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Regular Generalized Adjoint Semigroups of a Ring

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Abstract. In this paper, we characterize a ring with a generalized adjoint semigroup having a property **P** and such generalized adjoint semigroups, where **P** stands for orthodox, right inverse, inverse, pseudoinverse, E-unitary, and completely simple, respectively. Surprisingly, if R has a GA-semigroup with a property **P**, then the adjoint semigroup of R has the property **P**.

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1. Introduction

Based on the paper [7], we continue our study of generalized adjoint semigroups (GA-semigroup) of a ring. In the present paper we are concerned with the description of a ring R with a GA-semigroup having a property **P** and such GA-semigroups of R, where **P** stands for orthodox, right inverse, inverse, pseudoinverse, E-unitary, and completely simple, respectively.

Let R be a ring not necessarily with identity. The composition defined by $a \circ b = a + b + ab$ for any $a, b \in R$ is usually called the circle or adjoint multiplication of R. It is well-known that $(R, \circ)$ is a monoid with identity 0, called the circle or adjoint semigroup of R, denoted by R°. There are many interesting connections between a ring and its adjoint semigroup, which were studied in several papers, for example, [2, 4, 5, 6, 10, 11, 12, 15, 16]. Typical results are to describe the adjoint semigroup of a given ring and the ring with a given semigroup as its adjoint semigroup.

The circle multiplication of a ring satisfies the following generalized distributive laws:

$$a \circ (b + c - d) = a \circ b + a \circ c - a \circ d, \tag{1}$$

$$(b + c - d) \circ a = b \circ a + c \circ a - d \circ a, \tag{2}$$

which was observed in [1]. Thus as generalizations of the circle multiplication of a ring, a binary operation $\diamond$ (associative or nonassociative) on an Abelian group A satisfying the generalized distributive laws have been studied by several authors making use of different terminologies, for example, pseudo-rings, weak rings, quasirings, prerings. In [7], we call a binary operation $\diamond$ on R is called a generalized adjoint multiplication on R, if it satisfies the following conditions:

(i) the associative law: $x \diamond (y \diamond z) = (x \diamond y) \diamond z$;

(ii) the generalized distributive laws:

$$x \diamond (y + z) = x \diamond y + x \diamond z - x \diamond 0,$$
$$(y + z) \diamond x = y \diamond x + z \diamond x - 0 \diamond x;$$

(iii) the compatibility: $xy = x \diamond y - x \diamond 0 - 0 \diamond y + 0 \diamond 0$.

The semigroup $(R, \diamond)$ is called a generalized adjoint semigroup of R, abbreviated GA-semigroup and denoted by $R^\diamond$, which is a generalization of the multiplicative semigroup and the adjoint semigroup of a ring R. Essentially, the multiplicative and adjoint semigroup of R are exactly generalized adjoint semigroup of R with zero and identity, respectively ([7, Theorem 2.14]).

In Section 2, we prove that a GA-semigroup with central idempotents is a product of a multiplicative semigroup and an adjoint semigroup of ideals. The GA-semigroups of a strongly regular ring are determined.

The remaining sections are devoted to the description of the rings with a GA-semigroup having a property **P** and its such GA-semigroups in terms of the ring of a Morita context, where **P** stands for orthodox, right inverse, inverse, pseudoinverse, E-unitary, and completely simple, respectively. Surprisingly, we observe the following implication:

$$R^\bullet \text{ has the property } \mathbf{P} \Rightarrow R^\diamond \text{ has the property } \mathbf{P} \ \Rightarrow R^\circ \text{ has the property } \mathbf{P},$$

where $R^\bullet$ denotes the multiplicative semigroup of R.

Throughout, the set of idempotents of a semigroup or ring S will be denoted by $\mathcal{E}(S)$. For a ring R denote by $R^\bullet$ and R° the multiplicative and the adjoint semigroup of R, respectively. It is easy to see that an element e of a ring R is an idempotent of R° if and only if $e + e^2 = 0$, that is, $-e$ is an idempotent of $R^\bullet$, and hence $\mathcal{E}(R) = \mathcal{E}(R^\bullet) = -\mathcal{E}(R^\circ)$.

Although a ring R in this paper needs not contain identity, it is convenient to use a formal identity 1, which can be regarded as the identity of a unitary ring containing R, since R can be always embedded into a ring with identity 1; for example, we can write $a \circ b = (1+a)(1+b) - 1$ for any $a, b \in R$ and write $x^0 = 1$ for any $x \in R$ by making use of a formal 1.

A radical ring means a Jacobson radical ring. For the algebraic theory and terminology on semigroups we will refer to [3, 9, 13].

2. GA-semigroups with central idempotents

Recall that we call a GA-semigroups $R^\diamond$ of R affinely isomorphic to the GA-semigroup $S^\diamond$ of the ring S, notionally $R^\diamond \simeq S^\diamond$, if there exists a bijection ϕ from R onto S such that

$$\phi(x+y-z) = \phi(x) + \phi(y) - \phi(z) \text{ and } \phi(x \diamond y) = \phi(x) \diamond \phi(y)$$

for any $x, y, z \in M$. If $R^\diamond \simeq S^\diamond$, then $R \cong S$ ([7, Theorem 2.12]). $R^\diamond$ is called (centrally) 0-idempotent if the additive 0 of R is an (central) idempotent in $R^\diamond$ ([7]). One should note that (centrally) 0-idempotent is not an affinely isomorphic invariant. However, we have:

Lemma 2.1. ([7, Lemma 4.1]) *Every GA-semigroup containing an (central) idempotent is affinely isomorphic to a (centrally) 0-idempotent one.*

Lemma 2.2. ([7, Corollary 4.4]) *A GA-semigroup $R^\diamond$ is (centrally) 0-idempotent if and only if there exists an ideal extension $\tilde{R}$ of R and an idempotent $\varepsilon \in \tilde{R}$ (commuting with elements of R) such that $x \diamond y = (x+\varepsilon)(y+\varepsilon) - \varepsilon$ for any $x, y \in R$.*

Let $R_i^\diamond$, $i = 1, 2, \ldots, n$, be GA-semigroups of rings R_i. Then the direct product $\prod_{i=1}^{n} R_i^\diamond$ is a GA-semigroup of the ring $\prod_{i=1}^{n} R_i$, called the direct product of $R_i^\diamond$, $i = 1, 2, \ldots, n$.

Example 2.3. Let R be a direct sum of ideals R_0 and R_1. For any $x = a + b$ and $y = a' + b'$, $a, a' \in R_0$, $b, b' \in R_1$, define $x \diamond y = a'a + b \circ b'$. Then $R^\diamond$ is a GA-semigroup of R. Clearly, $R^\diamond \simeq R_0^\bullet \times R_1^\circ$.

Example 2.4. Let R be a zero ring, i.e., $R^2 = 0$. Define $x \diamond y = y$ for any $x, y \in R$. Then $R^\diamond$ is a GA-semigroup of R, called the right zero GA-semigroup of R. Symmetrically, one can define the left zero GA-semigroup of R.

Theorem 2.5. *$R^\diamond$ has a central idempotent if and only if $R^\diamond \simeq R_0^\bullet \times R_1^\circ$ for some ideals R_0 and R_1 of R such that $R = R_0 \oplus R_1$.*

Proof. The sufficiency is immediate. For the necessity, suppose that R contains a central idempotent e. Without loss of generality, we can assume 0 is a central idempotent in $R^\diamond$ by Lemma 2.1. Then we can complete the proof by taking $R_0 = \varepsilon R$ and $R_1 = (1-\varepsilon)R$ from Lemma 2.2. □

A duo ring is a ring in which one-sided ideals are ideals.

