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Characterizations of Reduced Polytopes in Finite-Dimensional Normed Spaces

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Abstract. A convex body R in a normed d-dimensional space M^d is called reduced if the M^d-thickness $\Delta(K)$ of each convex body $K \subset R$ different from R is smaller than $\Delta(R)$. We present two characterizations of reduced polytopes in M^d. One of them is that a convex polytope $P \subset M^d$ is reduced if and only if through every vertex v of P a hyperplane strictly supporting P passes such that the M^d-width of P in the perpendicular direction is $\Delta(P)$. Also two characterization of reduced simplices in M^d and a characterization of reduced polygons in M^2 are given.

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If every convex body properly contained in a convex body R of Euclidean space E^d has the thickness smaller than the thickness of R, we call R a *reduced* body. Results about reduced bodies were obtained successively in [8], [6], [4], [5], [14], [10], [13] and [11]. In [12] this notion is generalized for the d-dimensional real normed space M^d (called also d-dimensional Banach space or Minkowski space). A few properties of reduced bodies in M^d are established there. The present paper gives two characterizations of reduced polytopes in M^d and a characterization of reduced polygons in M^2.

The symbol $||\cdot||$ stands for the norm of the space M^d. As usual, we write bd for the boundary, and conv for the convex hull. By ab we denote the segment with endpoints a, b, and by $|ab|$ we denote the M^d-length $||b-a||$ of ab.

Let H_1 and H_2 be different parallel hyperplanes in M^d. Then $S = \mathrm{conv}(H_1 \cup H_2)$ is called a *strip*. If H_1 and H_2 are perpendicular (in Euclidean sense) to a direction m, then we say that S is a *strip of direction* m. We call H_1 and H_2 the bounding hyperplanes of S. If H_1 and H_2 support a convex body C, then S is called a *C-strip*. By the *M^d-width* $w(S)$ of S we have in mind the smallest value of $|h_1h_2|$ over all $h_1 \in H_1$ and $h_2 \in H_2$. If a point $x \in \mathrm{bd}(C)$ belongs to one of the bounding hyperplanes of a C-strip S, we say that S *passes through* x. The *C-strip of direction* m is denoted by $S(C, m)$. By the *M^d-width* $w(C, m)$ of a convex body $C \subset M^d$ in a direction m we mean $w(S(C, m))$. The number $\Delta(C) = \min_m w(C, m)$ is called the *M^d-thickness* of C. If a chord c_1c_2 of a convex body $C \subset M^d$ connects the opposite hyperplanes bounding a C-strip of M^d-width $\Delta(C)$ and if $|c_1c_2| = \Delta(C)$, we call c_1c_2 a *thickness chord of* C. Denote by $\Gamma(C, m)$ the M^d-length of a longest chord of C in direction m and put $\Gamma(C) = \min_m \Gamma(C, m)$.

Lemma 1. *For every convex body $C \subset M^d$ and every direction m we have $\Gamma(C, m) \leq w(C, m)$. For every convex body $C \subset M^d$ the equality $\Gamma(C) = \Delta(C)$ holds true.*

The proof of Lemma 1 for M^d is analogous to the short proof of the well known fact that Lemma 1 holds true in E^d: an easy proof of this fact can be found for instance in [15], pp. 157–158. Comp. also (1.5) of [7] and Theorem 3 of [1].

Let us recall the definition of a reduced body in M^d.

Definition. *A convex body $R \subset M^d$ is called reduced if $\Delta(K) < \Delta(R)$ for every convex body $K \subset R$ different from R.*

From Lemma 1 we obtain the following reformulation of the definition of a reduced body in terms of its chords (clearly, in particular, Claim holds true for reduced bodies in E^d).

Claim. *A convex body $R \subset M^d$ is reduced if and only if for every convex body $K \subset R$ different from R there is a direction m such that all chords of K in direction m are of M^d-length smaller than $\Gamma(R)$.*

For many problems concerning the M^d-thickness of convex bodies it is sufficient to consider only reduced bodies; still every convex body in M^d contains a reduced body of equal M^d-thickness. This is why our subject is also of applied nature. Such applications of reduced bodies in M^d are considered in [2] and [3].

We denote the body $\frac{1}{2}[C + (-C)]$ by C^*.

