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A Normed Space with the Beckman-Quarles Property

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Abstract. Beckman and Quarles proved that a unit distance preserving mapping from a Euclidean space $\mathbf{E}^n$ into itself is necessarily an isometry. In this paper, we give an example of a (non-strictly convex) normed space H for which every unit distance preserving function from H into itself is an isometry.

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1. Introduction

Beckman and Quarles [1] proved that a unit distance preserving mapping from a Euclidean space $\mathbf{E}^n$ into itself is necessarily an isometry. Since then, their result has been generalized and extended in various directions. See, for example, [2]–[10] for results and related questions. In this paper, we shall give an example of a (non-strictly convex) normed space H for which every unit distance preserving function from H into itself is an isometry.

Let C be a centrally symmetric convex set in $\mathbf{R}^n$. For this paper, we assume that C is bounded in the sense that there is some positive real number K such that for all $x = (x_1, \ldots, x_n)$ in C, we have $|x_i| \leq K$ for all $i = 1, 2, \ldots, n$, and that C is open in the usual topology of $\mathbf{R}^n$. Now, let

$$||p|| = \inf \{t > 0 : p \in tC\}$$

$$d(p, q) = ||p - q|| .$$

Then $||\cdots||$ is a norm, and d is a metric on $\mathbf{R}^n$, generating also the usual topology of $\mathbf{R}^n$, with the following properties:

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(1) d is translation invariant:

$$d\left(t_a\left(p\right), t_a\left(q\right)\right) = d\left(p, q\right)$$

for all $a \in \mathbf{R}^n$, where $t_a\left(z\right) = a + z$. In other words, if we use M_d to denote the metric space $\left(\mathbf{R}^n, d\right)$, then translations are isometries on M_d.

(2) The reflection σ about the origin is an isometry:

$$d\left(\sigma\left(p\right), \sigma\left(q\right)\right) = d\left(p, q\right)$$

for all p, q in M_d.

Examples of M_d include the Euclidean space $\mathbf{E}^n$ (C being the usual open unit ball defined by the inequality $x_1^2 + x_2^2 + \cdots + x_n^2 < 1$), and more generally, the l_p spaces (C being defined by $|x_1|^p + |x_2|^p + \cdots + |x_n|^p < 1$) for $1 \leq p$. We may also look at l_∞, where C is defined by $\max\left\{|x_1|, |x_2|, \ldots, |x_n|\right\} < 1$. For l_∞, there is a tiling of all of $\mathbf{R}^n$ by lattice translates of (the closure of) C. For $\mathbf{R}^2$, l_1 also has this property. Another d that has this property for $\mathbf{R}^2$ comes from taking $C = Q + (-Q)$, where Q is $\{(x, y) : x > 0,\ y > 0,\ x + y < 1\}$, the interior of the convex hull of $\{0, e_1, e_2\}$, where as usual, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Here C is the (interior of the) hexagon with vertices $\pm e_1, \pm e_2$, and $\pm\left(e_1 - e_2\right)$. This space will become our main object of study below, and we shall just refer it as H.

Now, according to the Beckman-Quarles theorem for the Euclidean plane $\mathbf{E}^n$, if $\varphi : \mathbf{E}^n \to \mathbf{E}^n$ is unit distance preserving, then φ is an isometry on $\mathbf{E}^2$. That is, condition

(1) for all $p, q \in \mathbf{E}^n$, $d\left(p, q\right) = 1 \Rightarrow d\left(\varphi\left(p\right), \varphi\left(q\right)\right) = 1$

implies condition

(2) for all $p, q \in \mathbf{E}^n$, $d\left(\varphi\left(p\right), \varphi\left(q\right)\right) = d\left(p, q\right)$.

We propose to say that M_d has the *Beckman-Quarles property* if every unit distance preserving map $\varphi : M_d \to M_d$ is an isometry on M_d.