Lemma 2.6. *Let $R^\diamond$ be a GA-semigroup of a duo ring R. If $R^\diamond$ contains idempotents, then $R^\diamond \simeq R_0^\bullet \times R_1^\circ \times R_2^\diamond \times R_3^\diamond$, where R_i, $i = 1, 2, 3$, are ideal of R such that $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$, $R_2^2 = R_3^2 = 0$, $R_2^\diamond$ is the left zero GA-semigroup of R_2, and $R_3^\diamond$ is the right zero GA-semigroup of R_3.*

Proof. By Lemma 2.1 we can assume that $R^\diamond$ is a 0-idempotent GA-semigroup. Put $R_0 = \varepsilon R\varepsilon$, $R_1 = (1-\varepsilon)R(1-\varepsilon)$, $R_2 = \varepsilon R(1-\varepsilon)$, and $R_3 = (1-\varepsilon)R\varepsilon$, where ε is as in Lemma 2.2. Note that $R_0 = \varepsilon R \cap R\varepsilon$. Then we have that R_0 is an ideal of R since R is a duo ring. Similarly, R_1, R_2, and R_3 are ideals of R, and $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$. The rest is routine. □

Corollary 2.7. *Let $R^\diamond$ be a GA-semigroup of a commutative π-regular ring. Then $R^\diamond \simeq R_0^\bullet \times R_1^\circ \times R_2^\diamond \times R_3^\diamond$, where R_i, $i = 0, 1, 2, 3$, are ideals of R such that $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$, $R_2^2 = R_3^2 = 0$, $R_2^\diamond$ is the left zero GA-semigroup of R_2, and $R_3^\diamond$ is the right zero GA-semigroup of R_3.*

Theorem 2.8. *Any GA-semigroup $R^\diamond$ of a strongly regular ring contains central idempotents, and so $R^\diamond \simeq R_0^\bullet \times R_1^\circ$ for some ideals R_0 and R_1 of R such that $R = R_0 \oplus R_1$.*

Proof. It follows from [7, Theorem 3.5], Lemma 2.6 and the fact that a strongly regular ring is a duo ring ([8, Theorem 3.2]). □

Corollary 2.9. *The following statements for a ring R are equivalent.*

(i) *R is a Boolean ring;*

(ii) *R has a GA-semigroup is a semilattice;*

(iii) *any GA-semigroup of R is a semilattice.*

Proof. (iii)⇒(ii) is trivial.

(ii)⇒(i): If a GA-semigroup $R^\diamond$ is a semilattice, then by Theorem 2.5, $R^\diamond \simeq R_0^\bullet \times R_1^\circ$, where R_0 and R_1 are ideals of R such that $R = R_0 \oplus R_1$. Since $R^\diamond$ is a semilattice, $R_0^\bullet$ and R_1° are semilattices, implying that R is a Boolean ring.

(i)⇒(iii): Let $R^\diamond$ be a GA-semigroup of R. Since R is a Boolean ring, by Theorem 2.8, $R^\diamond \simeq R_0^\bullet \times R_1^\circ$, where R_0 and R_1 are ideals of R such that $R = R_0 \oplus R_1$. Since R is a Boolean ring, $R^\diamond$ is a semilattice. □

3. Orthodox GA-semigroups

Given two rings S and T, denote by $\tilde{S}$ and $\tilde{T}$ the Dorroh extension of S and T, respectively. Let $\tilde{R} = \begin{pmatrix} \tilde{S} & U \\ V & \tilde{T} \end{pmatrix}$ be the ring of the Morita context with bimodules ${}_SU_T$ and ${}_TV_S$, which are considered as unitary $\tilde{S}$-$\tilde{T}$ and $\tilde{T}$-$\tilde{S}$ bimodules in a natural way, respectively. Let $R = \begin{pmatrix} S & U \\ V & T \end{pmatrix}$. Then R is an ideal of $\tilde{R}$. We call R the ring of the Morita context or a Morita ring, and denote by $\mathcal{M}(S, T, U, V)$. Let

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \tilde{R}$. Then the generalized adjoint multiplication induced by E_{11} is given by

$$\begin{aligned} A \diamond B &= AB + AE_{11} + E_{11}B \\ &= (A + E_{11})(B + E_{11}) - E_{11} \\ &= \begin{pmatrix} s \circ s' + uv' & (1+s)u + ut' \\ u(1+s') + tv' & uu' + tt' \end{pmatrix} \end{aligned}$$

for any $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix}, B = \begin{pmatrix} s' & u' \\ v' & t' \end{pmatrix} \in R$. The semigroup $R^\diamond$ is called the E_{11}-GA-semigroup of R, denoted by $\mathcal{M}_{11}^\diamond(S, T, U, V)$. It is clear that the E_{11}-GA-semigroup $\mathcal{M}_{11}^\diamond(S, T, U, V)$ is 0-idempotent ([7]).

Theorem 3.1. ([7, Theorem 4.3]) *Let $R^\diamond$ be a GA-semigroup of R. If $R^\diamond$ contains idempotents, then there exists a Morita ring $\mathcal{M}(S, T, U, V)$ such that $R \cong \mathcal{M}(S, T, U, V)$ and $R^\diamond \simeq \mathcal{M}_{11}^\diamond(S, T, U, V)$.*

A ring R is called adjoint regular if its adjoint semigroup R° is a regular semigroup ([5, 11]).

Lemma 3.2. *Let $R = \mathcal{M}(S, T, U, V)$ and let $R^\diamond$ be the E_{11}-GA-semigroup of R. If $R^\diamond$ is regular, then S is an adjoint regular ring and T is a regular ring.*

Proof. Let $R_0 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$. Then we have $R_0^\diamond = R_0^\circ \simeq S^\circ$ and $R_1^\diamond = R_0^\bullet \simeq T^\bullet$.

Suppose $R^\diamond$ is regular. For any $a \in R_0$, there exists $x \in R$ such that $a = a \diamond x \diamond a$. Noting that $0 \diamond a \diamond 0 = E_{11} a E_{11} = a$ and $0 \diamond 0 = 0$, we see that $a = a \diamond 0 \diamond x \diamond 0 \diamond a$, and $0 \diamond x \diamond 0 = E_{11} x E_{11} \in R_0$, whence $R_0^\diamond$ is regular and so S° is regular.