Lemma 2. *Let $C \subset M^d$ be a convex body. For every direction m we have $w(C^*, m) = w(C, m)$.*

Lemma 2 immediately follows from the well known fact that it holds true in E^d and from the definition of the M^d-width of C in direction m.

Lemma 3. *The M^d-thickness of every convex polytope $P \subset M^d$ is attained for a direction perpendicular to a facet of the polytope P^*.*

Proof. Since C^* is centrally-symmetric, from Lemma 2 we see that it is sufficient to show that the M^d-thickness of every centrally-symmetric convex polytope $Q \subset M^d$ is attained for a direction perpendicular to a facet of it. Let us show this. Take into account the largest ball B_Q of M^d contained in Q whose center is the center of Q. Of course, B_Q touches the boundary of Q at a point q, and q belongs to a facet F of Q. The hyperplane H containing F supports both Q and B_Q at q. The symmetric hyperplane also supports Q and B_Q. So the M^d-thickness of Q is attained for the direction perpendicular to H. □

Theorem 1. *A convex polytope $P \subset M^d$ is reduced if and only if through every vertex v of P a strip of M^d-width $\Delta(P)$ passes whose bounding hyperplane through v strictly supports P.*

Proof. Let $P \subset M^d$ be a convex polytope.

($\Rightarrow$) Assume that P is reduced. Consider a vertex v of P. There exists a sequence $H_1, H_2, \ldots$ of hyperplanes such that

(1) $H_1, H_2, \ldots$ are parallel,

(2) every H_i passes through an internal point of every edge of P whose endpoint is v,

(3) the distances between H_i and v tend to 0 as $i \to \infty$.

Let $i \in \{1, 2, \ldots\}$. The closed half-space bounded by H_i which does not contain v is denoted by H_i^+. Put $P_i = P \cap H_i^+$. Since P is reduced, $\Delta(P_i) < \Delta(P)$. By Lemma 3 there is a P_i-strip S_i of direction perpendicular to a facet of P_i^* such that $w(S_i) = \Delta(P_i)$. Of course, $v \notin S_i$. Denote by G_i the bounding hyperplane of S_i which separates v from P_i.

From (1) and (2) we deduce that between the faces of every two polytopes P_i and P_j from amongst $P_1, P_2, \ldots$ there is the following correspondence: for every k-dimensional face of P_i, a parallel k-dimensional face is in P_j. Consequently, for every facet of P_i^*, there is a parallel facet in P_j^*. This and the choice of $S_1, S_2, \ldots$ imply that between the directions perpendicular to the hyperplanes $G_1, G_2, \ldots$ there is only a finite number of different directions. This, the fact that $G_1, G_2, \ldots$ do not pass through v, and (3) lead to the conclusion that from the sequence $G_1, G_2, \ldots$ we can select a subsequence convergent to a hyperplane G which strictly supports P at v. Moreover, from (3) we deduce that $\lim_{i \to \infty} \Delta(P_i) = \Delta(P)$. Both these facts and $w(S_i) = \Delta(P_i)$ for $i \in \{1, 2, \ldots\}$ imply that the M^d-width of the P-strip bounded by G is equal to $\Delta(P)$.

($\Leftarrow$) Let $K \subset P$ be an arbitrary convex body different from P. Of course, there is a vertex v of P which does not belong to K. By the "only if" assumption of Theorem 1 there exists a P-strip of M^d-width equal to $\Delta(P)$ such that one of its bounding hyperplanes strictly supports P at v. Clearly, this hyperplane is disjoint with K. Consequently, $\Delta(K) < \Delta(P)$. The arbitrarines of K implies that P is reduced. □

The following Theorem 2 is formulated only in terms of chords of a polytope and of their directions. So it is often more intuitive and more convenient than Theorem 1 to apply for checking if a polytope is reduced.

Theorem 2. *A convex polytope $P \subset M^d$ is reduced if and only if for every vertex v of P there is a direction such that the longest chord of P in this direction is unique, has M^d-length $\Gamma(P)$ and v as its endpoint.*

Proof. Let $P \subset M^d$ be a convex polytope.