It is immediate that not all M_d have the Beckman-Quarles property. The simplist example is l_∞ on $\mathbf{R}^n$. Indeed, the map $\varphi : l_\infty \to l_\infty$ defined by

$$\varphi\left(x_1, x_2, \ldots, x_n\right) = \left(\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor\right),$$

is unit distance preserving, but is not an isometry. (Here, $\lfloor x \rfloor$ is the floor of x, i.e., the largest integer not larger than x.) The situation for l_1 on $\mathbf{R}^2$ is clear, too.

These examples make one wonder if the existence or non-existence of tiling of $\mathbf{R}^n$ by translates of C plays a determining factor. In this regard, we note that H has the Beckman-Quarles property, and it is the purpose of this article to supply a prove for this fact.

2. Main result

H has the Beckman-Quarles property.

In this section, we let $\varphi : H \to H$ be a unit distance preserving mapping, and fix it for our discussion. That is, for $p, q \in H$,

$$d\left(p, q\right) = 1 \Rightarrow d\left(\varphi\left(p\right), \varphi\left(q\right)\right) = 1.$$

The following claims follow immediately. We omit their proofs.

Lemma 1. *For $p \in H$, let $S_p = \{q \in H : d(p,q) = 1\}$ denote the unit circle centered at p. The following hold.*

(1) $\varphi(S_p) \subseteq S_{\varphi(p)}$ *for all* $p \in H$.

(2) *If* p, q, r *are vertices of a unit equilateral triangle, i.e.,* $d(p,q) = d(q,r) = d(r,p) = 1$, *then so are* $\varphi(p), \varphi(q), \varphi(r)$.

(3) *For all* $p \in H$, S_p *does not contain all three vertices of any unit equilateral triangle.* □

For any $p \in H$, we consider the six "vertices" of S_p as follows:

$$p_1 = p + e_1, \quad p_2 = p + e_2, \quad p_3 = p - e_1 + e_2,$$
$$p_4 = p - e_1, \quad p_5 = p - e_2, \quad p_6 = p + e_1 - e_2.$$

We call p_i and p_{i+1} adjacent vertices of S_p, where we also take p_7 to mean p_1. (In the following discussion, we shall assume that indices are treated with modulo 6, unless otherwise stated.) In terms of this notation, we note the following facts. Again, we omit the proofs.

Lemma 2. *Let* $p \in H$, *and let* $i \in \{1, 2, \ldots, 6\}$. *Then*

(1) $p = (p_i)_{i+3} = (p_i)_{i-3}$.

(2) $p_i = (p_{i-1})_{i+1} = (p_{i+1})_{i-1}$.

(3) *Let* $m^{i+j} \in [p, p_{i+j}]$ *for* $j \in J$, *where* $J = \{1,3,5\}$ *or* $\{2,4,6\}$. *Then* $\{m^{i+j} : j \in J\}$ *is the set of vertices of a unit equilateral triangle if and only if* $m^{i+j} = (p + p_{i+j})/2$ *for all* $j \in J$. *(Here,* $[p,q] = \{\lambda p + (1-\lambda) q : 0 \leq \lambda \leq 1\}$.*)*

(4) S_p *is the union of six line segments:* $S_p = \cup_{i=1}^{6} [p_i, p_{i+1}]$. □

Now, we can start to investigate the properties of unit distance preserving maps $\varphi : H \to H$.

Lemma 3. *Let* $\varphi(p) = q$. *The following hold.*

(1) *For each* $i = 1, 2, \ldots, 6$, $\varphi(p_i)$ *is a vertex of* S_q, *i.e., there is some* $j \in \{1, 2, \ldots, 6\}$ *such that* $\varphi(p_i) = q_j$.

(2) *Adjacent vertices of* S_p *are mapped onto adjacent vertices of* S_q. *That is, if* $\varphi(p_i) = q_j$, *then* $\{\varphi(p_{i-1}), \varphi(p_{i+1})\} \subseteq \{q_{j-1}, q_{j+1}\}$.