For any $t \in T$, let $A = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}$. Then there exists $B = \begin{pmatrix} a & u \\ v & b \end{pmatrix} \in R$ such that

$$A = A \diamond B \diamond A = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1+a & u \\ v & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} - E_{11} = \begin{pmatrix} a & ut \\ tv & tbt \end{pmatrix},$$

yielding $t = tbt$ for some $b \in T$. Thus T is a regular ring. □

Lemma 3.3. *If $a - a \diamond b \diamond a + a \diamond c \diamond a$ be regular in $R^\diamond$ for some $b, c \in R$, then a is regular in $R^\diamond$.*

Proof. Let $x = a - a \diamond b \diamond a + a \diamond c \diamond a$. Then $x = x \diamond y \diamond x$ for some $y \in R$. Let $z = y - b \diamond a \diamond y + c \diamond a \diamond y$. Then

$$\begin{aligned} x \diamond y \diamond x &= a \diamond (y - b \diamond a \diamond y + c \diamond a \diamond y) \diamond x \\ &= a \diamond z \diamond x \\ &= a \diamond (z - z \diamond a \diamond b + z \diamond a \diamond c) \diamond a. \end{aligned}$$

Thus

$$\begin{aligned} a &= a \diamond b \diamond a - a \diamond c \diamond a + a \diamond (z - z \diamond a \diamond b + z \diamond a \diamond c) \diamond a \\ &= a \diamond (b - c + z - z \diamond a \diamond b + z \diamond a \diamond c) \diamond a, \end{aligned}$$

as desired. □

Lemma 3.4. *Let $R = \mathcal{M}(S,T,U,V)$ with $VU = 0$. Then the E_{11}-GA-semigroup $R^\diamond$ is regular if and only if S is an adjoint regular ring, T is a regular ring and $\mathcal{E}(S)U = V\mathcal{E}(S) = 0$, and if so, then*

(i) *R is an adjoint regular ring;*

(ii) $\mathcal{E}(R^\diamond) = \left\{ \begin{pmatrix} -e-uv & u(1-f) \\ (1-f)v & f \end{pmatrix} \mid e \in \mathcal{E}(S), f \in \mathcal{E}(T), u \in U, v \in V \right\}$;

(iii) $\mathcal{E}(R) = \left\{ \begin{pmatrix} e+uv & uf \\ fv & f \end{pmatrix} \mid e \in \mathcal{E}(S), f \in \mathcal{E}(T), u \in U, v \in V \right\}$.

Proof. Suppose that $R^\diamond$ is regular. Then by Lemma 3.2 we see that S is an adjoint regular ring and T is a regular ring. Now for any $e \in \mathcal{E}(S^\circ)$ and $u \in U$ there exists $\begin{pmatrix} s & y \\ z & t \end{pmatrix}$ such that

$$\begin{aligned} \begin{pmatrix} e & u \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} e & u \\ 0 & 0 \end{pmatrix} \diamond \begin{pmatrix} s & y \\ z & t \end{pmatrix} \diamond \begin{pmatrix} e & u \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1+e & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+s & y \\ z & t \end{pmatrix} \begin{pmatrix} 1+e & u \\ 0 & 0 \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} e \circ s \circ e + uz(1+e) & (1 + e \circ s + uz)u \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

forcing that $(e \circ s)u = 0$, since $(uz)u = u(zu) = 0$. Observing $e = -e(e \circ s)$, we can see that $eu = -e(e \circ s)u = 0$, from which it follows that $\mathcal{E}(S^\circ)U = 0$. Since $\mathcal{E}(S) = -\mathcal{E}(S^\circ)$, we have that $\mathcal{E}(S)U = 0$. Symmetrically, $V\mathcal{E}(S) = 0$.

Conversely, suppose that S is an adjoint regular ring, T is a regular ring and $\mathcal{E}(S)U = V\mathcal{E}(S) = 0$. Let I be the ideal of S generated by $\mathcal{E}(S)$. Then I is adjoint regular by [5, Proposition 1] and $IU = VI = 0$. Let $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \in R$, and $s = s \circ s' \circ s$ for some $s' \in S$. Let $B = \begin{pmatrix} s' & 0 \\ 0 & 0 \end{pmatrix}$ and let $C = A - A \diamond B \diamond A + A \diamond B \diamond B \diamond A$. To prove that A is regular in $R^\diamond$, it suffices to prove that C is regular in $R^\diamond$ by Lemma 3.3. A straightforward computation gives

$$C = \begin{pmatrix} s \circ s' \circ s' \circ s & b \\ c & d \end{pmatrix}$$

for some $b \in U$, $c \in V$, and $d \in T$. Let $a = s \circ s' \circ s' \circ s$. Then $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $s \circ s'$ and $s' \circ s$ are idempotents of S°, we have that $a \in I$, and so $aU =$

$Va = 0$ and $a = a \circ a' \circ a$ for some $a' \in I$. Let $d' \in T$ such that $d = dd'd$ and let $x = a' + bd'c$. Then $xU = Vx = 0$. Let $D = \begin{pmatrix} x & -bd' \\ -d'c & d' \end{pmatrix}$. Then a straightforward calculation shows that

$$\begin{aligned} C \diamond D \diamond C &= \begin{pmatrix} 1+a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+x & -bd' \\ -d'c & d' \end{pmatrix} \begin{pmatrix} 1+a & b \\ c & d \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} (a \circ x - bd'c) \circ a & b \\ c & d \end{pmatrix}. \end{aligned}$$

But $(a \circ x - bd'c) \circ a = (a \circ a') \circ a = a$. It follows that $C \diamond D \diamond C = C$, as desired.

To prove (i), let $S_1 = I + UV$. Then S_1 is an ideal of S, whence S_1 is an adjoint regular ring by [5, Proposition 1] and clearly $S_1U = VS_1 = 0$. Let $R_1 = \begin{pmatrix} S_1 & U \\ V & T \end{pmatrix}$. Then R_1 is an ideal of R and $R/R_1 \cong S/S_1$ is a radical ring since S/I is a radical ring by [6, Lemma 7] or [5, Theorem 3]. To prove R is adjoint regular, it is sufficient to prove R_1 is adjoint regular by [5, Theorem 3]. If $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \in R_1$, then $s = s \circ s' \circ s$ for some $s' \in S_1$. Since a regular ring is adjoint regular by [5, Theorem 1] ([6, Theorem 4], [11, Proposition 2.3]), we have that T is an adjoint regular ring, implying that $t = t \circ t' \circ t$ for some $t' \in T$. Let $x = s' + u(1+t')v$. Then $xU = Vx = 0$. Let $B = \begin{pmatrix} x & -u(1+t') \\ -(1+t')v & t' \end{pmatrix}$. Then

$$\begin{aligned} &A \circ B \circ A \\ &= \begin{pmatrix} 1+s & u \\ v & 1+t \end{pmatrix} \begin{pmatrix} 1+x & -u(1+t') \\ -(1+t')v & 1+t' \end{pmatrix} \begin{pmatrix} 1+s & u \\ v & 1+t \end{pmatrix} - 1 \\ &= \begin{pmatrix} (s \circ x - u(1+t')v) \circ s & u \\ v & t \end{pmatrix}. \end{aligned}$$

But $(s \circ x - u(1+t')v) \circ s = (s \circ s') \circ s = s$. It follows that $A \circ B \circ A = A$, proving (i).

For $e \in \mathcal{E}(S)$ and $f \in \mathcal{E}(T)$, if $uf = fu = 0$, then it is easy to verify that $\begin{pmatrix} -e-uv & u \\ v & f \end{pmatrix} \in \mathcal{E}(R^{\diamond})$. Conversely, let $E = \begin{pmatrix} s & u \\ v & t \end{pmatrix}$. If $E \in \mathcal{E}(R^{\diamond})$, then

$$E = E \diamond E = \begin{pmatrix} s \circ s + uv & (1+s)u + ut \\ v(1+s) + tv & t^2 \end{pmatrix}, \tag{3}$$

yielding $s = s \circ s + uv$. Thus

$$s + s^2 = -uv. \tag{4}$$

By (4) $(s+s^2)^2 = u(vu)v = 0$, that is, $((-s)-(-s)^2)^2 = 0$. By the $R^{\bullet}$-version of [7, Lemma 4.5] there exists an idempotent $e' \in \mathcal{E}(R)$ such that $s^2 = s^2e' = e's^2$. Noting that $(s+s^2)u = -u(vu) = 0$ by (4), we have that $su = -s^2u = -s^2e'u = 0$, and dually we have that $vs = 0$. Since $s^2 + s^3 = -suv = 0$ by (4), one can deduce that $s^2 \in \mathcal{E}(R)$. Let $e = s^2$. Then $s = -e - uv$ by (4). Putting $f = t$, we have $f \in \mathcal{E}(T)$ from (3). Since $su = vs = 0$, we obtain that $ut = tv = 0$ from (3). Thus $E = \begin{pmatrix} -e - uv & u \\ v & f \end{pmatrix}$ with $e \in \mathcal{E}(S)$, $f \in \mathcal{E}(T)$, and $uf = fv = 0$, proving (ii).