($\Rightarrow$) Assume that P is reduced and let v be a vertex of P. By Theorem 1 there is a strip S of M^d-width $\Delta(P)$ which passes through v such that its bounding hyperplane H_1 through v strictly supports P. Denote by H_2 the opposite bounding hyperplane of S. By Lemma 3 of [12] there is a thickness chord vw of P with $w \in H_2$. Since H_1 strictly supports P, there is no translate of vw in P. So vw is the unique chord of P of its own direction and of the M^d-length $\Delta(P)$. By Lemma 1 we have $\Delta(P) = \Gamma(P)$.

($\Leftarrow$) Let $K \subset P$ be an arbitrary convex body different from P. Of course, K does not contain a vertex v of P. By the "only if" assumption of Theorem 2 there exists a direction m and a chord vw of P of direction m and M^d-length $\Gamma(P)$ such that no translate of this chord is contained in P. Since moreover $v \notin K$, we conclude that K does not contain a chord of the direction m and of the M^d-length at least $|vw|$. Consequently, by Claim we see that the polytope P is reduced. $\square$

Theorems 1 and 2 are not true for a convex body C instead of a polytope P when we take an extreme point of C in the part of a vertex of P in their formulations. Just take into account the extreme point p of the reduced body $Z \subset E^3$ presented in Figure 2 of [10].

Example 1. Consider the space M^3 with the norm $\max_{i=1}^3 |x_i|$. Its unit ball B is a cube. Denote the successive vertices of the bottom base by a, b, c, d and the opposite vertices by a', b', c', d' (see Figure 1). Of course, B itself is a reduced body and $\Delta(B) = 2$. In Figures 1–3 we see polytopes T, A and D which are reduced. In each case this can be easily checked applying Theorem 1 or Theorem 2. We can also use the definition of a reduced body or the reformulated definition presented in Claim. For the tetrahedron $T = acb'd'$ we have $\Delta(T) = \frac{4}{3}$. More general, also the tetrahedron $acb''d''$ is reduced, where $b''d''$ is any "vertical" (i.e. parallel to ac') translation of $b'd'$ such that the M^3-length of the segment connecting the centers of ac and $b''d''$ is at least 1. For the pyramid A with base $abcd$ and vertex o we have $\Delta(A) = 1$. The bottom base of the right prism D is the convex hull of two intersecting segments: of a horizontal translate a_1c_1 of ac and of a horizontal translate b_2d_2 of bd. The top base $c_1'd_2'a_1'b_2'$ is a translate of the bottom base by 2 units "vertically" up. Of course, $\Delta(D) = 2$. A natural task is to describe all reduced bodies in the space with the norm $\max_{i=1}^d |x_i|$. Since its unit ball is a d-dimensional cube, from Corollary 1 of [12] we see that they are some polytopes with at most 2^d vertices.

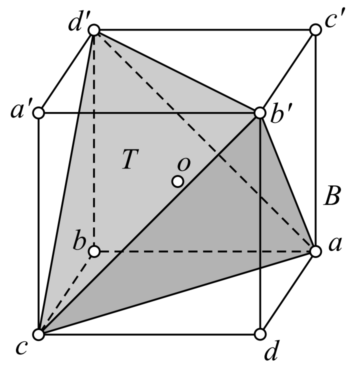

Figure 1

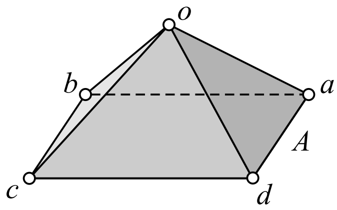

Figure 2

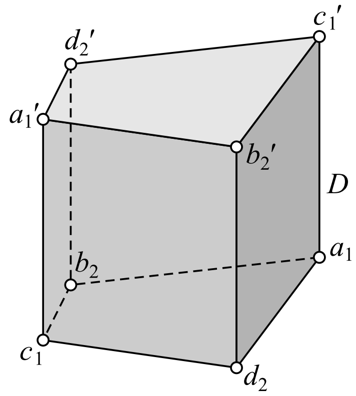

Figure 3

By the way, we let the reader to show that *if* R_i *is a reduced body in a normed space* $M_i^{d_i}$ *with the unit ball* B_i*, where* $i \in \{1, \dots, k\}$*, and if* $\Delta(R_1) = \dots = \Delta(R_k)$*, then* $\Pi_{i=1}^k R_i$ *is a reduced body in the space* $\Pi_{i=1}^k M_i^{d_i}$ *with the norm whose unit ball is* $\Pi_{i=1}^k B_i$*.*

Recall that the paper [9] asks if there exist reduced polytopes in E^d, where $d \geq 3$. In [14] and [13] it is proved that all simplices in every E^d, where $d \geq 3$, are not reduced. From [12] we also know that in every M^2 for every direction there exists a reduced triangle whose side is parallel to this direction. We see that a natural question is in which spaces M^d there exist reduced simplices.