(3) *There is a* $j \in \{1, 2, \ldots, 6\}$ *such that either* $\varphi(p_i) = q_{j+i}$ *for all* i, *or* $\varphi(p_i) = q_{j-i}$ *for all* i.

Proof. (1) Suppose that $r^i = \varphi(p_i)$ is not a vertex of S_q. Since $r^i \in S_q$, there must be some j such that r^i belongs to the relative interior of the line segment $[q_j, q_{j+1}]$. Thus, there is some $\lambda \in (0,1)$ such that

$$r^i = \lambda q_j + (1-\lambda) q_{j+1}.$$

To facilitate our later discussion, we write

$$r^{i+k} = \lambda q_{j+k} + (1-\lambda)\, q_{j+k+1},$$

for all k. Also, for the same reason, we let

$$m^k = \frac{p + p_k}{2}$$

for all k. Now, as

$$\{p_{i-1}, p_{i+1}\} = S(p) \cap S(p_i),$$

we have

$$\{\varphi(p_{i-1}), \varphi(p_{i+1})\} \subseteq S(q) \cap S\left(r^i\right) = \left\{r^{i-1}, r^{i+1}\right\}.$$

Suppose now that $\varphi(p_{i-1}) \neq \varphi(p_{i+1})$. Then $\{\varphi(p_{i-1}), \varphi(p_{i+1})\} = \{r^{i-1}, r^{i+1}\}$. As

$$[p, p_i] = S(p_{i-1}) \cap S(p_{i+1}),$$

we have

$$\varphi([p, p_i]) \subseteq S(\varphi(p_{i-1})) \cap S(\varphi(p_{i+1})) = S\left(r^{i-1}\right) \cap S\left(r^{i+1}\right) = \left\{q, r^i\right\}.$$

In particular, $\varphi(m^i) = q$ or r^i.

If $\varphi(m^i) = q$, then as

$$m^{i+2} \in S(p_{i+1}) \cap S\left(m^i\right), \text{ and } m^{i-2} \in S(p_{i-1}) \cap S\left(m^i\right),$$

we have

$$\varphi\left(m^{i+2}\right) \in S(\varphi(p_{i+1})) \cap S(q), \text{ and } \varphi\left(m^{i-2}\right) \in S(\varphi(p_{i-1})) \cap S(q).$$

In particular,

$$\begin{aligned}\left\{\varphi\left(m^{i+2}\right), \varphi\left(m^{i-2}\right)\right\} &\subseteq (S(\varphi(p_{i+1})) \cup S(\varphi(p_{i-1}))) \cap S(q) \\ &= \left(S\left(r^{i-1}\right) \cup S\left(r^{i+1}\right)\right) \cap S(q) \\ &= \left\{r^i, r^{i-2}, r^{i+2}\right\}.\end{aligned}$$

But this is impossible, as $d(m^{i+2}, m^{i-2}) = 1$ but none of $d(r^{i-2}, r^i)$, $d(r^{i-2}, r^{i+2})$, and $d(r^i, r^{i+2})$ is 1. A similar calculation shows that $\varphi(m^i) = r^i$ also leads to a contradiction.

Thus, we must have $\varphi(p_{i-1}) = \varphi(p_{i+1})$. The above argument may be repeated at the vertices p_{i-1} and p_{i+1} in place of p_i, and we conclude that $\varphi(p_i) = \varphi(p_{i+2}) = \varphi(p_{i+4}) = r_i$ and $\varphi(p_{i-1}) = \varphi(p_{i+1}) = \varphi(p_{i+3})$. But then as $m^{i+k+1} \in S(p_{i+k})$, we have $\varphi\left(m^{i+k+1}\right) \in S(\varphi(p_{i+k})) = S(\varphi(p_i))$ for $k = 0, 2, 4$. But then m^{i+1}, m^{i+3}, m^{i+5} are vertices of a unit equilateral triangle whose images under φ belong to the same $S(\varphi(p_i))$, which is impossible, in view of Lemma 1 (2) and (3).