For $e \in \mathcal{E}(S)$ and $f \in \mathcal{E}(T)$, if $u(1-f) = (1-f)u = 0$, then it is easy to verify that $\begin{pmatrix} e + uv & u \\ v & f \end{pmatrix} \in \mathcal{E}(R)$. Conversely, let $E = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \in \mathcal{E}(R)$. Then

$$E = E^2 = \begin{pmatrix} s^2 + uv & su + ut \\ vs + tv & t^2 \end{pmatrix}, \tag{5}$$

yielding $s = s^2 + uv$, that is $s - s^2 = uv$. Similar to the proof of paragraph above, we have $s^2 \in \mathcal{E}(R)$ and $su = vs = 0$. Let $e = s^2$ and $f = t$. Then $s = e + uv$ and $e \in \mathcal{E}(S)$ and $f \in \mathcal{E}(R)$. From (5) we get $u = uf$ and $v = fv$, proving (iii). ☐

An orthodox semigroup means a regular semigroup whose idempotents constitute a subsemigroup. A band is called regular if $xyzx = xyxzx$ for any $x, y, z \in \mathcal{E}(S)$ ([9]).

It is easy to see that $R^\bullet$ is an orthodox semigroup if and only if R is a strongly regular ring. In [6], we characterize the ring such that R° is orthodox, and we particularly prove that such a ring is a generalized radical ring such that $\mathcal{E}(R^\circ)$ is a regular band ([6, Theorem 14]), where a generalized radical ring means a ring whose adjoint semigroup is a union of groups ([2]).

Lemma 3.5. *Let $R = \mathcal{M}(S, T, U, V)$. Then E_{11}-GA-semigroup $R^\diamond$ is orthodox if and only if S° is an orthodox semigroup, T is a strongly regular ring, $\mathcal{E}(S)U = V\mathcal{E}(S) = UV = VU = 0$. Moreover, if $R^\diamond$ is orthodox, then $R^\diamond$ are a union of groups and $\mathcal{E}(R^\diamond)$ is a regular band.*

Proof. Suppose $R^\diamond$ is an orthodox semigroup. Then by Lemma 3.4, S° and $T^\bullet$ are both orthodox semigroups, and so T is a strongly regular ring. For any $x \in U$ and $y \in V$ it is easy to see that $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ are both idempotents of $R^\diamond$. Since $\mathcal{E}(R^\diamond)$ is a semigroup, $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \diamond \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & yx \end{pmatrix}$ is an idempotent of $R^\diamond$, whence

$$\begin{aligned} \begin{pmatrix} 0 & x \\ y & yx \end{pmatrix} &= \begin{pmatrix} 0 & x \\ y & yx \end{pmatrix} \diamond \begin{pmatrix} 0 & x \\ y & yx \end{pmatrix} \\ &= \begin{pmatrix} 1 & x \\ y & yx \end{pmatrix} \begin{pmatrix} 1 & x \\ y & yx \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} xy & x + xyx \\ y + yxy & yx + (yx)^2 \end{pmatrix}, \end{aligned}$$

and so $xy = 0$. Noting that yx is a nilpotent element of T, we see that $yx = 0$ since T is a strongly regular ring. Thus $UV = VU = 0$. Since $R^\diamond$ is regular, we have $\mathcal{E}(S)U = V\mathcal{E}(S) = 0$ by Lemma 3.4.

Conversely, suppose S° is an orthodox semigroup, T is a strongly regular ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UV = VU = 0$. It suffices to prove that $R^\diamond$ is a union of groups and $\mathcal{E}(R^\diamond)$ is a regular band. By Lemma 3.4,

$$\mathcal{E}(R^\diamond) = \left\{ \begin{pmatrix} e & u(1-f) \\ (1-f)v & f \end{pmatrix} \mid e \in \mathcal{E}(S^\circ), f \in \mathcal{E}(T), u \in U, v \in V \right\}. \quad (6)$$

For any $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \in R$, there exist $s' \in S$, $e \in \mathcal{E}(S^\circ)$, $t \in T$ and $f \in \mathcal{E}(T)$ such that $s \circ s' = s' \circ s = e$, $e \circ s = s \circ e = s$, and $e \circ s' = s' \circ e = s$ since S° is a union of groups by [6, Theorem 14], and $tt' = t't = f$ and $ft = t$ since T is a strongly regular ring. Let

$$B = \begin{pmatrix} e & (1+s')u(1-f) \\ (1-f)v(1+s') & f \end{pmatrix},$$
$$C = \begin{pmatrix} s' & (1+s'\circ s')u(1-f) - (1+s')ut' \\ -t'v(1+s') & t' \end{pmatrix}.$$

Then $B \in \mathcal{E}(R)$ by (6), and a computation yields that $A \diamond B = B \diamond A = A$ and $A \diamond C = B$, whence $R^\diamond$ is completely regular and so it is a union of groups by [3, Theorem 4.3].

Now we have to prove that $\mathcal{E}(R^\diamond)$ is a regular band. For $E, E', E'' \in \mathcal{E}(R^\diamond)$, if $E = \begin{pmatrix} e & u \\ v & f \end{pmatrix}$, $E' = \begin{pmatrix} e' & u' \\ v' & f' \end{pmatrix}$, $E'' = \begin{pmatrix} e'' & u'' \\ v'' & f'' \end{pmatrix}$, then $uf = fv = u'f' = f'v' = u''f'' = f''v'' = 0$ by (6). Observe that

$$\begin{aligned} E \diamond E' &= \begin{pmatrix} 1+e & u \\ v & f \end{pmatrix} \begin{pmatrix} 1+e' & u' \\ v' & f' \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} e\circ e' & u' + uf' \\ v + fv' & ff' \end{pmatrix}. \end{aligned} \quad (7)$$