Corollary 1. *A simplex* $S \subset M^d$ *is reduced if and only if* S^* *contains a concentric ball of* M^d *which touches all facets of* S^* *which are parallel to the facets of* S*.*

Proof. Assume that $S \subset M^d$ is a reduced simplex. By Theorem 1 through every vertex v_i of S, where $i \in \{0, \dots, d\}$, an S-strip of M^d-width $\Delta(S)$ passes whose one bounding hyperplane strictly supports S at v_i; the opposite hyperplane supports S at points (or at one point) of the opposite facet. Lemma 2 implies that S^* contains a concentric ball of M^3 touching all facets of S^* parallel to the facets of S.

Assume that S^* contains a concentric ball B_{S^*} of M^d touching all pairs of opposite facets F_i, F_i' of S^* parallel to the facets of S, where $i \in \{0, \dots, d\}$. For every $i \in \{0, \dots, d\}$, we provide parallel hyperplanes containing facets F_i and F_i'; the M^d-width of the strip between these hyperplanes is equal to $\Delta(B_{S^*}) = \Delta(S^*)$. From Lemma 2 we deduce that for every vertex of S an S-strip of M^d-width $\Delta(S^*) = \Delta(S)$ passes whose one bounding hyperplane strictly supports S at this vertex. By Theorem 1 the simplex S is reduced. □

In particular, Corollary 1 implies that the simplex T with vertices $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$ and $(-1, 1, -1)$ in the space M^3 with the norm $(\sum_{i=1}^3 |x_i|^p)^{1/p}$ (i.e. in l_3^p) is reduced if and only if $p \geq \frac{\ln 3}{\ln 1.5}$ (≈ 2.7095). We omit an easy calculation showing this. It is based on the fact that T^* is a cuboctahedron and that the (Euclidean) distance of opposite triangular facets of T^* is $\frac{2}{3}\sqrt{3}$ times the distance between opposite square facets.

Applying Theorem 1 and the fact that the strips of M^d-width $\Delta(S)$ constructed in the second part of the proof of Corollary 1 are parallel to the facets of S, we obtain the following

Corollary 2. *A simplex $S \subset M^d$ is reduced if and only if for every facet of S the M^d-width of S in the perpendicular direction is equal to $\Delta(S)$.*

Lemma 4. *For a planar convex body C consider two different C-strips $S(C, n_1)$ and $S(C, n_2)$. Let $x_i y_i$ be a segment contained in C and connecting the lines bounding $S(C, n_i)$, where $i = 1, 2$. Then the segments $x_1 y_1$ and $x_2 y_2$ intersect.*

If the oriented positive angle from a direction n_1 to a different direction n_2 in M^2 is smaller than π, then we write $n_1 \prec n_2$.

Corollary 3. *For every side of any reduced polygon $P \subset M^2$ the M^2-width in the perpendicular direction is equal to $\Delta(P)$.*

Proof. Let $t_1 t_2$ be a side of P, where t_2 is after t_1 when we go counterclockwise on $\mathrm{bd}(P)$ (see Figure 4).