(2) follows immediately from (1). Indeed, if $S(p_i) = q_j$, then as $\{p_{i-1}, p_{i+1}\} = S(p) \cap S(p_i)$, we have $\{\varphi(p_{i-1}), \varphi(p_{i+1})\} \subseteq S(q) \cap S(q_j) = \{q_{j-1}, q_{j+1}\}$.

(3) By (2), the sequence of points $\varphi(p_1), \varphi(p_2), \ldots, \varphi(p_6), \varphi(p_1)$ forms a "close chain" of adjacent vertices of $S(q)$. As we see in the last stage of the proof of (1), $\varphi(p_1), \varphi(p_3)$, and $\varphi(p_5)$ cannot be the same vertex of $S(q)$. Similarly, $\varphi(p_2), \varphi(p_4)$, and $\varphi(p_6)$ cannot be the same vertex of $S(q)$. It follows that the set $V = \{\varphi(p_1), \ldots, \varphi(p_6)\}$ contains at least four (consecutive) vertices of $S(q)$.

If V has exactly four elements, then there is some i and some j such that

$$\begin{array}{ll} \varphi(p_i) = q_j, & \varphi(p_{i+1}) = \varphi(p_{i-1}) = q_{j+1}, \\ \varphi(p_{i+2}) = \varphi(p_{i-2}) = q_{j+2}, & \varphi(p_{i+3}) = q_{j+3}. \end{array}$$

But then $m^{i+1}, m^{i+3} \in S(p_{i+2})$ implies that $\varphi(m^{i+1}), \varphi(m^{i+3}) \in S(q_{j+2})$. Similarly, $\varphi(m^{i-1}) \in S(\varphi(p_{i-2})) = S(q_{j+2})$ as well. But then the vertices $m^{i-1}, m^{i+1}, m^{i+3}$ of a unit equilateral triangle are mapped into the same $S(q_{j+2})$. This is impossible.

So, V has at least five elements. Thus, there are i and j such that

$$\begin{array}{rcl} \varphi(p_i) & = & q_j, \varphi(p_{i+1}) = q_{j+1}, \ldots, \varphi(p_{i+4}) = q_{j+4}, \text{ or} \\ \varphi(p_i) & = & q_j, \varphi(p_{i+1}) = q_{j-1}, \ldots, \varphi(p_{i+4}) = q_{j-4} \end{array}$$

But as $q_j, \ldots, q_{j+6}$ forms a "closed chain" of adjacent vertices, the first possibility implies $\varphi(p_{i+5}) = q_{j+5}$ and the second $\varphi(p_{i+5}) = q_{j-5}$. □

In view of Lemma 3 (3), we introduce the following definition. We say that p is a point of type k^+ if $\varphi(p_i) = (\varphi(p))_{k+i}$ for all i, and of type k^- if $\varphi(p_i) = (\varphi(p))_{k-i}$ for all i. Then Lemma 3 (3) implies that each point of H is of type k^+ or type k^- for some k.

Lemma 4. *Let $p \in H$ be of type k^+ (respectively, k^-). The following hold:*

(1) *For all i, p_i is of type k^+ (respectively, k^-). Hence, q is of type k^+ for all $q \in p + \mathbf{Z}^2$ (respectively, k^-).*

(2) *For all i, $\varphi([p, p_i]) \subseteq [\varphi(p), (\varphi(p))_{i+k}]$ (respectively, $\varphi([p, p_i]) \subseteq [\varphi(p), (\varphi(p))_{k-i}]$).*