Since $\mathcal{E}(S^\circ)$ is a band, we have that $e \circ e' \in \mathcal{E}(S^\circ)$. Since T is a strongly regular ring, idempotents are contained in the center of T, and so $ff' \in \mathcal{E}(T)$. Moreover, $(u' + uf')ff' = u'f'f + uff' = 0$ and similarly $ff'(v + fv') = 0$. It follows from (6) and (7) that $E \circ E' \in \mathcal{E}(R^\diamond)$. Thus $\mathcal{E}(R^\diamond)$ is a band. Now we need to prove $\mathcal{E}(R^\diamond)$ is regular. By (7), we have that

$$\begin{aligned} &E \diamond E' \diamond E'' \\ &= \begin{pmatrix} 1+e\circ e' & u'+uf' \\ v+fv' & ff' \end{pmatrix} \begin{pmatrix} 1+e'' & u'' \\ v'' & f'' \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} e\circ e'\circ e'' & u'' + u'f'' + uf'f'' \\ v + fv' + ff'v'' & ff'f'' \end{pmatrix}, \end{aligned} \quad (8)$$

and by (8) we have

$$\begin{aligned} &E \diamond E' \diamond E'' \diamond E \\ &= \begin{pmatrix} 1+e\circ e'\circ e'' & u''+u'f''+uf'f'' \\ v+fv'+ff'v'' & ff'f'' \end{pmatrix} \begin{pmatrix} 1+e & u \\ v & f \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} e\circ e'\circ e''\circ e & u+u''f+u'f''f \\ v+fv'+ff'v'' & ff'f'' \end{pmatrix}. \end{aligned} \tag{9}$$

Replacing x'' by x in (8), $x \in \{u, v, e, f\}$, we get that

$$E \diamond E' \diamond E = \begin{pmatrix} e\circ e'\circ e & u+u'f \\ v+fv' & ff' \end{pmatrix}, \tag{10}$$

and replacing x' by x'' in (10), $x \in \{u, v, e, f\}$, we get that

$$E \diamond E'' \diamond E = \begin{pmatrix} e\circ e''\circ e & u+u''f \\ v+fv'' & ff'' \end{pmatrix}. \tag{11}$$

Thus by (7) and (11) we have that

$$\begin{aligned} &E \diamond E' \diamond E \diamond E'' \diamond E \\ &= \begin{pmatrix} 1+e\circ e' & u'+uf' \\ v+fv' & ff' \end{pmatrix} \begin{pmatrix} 1+e\circ e''\circ e & u+u''f \\ v+fv'' & ff'' \end{pmatrix} - E_{11} \\ &= \begin{pmatrix} e\circ e'\circ e\circ e''\circ e & u+u''f+u'ff'' \\ v+fv'+ff'v'' & ff'f'' \end{pmatrix}. \end{aligned} \tag{12}$$

Since $\mathcal{E}(S^\circ)$ is a regular band by [6, Theorem 14], we have that $E \diamond E' \diamond E'' \diamond E = E \diamond E' \diamond E \diamond E'' \diamond E$ by (9) and (12). Hence $\mathcal{E}(R^\diamond)$ is a regular band, and so $R^\diamond$ is an orthodox semigroup. □

Theorem 3.6. *The following statements are equivalent for a GA-semigroup $R^\diamond$ of R.*

(i) *$R^\diamond$ is orthodox;*

(ii) *$R^\diamond$ is a union of groups and $\mathcal{E}(R^\diamond)$ is a regular band;*

(iii) *$R^\diamond \simeq \mathcal{M}_{11}^\diamond(S, T, U, V)$, where S° is an orthodox semigroup, T is a strongly regular ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UV = VU = 0$.*

Proof. If follows from Lemma 3.5 and Theorem 3.1. □

Theorem 3.7. *The following statements are equivalent for a ring R.*

(i) *R has an orthodox GA-semigroup;*

(ii) *R° is an orthodox semigroup;*

(iii) *R° is a union of groups and $\mathcal{E}(R^\circ)$ is a regular band.*

Proof. (ii)⇔(iii) follows from [6, Theorem 14] and (ii)⇒(i) is trivial. It remains to prove (i)⇒(ii). Suppose that a GA-semigroup $R^\diamond$ is orthodox. Then by Theorem 3.1 and Theorem 3.6 we can assume that $R = \mathcal{M}(S,T,U,V)$ and $R^\diamond = \mathcal{M}_{11}^\diamond(S,T,U,V)$, where S° is an orthodox semigroup, T is a strongly regular ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UV = VU = 0$. By Lemma 3.4, R is an adjoint regular ring with

$$\mathcal{E}(R^\circ) = \left\{ \begin{pmatrix} e & u \\ v & f \end{pmatrix} | e \in \mathcal{E}(S^\circ), f \in \mathcal{E}(T^\circ), u(1+f) = (1+f)v = 0 \right\}. \tag{13}$$

If $A = \begin{pmatrix} e & u \\ v & f \end{pmatrix}$ and $A' = \begin{pmatrix} e' & u' \\ v' & f' \end{pmatrix}$ are both idempotents of R°, then

$$\begin{aligned} A \circ A' &= (1+A)(1+A') - 1 \\ &= \begin{pmatrix} 1+e & u \\ v & 1+f \end{pmatrix} \begin{pmatrix} 1+e' & u' \\ v' & 1+f' \end{pmatrix} - 1 \\ &= \begin{pmatrix} e \circ e' & u' + u(1+f') \\ v + (1+f)v' & f \circ f' \end{pmatrix}. \end{aligned}$$

Since S° is orthodox, $e \circ e' \in \mathcal{E}(S^\circ)$. Since T is a strongly regular ring, $f \circ f' = f' \circ f \in \mathcal{E}(T^\circ)$. Observing that

$$(u' + u(1+f'))\,(1 + f' \circ f) = u'(1+f')(1+f) + u(1+f)(1+f') = 0$$

and similarly $(1 + f' \circ f)(v + (1+f)v') = 0$, we see that $A \circ A' \in \mathcal{E}(R^\circ)$ by (13). Hence $\mathcal{E}(R^\circ)$ is a band. It follows that R° is orthodox. □

4. Inverse GA-semigroups

Recall that a semigroup is called a right inverse semigroup if its every principal left ideal has a unique idempotent generator. According to [18, Theorem 1], a semigroup S is a right inverse semigroup if and only if S is a regular semigroup in which the set $\mathcal{E}(S)$ of all idempotents is a right regular band, that is $xy = yxy$ for any $x, y \in \mathcal{E}(S)$. A semigroup is inverse if it is left and right inverse.

It is clear that $R^\bullet$ is inverse if and only if R is a strongly regular ring. A ring with the inverse adjoint semigroup was studied by [4, 6, 10, 11, 12, 16] and a ring with the right inverse adjoint semigroup was described in [6].

Lemma 4.1. *Let $R = \mathcal{M}(S,T,U,V)$. Then the E_{11}-GA-semigroup $R^\diamond$ is right inverse if and only if S° is a right inverse semigroup, T is a strongly regular ring, and $\mathcal{E}(S)U = V = 0$, and if so, then R° is right inverse.*

Proof. Suppose that $R^\diamond$ is right inverse. Then S° is a right inverse semigroup and T is a strongly regular ring by Lemma 3.2. Noting that $R^\diamond$ is orthodox, we have that $\mathcal{E}(S)U = 0$ by Lemma 3.5. For any $v \in V$, let $A = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$. Then $A \in \mathcal{E}(R^\diamond)$, and hence

$$A = A \diamond 0 = 0 \diamond A \diamond 0 = 0,$$

yielding $v = 0$. It follows that $V = 0$.