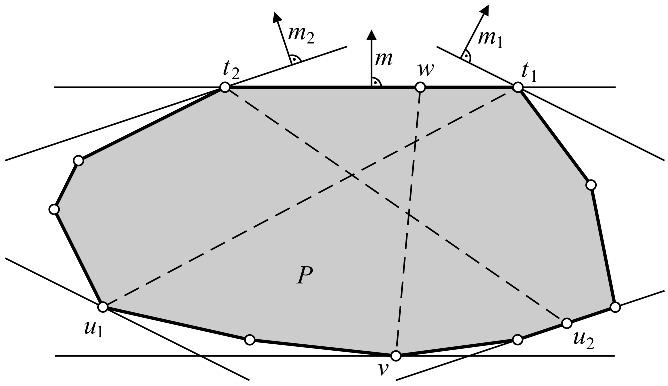

Figure 4

Consider the strip $S(P, m)$ whose bounding line contains $t_1 t_2$. By Theorem 1 through t_i a strip $S(P, m_i)$ of M^2-width $\Delta(P)$ passes whose bounding line through t_i strictly supports P, where $i \in \{1, 2\}$. Saying about directions m, m_1 and m_2 above we have in mind the outer directions of P perpendicular to the mentioned supporting lines. We have $m_1 \prec m \prec m_2$. Since $w(S(P, m_i)) = \Delta(P)$ and since both the bounding lines of $S(P, m_i)$ support P, by Lemma 3 of [12] we see that there is a thickness chord $t_i u_i$ of P such that u_i is in the other bounding line of $S(P, m_i)$ than the line containing t_i, where $i \in \{1, 2\}$. The bounding line of $S(P, m)$ not containing $t_1 t_2$ passes through a vertex v of P. Theorem 1 of [12] guarantees that v is an endpoint of a thickness chord vw of P.

From $m_1 \prec m \prec m_2$ and from Lemma 4 we conclude that when we go counterclockwise on $\mathrm{bd}(P)$, after u_1 we meet first v and next u_2 (in particular, it is possible that $u_1 = v$ or $v = u_2$). Hence, again from Lemma 4, the segment vw intersects both t_1u_1 and t_2u_2. This and $w \in \mathrm{bd}(P)$ imply $w \in t_1t_2$. So vw connects opposite lines bounding $S(P,m)$. Since vw is a thickness chord of P, we get $w(S(P,m)) = \Delta(P)$, which ends the proof. $\square$

The following Example 2 shows that Corollary 3 is not true in M^3 if in the part of sides of a polygon we have in mind facets of a polytope. It also shows that Corollaries 1 and 2 are not true for arbitrary convex polytopes in place of simplices.

Example 2. Let the unit ball B of M^3 be the convex hull of a regular octagon T and of a perpendicular segment wz whose centers coincide (see Figure 5). Take into account two different squares W and Z with vertices at the vertices of T. Translate Z in the direction parallel to wz such that the new position Z' of Z is in a distance from W larger than the distance between w and z. The polytope $P = \mathrm{conv}(W \cup Z')$ presented in Figure 6 is reduced. This easily follows from Theorem 1 or from Theorem 2. The width of P in the direction perpendicular to W is over $\Delta(P)$.

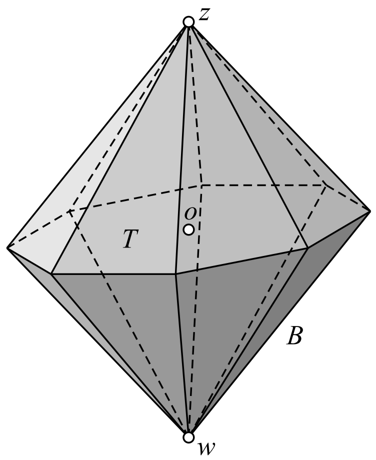

Figure 5

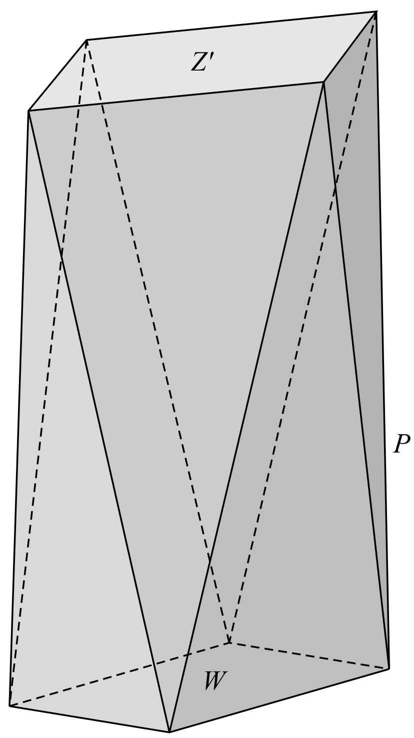

Figure 6

Proposition. *A convex polygon $P \subset M^2$ is reduced if and only if the polygon P^* is circumscribed about a ball of M^2.*

Proposition follows from Corollary 3. The reduced polytope P presented in Example 2 (see Figure 6) shows that Proposition does not hold true in higher dimensions.

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