(3) *For all i, $\varphi\left(\frac{p+p_i}{2}\right) = \frac{\varphi(p) + (\varphi(p))_{i+k}}{2}$ and $\varphi\left(\frac{p_i+p_{i+1}}{2}\right) = \frac{(\varphi(p))_{i+k} + (\varphi(p))_{i+k+1}}{2}$ (respectively, $\varphi\left(\frac{p+p_i}{2}\right) = \frac{\varphi(p) + (\varphi(p))_{k-i}}{2}$ and $\varphi\left(\frac{p_i+p_{i+1}}{2}\right) = \frac{(\varphi(p))_{k-i} + (\varphi(p))_{k-i-1}}{2}$).*

Proof. (1) If p is of type k^+, then using Lemma 2 (1) and (2), we get

$$\begin{array}{rcl} \varphi\left((p_i)_{i+3}\right) & = & \varphi(p) = \left[(\varphi(p))_{i+k}\right]_{i+k+3} = [\varphi(p_i)]_{k+i+3}, \\ \varphi\left((p_i)_{i+2}\right) & = & \varphi(p_{i+1}) = (\varphi(p))_{k+i+1} = \left[(\varphi(p))_{k+i}\right]_{k+i+2} = [\varphi(p_i)]_{k+i+2}. \end{array}$$

Thus, $\varphi\left((p_i)_j\right) = [\varphi(p_i)]_{k+j}$ for two consecutive integers $j = i+2$ and $i+3$. By Lemma 3 (3), we see that the same must be true for all integers j. Thus, p_i is of type k^+.

A similar argument works for the case when p is of type k^-. Simply observe that if for all i, $\varphi(p_i) = (\varphi(p))_{k-i}$, then

$$\begin{aligned}
\varphi\left((p_i)_{i+3}\right) &= \varphi(p) = \left[(\varphi(p))_{k-i}\right]_{k-i-3} = [\varphi(p_i)]_{k-(i+3)}, \\
\varphi\left((p_i)_{i+2}\right) &= \varphi(p_{i+1}) = (\varphi(p))_{k-i-1} = \left[(\varphi(p))_{k-i}\right]_{k-i-2} = [\varphi(p_i)]_{k-(i+2)}.
\end{aligned}$$

Now, the statement for $q \in p + \mathbf{Z}^2$ follows from an inductive application of the argument, each step from a point q in $p+\mathbf{Z}^2$ to its neighboring points q_i, $1 \le i \le 6$.
(2) Since $[p, p_i] = S(p_{i-1}) \cap S(p_{i+1})$, we have

$$\begin{aligned}
\varphi([p, p_i]) &\subseteq S(\varphi(p_{i-1})) \cap S(\varphi(p_{i+1})) \\
&= S\left([\varphi(p)]_{k+i-1}\right) \cap S\left([\varphi(p)]_{k+i+1}\right) \\
&= \left[p, (\varphi(p))_{k+i}\right].
\end{aligned}$$

The proof for type k^- points is similar.
(3) The points $\dfrac{p+p_j}{2}$, $j \in J$, are vertices of a unit equilateral triangle, where $J = \{1,3,5\}$ or $\{2,4,6\}$. The first result follows from Lemma 1 (2), Lemma 2 (3) and Lemma 4 (2). Next, using this, together with Lemma 2 (2) and Lemma 4 (1), we have

$$\begin{aligned}
\varphi\left(\frac{p_i + p_{i+1}}{2}\right) &= \varphi\left(\frac{p_i + (p_i)_{i+2}}{2}\right) \\
&= \frac{\varphi(p_i) + (\varphi(p_i))_{i+2+k}}{2} \\
&= \frac{[\varphi(p)]_{k+i} + \left[(\varphi(p))_{k+i}\right]_{k+i+2}}{2} \\
&= \frac{[\varphi(p)]_{k+i} + [\varphi(p)]_{k+i+1}}{2}.
\end{aligned}$$

The case for type k^- points is similar. □

Before we prove that a unit distance preserving function φ on H is necessarily an isometry, let us note the following isometries on H:

Let $\rho : H \to H$ be defined by

$$\rho(x, y) = (-y, x+y).$$

Then it is easy to check that ρ is linear, and for the origin $0 = (0,0)$, we have $\rho(0_i) = 0_{i+1}$ for $i = 1, 2, \ldots, 6$. But then if $p = (x, y)$ is any point in H, and $p \neq 0$, there is a unique $t > 0$ such that $tp \in S(0)$. Indeed, $t = 1/d(p, 0)$. Hence there is some j and some $\lambda \in [0,1]$ such that $tp = \lambda 0_j + (1-\lambda) 0_{j+1}$. Thus,

$$\rho(p) = \frac{\lambda}{t}\rho(0_j) + \frac{1-\lambda}{t}\rho(0_{j+1}) = \frac{1}{t}(\lambda 0_{j+1} + (1-\lambda) 0_{j+2}),$$

and so for the norm in H, $||\rho(p)|| = 1/t = ||p||$. Now, since ρ is linear and d is translation invariant, ρ must be an isometry.

It follows that powers ρ^k of ρ are isometries, $k \in \mathbf{Z}$. Note also that $\rho^{-k} = \rho^{6-k}$ for all k. Furthermore, for each k and each i, $\rho^k(0_i) = 0_{i+k}$.

It is straightforward to see that $\sigma : H \to H$, defined by $\sigma(x, y) = (y, x)$, is an isometry. Furthermore, $\sigma(0_i) = 0_{3-i}$ for each i. Then we obtain the isometries $\sigma\rho^k$, $k \in \mathbf{Z}$. Again, we note that for all k and i, $\sigma\rho^k(0_i) = \sigma(0_{i+k}) = 0_{3-i-k}$. As well, $\rho^k\sigma = \sigma\rho^{-k}$ for all k.

Now, we are ready to prove our main theorem.

Theorem. *φ is an isometry on H.*

Proof. By composing φ with a translation, the reflection σ, and an appropriate power of ρ if necessary, we may assume that $\varphi(0) = 0$, and $\varphi(0_i) = 0_i$ for all i. (Thus, 0 and hence all points of $\mathbf{Z}^2$ are of type 0^+.) It suffices to show that φ must be the identity map. This will not only show that all unit distance preserving mappings are isometries, but also delineate all isometries on H.

Since $\varphi(0_i) = 0_i$ for all i and 0 is of type 0^+, all points of $\mathbf{Z}^2$ are of type 0^+, and φ fixes all points of $\mathbf{Z}^2$. So, using Lemma 4 (2), we see that at each point p in $\mathbf{Z}^2$, φ maps $[p, p_i]$ into $[p, p_i]$, and each edge of $S(p)$ into itself. As well, by Lemma 4 (3), φ fixes all midpoints of the segments $[p, p_i]$, $1 \le i \le 6$, and all midpoints of the edges of $S(p)$. We find it convenient to express this in an alternative way. The above amounts to saying that

(A) φ maps every horizontal segment $[p, p + e_1]$ into itself, every vertical segment $[p, p + e_2]$ into itself, and every slant segment $[p, p + (e_1 - e_2)]$ into itself, where $p \in \mathbf{Z}^2$, and

(B) φ fixes all points of $\left(\frac{1}{2}\mathbf{Z}\right)^2$.

Now, we use induction. Suppose that for some positive integer n, we have

(A_n) φ maps every horizontal segment $\left[p, p + \frac{e_1}{2^{n-1}}\right]$ into itself, every vertical segment $\left[p, p + \frac{e_2}{2^{n-1}}\right]$ into itself, and every slant segment $\left[p, p + \frac{e_1 - e_2}{2^{n-1}}\right]$ into itself, where $p \in \left(\frac{1}{2^{n-1}}\mathbf{Z}\right)^2$, and

(B_n) φ fixes all points of $\left(\frac{1}{2^n}\mathbf{Z}\right)^2$.