Conversely, suppose that $R = \mathcal{M}(S,T,U,0)$ such that S° is a right inverse semigroup, T is a strongly regular ring and $\mathcal{E}(S)U = V = 0$. Then the E_{11}-GA-semigroup $R^\diamond$ is a regular semigroup with

$$\mathcal{E}(R^\diamond) = \left\{ \begin{pmatrix} e & u \\ 0 & f \end{pmatrix} \mid e \in \mathcal{E}(S^\circ), f \in \mathcal{E}(T), u \in U \text{ and } uf = 0 \right\} \tag{14}$$

by Lemma 3.4. For $E, E' \in \mathcal{E}(R^\diamond)$, let $E = \begin{pmatrix} e & u \\ 0 & f \end{pmatrix}$ and $E' = \begin{pmatrix} e' & u' \\ 0 & f' \end{pmatrix}$. Then $uf = u'f' = 0$ by (14). Observing that

$$\begin{aligned}
E \diamond E' &= \begin{pmatrix} 1+e & u \\ 0 & f \end{pmatrix}\begin{pmatrix} 1+e' & u' \\ 0 & f' \end{pmatrix} - E_{11} = \begin{pmatrix} e\diamond e' & u'+uf' \\ 0 & ff' \end{pmatrix}, \\
E' \diamond E \diamond E' &= \begin{pmatrix} 1+e' & u' \\ 0 & f' \end{pmatrix}\begin{pmatrix} 1+e\diamond e' & u'+uf' \\ 0 & ff' \end{pmatrix} - E_{11} \\
&= \begin{pmatrix} e'\diamond e\diamond e' & u'+uf'+u'ff' \\ 0 & f'ff' \end{pmatrix} \\
&= \begin{pmatrix} e\diamond e' & u'+uf' \\ 0 & ff' \end{pmatrix} \\
&= E \diamond E',
\end{aligned}$$

we see that $\mathcal{E}(R^\diamond)$ is a right regular band. It follows that $R^\diamond$ is a right inverse semigroup.

We now proceed to prove R° is right inverse. It suffices to prove that $\mathcal{E}(R^\circ)$ is right regular since R° is regular by Lemma 3.4. Note that

$$\mathcal{E}(R^\circ) = \left\{ \begin{pmatrix} e & u \\ 0 & f \end{pmatrix} | e \in \mathcal{E}(S^\circ), f \in \mathcal{E}(T^\circ), u(1+f) = 0 \right\}$$

by Lemma 3.4. For $F, F' \in \mathcal{E}(R^\circ)$, let $F = \begin{pmatrix} e & u \\ 0 & f \end{pmatrix}$ and $F' = \begin{pmatrix} e' & u' \\ 0 & f' \end{pmatrix}$. Then

$$F \circ F' = \begin{pmatrix} e\circ e' & u'+u(1+f') \\ 0 & f\circ f' \end{pmatrix},$$

$$\begin{aligned}
F' \circ F \circ F' &= \begin{pmatrix} e' & u' \\ 0 & f' \end{pmatrix} \circ \begin{pmatrix} e\circ e' & u'+u(1+f') \\ 0 & f\circ f' \end{pmatrix} \\
&= \begin{pmatrix} e'\circ e\circ e' & u'+u'+u(1+f')+u'(f\circ f') \\ 0 & f'\circ f\circ f' \end{pmatrix} \\
&= F \circ F',
\end{aligned}$$

since $\mathcal{E}(S^\circ)$ is right regular, $\mathcal{E}(T^\circ)$ is a semilattice, and $u'(1+f\circ f') = 0$. It follows that $\mathcal{E}(R^\circ)$ is right regular, as required. □

Theorem 4.2. *A ring R has a right inverse GA-semigroup if and only if R° is right inverse. Moreover, a GA-semigroup $R^\diamond$ of R is right inverse if and only if $R^\diamond \simeq \mathcal{M}_{11}^\diamond(S,T,U,0)$, where S° is right inverse, T is a strongly regular ring, and $\mathcal{E}(S)U = 0$.*

Proof. It follows from Lemma 4.1 and Theorem 3.1. □

Lemma 4.3. *For $e \in \mathcal{E}(R^\diamond)$ and $x \in R$, we have*

$$e + e \diamond x - e \diamond x \diamond e \in \mathcal{E}(R^\diamond) \text{ and } e + x \diamond e - e \diamond x \diamond e \in \mathcal{E}(R^\diamond).$$

Proof. Let $a = e + e \diamond x - e \diamond x \diamond e$, then $a \diamond e = e$ and $e \diamond a = a$, whence

$$a \diamond a = a \diamond (e \diamond a) = (a \diamond e) \diamond a = e \diamond a = a.$$

The other can be proved dually. □

Lemma 4.4. *If idempotents of $R^\diamond$ commute, then idempotents are central in $R^\diamond$.*

Proof. Suppose idempotents of $R^\diamond$ commute. For any $e \in \mathcal{E}(R^\diamond)$ and $x \in R$, let $a = e + e \diamond x - e \diamond x \diamond e$. Then $a \in \mathcal{E}(R^\diamond)$ by Lemma 4.3, and so $e \diamond a = a \diamond e$, yielding $e \diamond x = e \diamond x \diamond e$. Dually, $x \diamond e = e \diamond x \diamond e$. Thus $e \diamond x = x \diamond e$ for any $x \in R$. □

Lemma 4.5. *Let $R = \mathcal{M}(S,T,U,V)$. Then the E_{11}-GA-semigroup $R^\diamond$ is inverse if and only if S° is an inverse semigroup, T is a strongly regular ring, and $U = V = 0$.*

Proof. The lemma follows from Lemma 4.1 and its left-hand version. □

Theorem 4.6. *The following statements are equivalent for a GA-semigroup $R^\diamond$ of R.*

(i) *$R^\diamond$ is inverse;*

(ii) *$R^\diamond$ is a regular semigroup in which idempotents are all central;*

(iii) *$R^\diamond \simeq R_0^\bullet \times R_1^\circ$, where R_0 and R_1 are ideals of R such that $R = R_0 \oplus R_1$, R_0 is a strongly regular ring and R_1° is inverse.*

Proof. (i)⇔(ii) follows from Lemma 4.4 and [3, Theorem 1.17]. Since the idempotents are central in a ring with inverse adjoint semigroup by [4, Theorem 6] or [6, Theorem 17], (iii)⇒(ii) is clear. Now suppose $R^\diamond$ is inverse. Then by Theorem 3.1 and Lemma 4.5, $R^\diamond \simeq \mathcal{M}_{11}^\diamond(S,T,0,0)$, where S° is inverse and T is a strongly regular ring. It is clear that $\mathcal{M}_{11}^\diamond(S,T,0,0) \simeq S^\circ \times T^\bullet$, proving (i)⇒(iii). □

Theorem 4.7. *A ring R has an inverse GA-semigroup if and only if R° is inverse.*

Proof. The sufficiency is trivial. For the necessity, if a GA-semigroup $R^\diamond$ is inverse, then by Theorem 4.6, we have that $R = R_0 \oplus R_1$, where R_0 and R_1 are ideals of R such that R_0 is a strongly regular ring and R_1° is inverse. Clearly $R^\circ \cong R_0 \oplus R_1$. Since the adjoint semigroup of a strongly regular ring is inverse by [4, Theorem 6] or [6, Theorem 17], we have R° is inverse. □

Recall that a regular semigroup is called pseudoinverse if and only if eSe is inverse for any $e \in \mathcal{E}(S)$, if and only if idempotents of eSe commute for any $e \in \mathcal{E}(S)$ ([9, IX.3]). We note that $R^\bullet$ is pseudoinverse if and only if R is a strongly regular ring. Since R° has identity, R° is pseudoinverse if and only if it is inverse. In [6], we described the ring such that $e \circ R \circ e$ is inverse for any idempotent $e \neq 0$.

