frac{(n+2)}{ns}\frac{\partial}{\partial s}, \quad K_{T^{\#}}\left(\frac{\partial}{\partial u_j}\right) = \lambda_j \frac{\partial}{\partial u_j}, \quad \lambda_j = \frac{(n+2)}{ns^2} \tag{7.4}$$

for $j = 2,\dots,n$. Hence we obtain

$$\lambda_j - \left(\frac{1-n}{n}\right)\hat{\theta}_k = \frac{3}{ns^2} \longrightarrow 0 \quad \text{as } s \to \infty. \tag{7.5}$$

This example shows that the estimate given in statement (1) of Theorem 4.2 is optimal.

**Example 7.2.** Consider the graph hypersurface $M$ in $\mathbf{R}^{n+1}$:

$$\left(u_2,\dots,u_n, e^{u_1}, u_1 - \frac{1}{2}\sum_{j=2}^{n} u_j^2\right) \tag{7.6}$$

with affine normal $\xi = (0,\dots,0,-1)$ and Calabi metric $h = du_1^2 + \cdots + du_n^2$. Obviously, we have $\hat{\theta}_2 = \cdots = \hat{\theta}_n = 0$. It follows from (7.6) that

$$K\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1}\right) = \frac{\partial}{\partial u_1}, \quad K\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_j}\right) = K\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = 0 \tag{7.7}$$

for $i, j = 2,\dots,n$. Thus we have

$$T^{\#} = \frac{1}{n}\frac{\partial}{\partial u_1}, \quad K_{T^{\#}}\left(\frac{\partial}{\partial u_j}\right) = 0 \tag{7.8}$$

for $j = 2,\dots,n$.

The last example illustrates that the estimate given in statement (2) of Theorem 4.2 is optimal as well.

**Acknowledgement.** The author is very grateful to the referee for several valuable suggestions.

## References

[1] Chen, B.-Y.: *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*. Glasg. Math. J. **41** (1999), 33–41. Zbl 0962.53015

[2] Chen, B.-Y.: *Riemannian geometry of Lagrangian submanifolds*. Taiwanese J. Math. **5** (2001), 681–723. Zbl 1002.53053

[3] Dillen, F.; Vrancken, L.: *Affine differential geometry of hypersurface*. Geometry and Topology of Submanifolds II, (Avignon, 1988), 144–164, World Sci. Publishing, Teaneck, NJ, 1990. Zbl 0727.53017

[4] Li, A. M.; Simon, U.; Zhao, G.: *Global Affine Differential Geometry of Hypersurfaces*. De Gruyter Expositions in Mathematics **11**, Walter de Gruyter, Berlin-New York 1993. Zbl 0808.53002

[5] Nomizu, K.; Sasaki, T.: *Affine Differential Geometry. Geometry of Affine Immersions*. Cambridge Tracts in Mathematics **111**, Cambridge University Press, 1994. Zbl 0834.53002

[6] Simon, U.; Schwenk-Schellschmidt, A.; Viesel, H.: *Introduction to the Affine Differential Geometry of Hypersurfaces*. Lecture Notes of the Science University of Tokyo, 1991. Zbl 0780.53002

Received April 28, 2004; revised version Mai 12, 2005