Lemma 4.8. *Let $R = \mathcal{M}(S, T, U, V)$. Then the E_{11}-GA-semigroup $R^\diamond$ is pseudoinverse if and only if S° is an inverse semigroup, T is a strongly regular ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = VU = 0$, and if so, then R° is inverse.*

Proof. Suppose $R^\diamond$ is pseudoinverse. Then T is a regular ring by Lemma 3.2. For any $f \in \mathcal{E}(T)$ let $A = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$. Then $A \in \mathcal{E}(R^\diamond)$ and $A \diamond R \diamond A = \begin{pmatrix} S & Uf \\ fV & fTf \end{pmatrix}$. Since $A \diamond R \diamond A$ is inverse, we have by Lemma 4.5 that S° is an inverse semigroup, fTf is a strongly regular ring and $Uf = fV = 0$. Thus T is a strongly regular ring and so $UT = TV = 0$. Noting that $VU \subseteq T$, one sees that $(VU)^2 \subseteq V(UT) = 0$, implying that $VU = 0$, since T is a strongly regular ring. Since $R^\diamond$ is regular, $\mathcal{E}(S)U = V\mathcal{E}(S) = 0$ by Lemma 3.4.

Now we prove the sufficiency. By Lemma 3.4,

$$\mathcal{E}(R^\diamond) = \left\{ \begin{pmatrix} -e - uv & u \\ v & f \end{pmatrix} | e \in \mathcal{E}(S), f \in \mathcal{E}(T), u \in U, v \in V \right\}.$$

For any $u \in U$ and $v \in V$ let

$$\mathcal{E}_{u,v} = \left\{ \begin{pmatrix} -e - uv & u \\ v & f \end{pmatrix} | e \in \mathcal{E}(S) \text{ and } f \in \mathcal{E}(T) \right\}.$$

Then $\mathcal{E}(R) = \bigcup_{(u,v) \in U \times V} \mathcal{E}_{u,v}$. For any $A, B \in \mathcal{E}(R^\diamond)$, straightforward computation shows that $A \diamond B = B \diamond A$ if and only if $A, B \in \mathcal{E}_{u,v}$ for some $u \in U$ and $v \in V$. It follows that the commutativity defines an equivalence relation on $\mathcal{E}(R^\diamond)$ whose set of equivalence classes consists of $\mathcal{E}_{u,v}$, $(u, v) \in U \times V$. Now for any $B \in \mathcal{E}_{u,v}$, idempotents of $B \diamond R \diamond B$ commute with B, implying that idempotents of $B \diamond R \diamond B$ commute. Thus $R^\diamond$ is pseudoinverse. We need to prove that R° is inverse. To do this, we observe that R° is regular and $\mathcal{E}(R) = \begin{pmatrix} \mathcal{E}(S) & 0 \\ 0 & \mathcal{E}(T) \end{pmatrix}$ by Lemma 3.4. Thus $\mathcal{E}(R^\circ) = \begin{pmatrix} \mathcal{E}(S^\circ) & 0 \\ 0 & \mathcal{E}(T^\circ) \end{pmatrix}$ and $\mathcal{E}(R^\circ)$ is a semilattice. It follows that R° is inverse by [3, Theorem 1.17]. □

Theorem 4.9. *A GA-semigroup $R^\diamond$ of a ring is a pseudoinverse semigroup if and only if $R^\diamond \simeq \mathcal{M}_{11}^\diamond(S,T,U,V)$, where S° is inverse, T is a strongly regular ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = VU = 0$.*

Proof. It is an immediate consequence of Theorem 3.1 and Lemma 4.8. □

Theorem 4.10. *The following statements for a ring R are equivalent.*

(i) *R has a pseudoinverse GA-semigroup;*

(ii) *R has an inverse GA-semigroup;*

(iii) *R° is inverse.*

Proof. (i)⇔(iii) follows from Lemma 4.8 and (ii)⇔(iii) is Theorem 4.7. □

Theorem 4.11. *Every GA-semigroup of R is inverse (orthodox, pseudoinverse) if and only if R is a strongly regular ring.*

Proof. If $R^\bullet$ is inverse (orthodox, pseudoinverse), then R is a strongly regular ring. Conversely, if R is a strongly regular ring, then by Theorem 2.8, $R^\diamond \simeq R_0^\bullet \times R_1^\circ$ for some ideals R_0 and R_1 of R such that $R = R_0 \oplus R_1$. Note that R_0 and R_1 are strongly regular rings. Then we have that $R_0^\bullet$ and R_1° are inverse semigroups, and so $R^\diamond$ is an inverse semigroup. □

5. *E*-unitary GA-semigroups

Recall that a regular semigroup S is called E-unitary if for any $a \in S$ and $e \in \mathcal{E}(S)$, $ae \in \mathcal{E}(S)$ implies $a \in \mathcal{E}(S)$, and this is equivalent to that $ea \in \mathcal{E}(S)$ implies $a \in \mathcal{E}(S)$ for any $e \in \mathcal{E}(S)$ and $a \in S$ ([9]). Clearly, $R^\bullet$ is E-unitary if and only if R is a Boolean ring, for $a0 = 0$ for any $a \in R$. In [6], we presented a characterization of the rings with E-unitary adjoint semigroups.

Lemma 5.1. *Let $R = \mathcal{M}(S,T,U,V)$. Then the E_{11}-GA-semigroup $R^\diamond$ is E-unitary if and only if S° is E-unitary, T is a Boolean ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = UV = VU = 0$.*

Proof. Suppose that $R^\diamond$ is E-unitary. Then S° and $T^\bullet$ are clearly E-unitary by Lemma 3.4, and hence T is a Boolean ring. Let $A = \begin{pmatrix} 0 & u \\ v & t \end{pmatrix}$. Then $A \diamond 0 = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in \mathcal{E}(R^\diamond)$, and so $\begin{pmatrix} 0 & u \\ v & t \end{pmatrix} \in \mathcal{E}(R^\diamond)$, that is,

$$\begin{pmatrix} 0 & u \\ v & t \end{pmatrix} = A = A \diamond A = \begin{pmatrix} 1 & u \\ v & t \end{pmatrix}\begin{pmatrix} 1 & u \\ v & t \end{pmatrix} - E_{11} = \begin{pmatrix} uv & u+ut \\ v+tv & vu+t^2 \end{pmatrix},$$

from which it follows that $UV = UT = TV = 0$ and so $VU = 0$ since $(VU)^2 = 0$ and T is a Boolean ring. Since $R^\diamond$ is regular, $\mathcal{E}(S)U = V\mathcal{E}(S) = 0$ by Lemma 3.4. The necessity is proved.

Now we prove the sufficiency. By Lemma 3.4,

$$\mathcal{E}(R^\diamond) = \begin{pmatrix} \mathcal{E}(S^\circ) & U \\ V & T \end{pmatrix} \tag{15}$$

since T is a Boolean ring. For any $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix} \in R$ and $E = \begin{pmatrix} e & y \\ z & f \end{pmatrix} \in \mathcal{E}(R^\diamond)$, if $A \diamond E = \begin{pmatrix} s \circ e & (1+s)y \\ v & tf \end{pmatrix} \in \mathcal{E}(R^\diamond)$, then $s \circ e \in \mathcal{E}(S^\circ)$ by (15), and so $s \in \mathcal{E}(S^\circ)$ since S° is E-unitary. Therefore, $A \in \mathcal{E}(R^\diamond)$ by (15). □

Lemma 5.2. *If R° is E-unitary, then R is a direct sum of a Boolean ring and a radical ring.*

Proof. By [6, Theorem 23], R is an extension of a Boolean ring by a radical ring. Let B be a Boolean ideal of R such that R/B is a radical ring. Observing that idempotents of R are central, we see that B is contained in the center of R. For any $a \in R$ there exists $b \in R$ such that $a \circ b \in B$. Let $e = a \circ b$. For any $f \in B$, we have that

$$f(1-e)a \circ f(1-e)b = f(1-e)(a \circ b) = 0.$$

Since $f(1-e)a \in B$, we have that $f(1-e)a = 0$. Thus $(1-e)a \in Ann_R(B)$, whence $a = ea + (1-e)a \in B + Ann_R(B)$. Since B is semiprime, $B \cap Ann_R(B) = 0$. Hence $R = B \oplus Ann_R(B)$ and $Ann_R(B)$ is a radical ring since R/B is a radical ring. □

Theorem 5.3. *A GA-semigroup $R^\diamond$ of a ring R is E-unitary if and only if $R^\diamond \simeq \mathcal{M}^\diamond_{11}(S,T,U,V)$, where S is a direct sum of a Boolean ring with a radical ring, T is a Boolean ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = UV = VU = 0$.*

Proof. It follows from Theorem 3.1, Lemma 5.1, and Lemma 5.2. □

Theorem 5.4. *The following conditions are equivalent for a ring R.*

(i) *R has an E-unitary GA-semigroup;*

(ii) *R is a direct sum of a Boolean ring with a radical ring;*

(iii) *R° is E-unitary.*

Proof. (ii)⇒(iii) and (iii)⇒(i) are clear. Suppose that a GA-semigroup $R^\diamond$ of R is E-unitary. Then by Theorem 5.3 and [7, Theorem 2.12], $R \cong \mathcal{M}(S,T,U,V)$, where S is a direct sum of a Boolean ring S_1 with a radical ring S_2, T is a Boolean ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = UV = VU = 0$. Thus

$$R \cong \begin{pmatrix} S_1 \oplus S_2 & U \\ V & T \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & T \end{pmatrix} \oplus \begin{pmatrix} S_2 & U \\ V & 0 \end{pmatrix},$$

and clearly $\begin{pmatrix} S_1 & 0 \\ 0 & T \end{pmatrix}$ is a Boolean ideal and $\begin{pmatrix} S_2 & U \\ V & 0 \end{pmatrix}$ is the radical of $\mathcal{M}(S,T,U,V)$, proving (i)⇒ (ii). □

Corollary 5.5. *A ring has a GA-semigroup which is a band if and only if it is a direct sum of a Boolean ring with a zero ring.*

Proof. Suppose $R^\diamond$ is a band. Then $R^\diamond$ is E-unitary, and so by Theorem 5.3, $R^\diamond \simeq \mathcal{M}_{11}^\diamond(S,T,U,V)$, where S is a direct sum of a Boolean ring with a radical ring, T is a Boolean ring, and $\mathcal{E}(S)U = V\mathcal{E}(S) = UT = TV = UV = VU = 0$. Since $R^\diamond$ is a band, S is a Boolean ring, and so $SU = VS = 0$. Therefore, by [7, Theorem 2.12], $R \cong \mathcal{M}(S,T,U,V) = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \oplus \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}$. Clearly $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is a Boolean ideal and $\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}$ is an ideal such that $\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}^2 = 0$.

Suppose R is a Boolean ideal S and an ideal T such that $T^2 = 0$. Then the direct product of $S^\bullet$ and the right zero GA-semigroup of T gives a GA-semigroup of R which is a band. □

6. Completely simple GA-semigroups

If $R^\bullet$ is completely 0-simple, then R is a division ring, while if R° is completely 0-simple or simple, then R is a division ring or a radical ring ([17, 10]).

If $a \in R$ is a unit in R°, we denote by a^- the inverse of a, i.e., the quasi-inverse of a ([14]).

Lemma 6.1. *Let $R = \mathcal{M}(S,T,U,V)$. Then the E_{11}-GA-semigroup $R^\diamond$ is completely simple if and only if S is a radical ring and $T = 0$, and if so, then R is a radical ring.*

Proof. Suppose that $R^\diamond$ is completely simple. Then $R^\diamond$ is regular, and so S° is a regular semigroup and T is a regular ring by Lemma 3.2. Noting that 0 is a primitive idempotent of $R^\diamond$, we have that $0 \diamond R \diamond 0$ is a group by [3, Lemma 2.47]. Since $S^\circ \simeq 0 \diamond R \diamond 0$, S is a radical ring. If f is an idempotent of T, then $E_{11}f = fE_{11} = 0$, and so $0 \diamond f = f \diamond 0 = 0$, which implies that $f = 0$ since 0 is primitive. Thus $T = 0$.

Conversely, for any $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \in R$, we have

$$\begin{pmatrix} 0 & 0 \\ z(1+x^-) & 0 \end{pmatrix} \diamond \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \diamond \begin{pmatrix} a^- \circ x & (1+a^-)y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ z & 0 \end{pmatrix},$$

whence $R^\diamond$ is a simple semigroup. By [9, Proposition II.4.7], it is sufficient to prove that 0 is a primitive idempotent in $R^\diamond$. Let $A = \begin{pmatrix} e & u \\ v & 0 \end{pmatrix}$ be an idempotent of $R^\diamond$ such that $0 \diamond A = A \diamond 0 = A$. Then

$$\begin{pmatrix} e & 0 \\ v & 0 \end{pmatrix} = \begin{pmatrix} e & u \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & u \\ v & 0 \end{pmatrix},$$

yielding $u = v = 0$. Thus $A = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$ and so e is an idempotent of S°, forcing $e = 0$ since S is a radical ring. Hence $A = 0$. It follows that 0 is a primitive idempotent in $R^\diamond$. We now prove that R is a radical ring. Observing that R° is regular and $\mathcal{E}(R) = 0$ by Lemma 3.4, we see that R° is a group, that is, R is a radical ring. □

Theorem 6.2. *A GA-semigroup $R^\diamond$ of R is a completely simple semigroup if and only if $R^\diamond \simeq \mathcal{M}^\diamond_{11}(S, 0, U, V)$, where S is a radical ring.*

Proof. It follows from Theorem 3.1 and Lemma 6.1. □

Corollary 6.3. *A ring has a completely (0-)simple GA-semigroup if and only if R is a (division) radical ring.*

Proof. Let $R^\diamond$ be a GA-semigroup of R. If $R^\diamond$ is completely 0-simple, then by [7, Theorem 2.14], $R^\bullet \simeq R^\diamond$ is completely 0-simple, and so R is a division ring. If $R^\diamond$ is completely simple, then $R^\diamond \simeq \mathcal{M}^\diamond_{11}(S, 0, U, V)$, where S is a radical ring by Theorem 6.2, and so $R \cong \mathcal{M}(S, 0, U, V)$ is a radical ring by Lemma 6.1 and [7, Theorem 2.12]. The sufficiency is clear. □

We conclude that

$$\begin{aligned} R^\bullet \text{ has the property } \mathbf{P} &\Rightarrow R^\diamond \text{ has the property } \mathbf{P} \\ &\Rightarrow R^\circ \text{ has the property } \mathbf{P}, \end{aligned}$$

where **P** stands for orthodox, right inverse, inverse, pseudoinverse, E-unitary, and completely simple, respectively.

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