Now, let $p \in \left(\frac{1}{2^n}\mathbf{Z}\right)^2$, and consider the line segments $\left[p, p + \frac{e_1}{2^n}\right]$, $\left[p, p + \frac{e_2}{2^n}\right]$, and $\left[p, p + \frac{e_1 - e_2}{2^n}\right]$. The points $u = p - \left(1 - \frac{1}{2^n}\right)e_1 + e_2$ and $v = p + e_1 - e_2$ are in $\left(\frac{1}{2^n}\mathbf{Z}\right)^2$, and hence by ($\mathrm{B}_n$), are fixed by φ. Hence, as

$$\left[p, p + \frac{e_1}{2^n}\right] = S(u) \cap S(v),$$

we have

$$\varphi\left(\left[p, p + \frac{e_1}{2^n}\right]\right) \subseteq S(\varphi(u)) \cap S(\varphi(v)) = S(u) \cap S(v) = \left[p, p + \frac{e_1}{2^n}\right].$$

So, φ maps the horizontal line segment $\left[p, p + \frac{e_1}{2^n}\right]$ into itself. The horizontal and slant segments can be treated in a similar manner, and this establishes (A_{n+1}).

Next, let p be a point of $\left(\frac{1}{2^{n+1}}\mathbf{Z}\right)^2$. We want to show that φ fixes p. In view of (B_n), we may assume that $p \notin \left(\frac{1}{2^n}\mathbf{Z}\right)^2$. First, we consider the case $p = \left(\frac{h}{2^{n+1}}, \frac{k}{2^{n+1}}\right)$, where h is odd, and k is even. Let $p' = \left(\frac{h-1}{2^{n+1}} + 1, \frac{k+1}{2^{n+1}}\right)$, $q = \left(\frac{h-1}{2^{n+1}}, \frac{k}{2^{n+1}} + 1\right)$, $q' = \left(\frac{h}{2^{n+1}} + 1, \frac{k+1}{2^{n+1}} - 1\right)$, and $r = \left(\frac{h-1}{2^{n+1}} + 1, \frac{k+2}{2^{n+1}} - 1\right)$. Then q and r are in $\left(\frac{1}{2^n}\mathbf{Z}\right)^2$ and so q and r are fixed by φ. Also, $q' = \frac{1}{2}\left(r + \left(r + \frac{e_1-e_2}{2^n}\right)\right) \in \left[r, r + \frac{e_1-e_2}{2^n}\right]$. By ($\mathrm{A}_{n+1}$), we see that $\varphi(q') \in \left[r, r + \frac{e_1-e_2}{2^n}\right]$. Similarly, we see that $\varphi(p) \in \left[p - \frac{e_1}{2^{n+1}}, p + \frac{e_1}{2^{n+1}}\right]$, and $\varphi(p') \in \left[p' - \frac{e_2}{2^{n+1}}, p' + \frac{e_2}{2^{n+1}}\right]$. So, if $\varphi(p) = \lambda\left(p - \frac{e_1}{2^{n+1}}\right) + (1-\lambda)\left(p + \frac{e_1}{2^{n+1}}\right)$, where $\lambda \in [0,1]$, then as $p' \in S(p) \cap S(q)$, we have $\varphi(p') \in S(\varphi(p)) \cap S(\varphi(q)) = S(\varphi(p)) \cap S(q)$, and this forces $\varphi(p') = \lambda\left(p' - \frac{e_2}{2^{n+1}}\right) + (1-\lambda)\left(p' + \frac{e_2}{2^{n+1}}\right)$. But then as $q' \in S(p) \cap S(p')$, we have $\varphi(q') \in S(\varphi(p)) \cap S(\varphi(p'))$. But $\varphi(q') \in \left[r, r + \frac{e_1-e_2}{2^n}\right]$, too. A simple calculation shows that this is possible only if $\lambda = 1/2$. Thus, $\varphi(p) = p$, i.e., φ fixes p. It is clear that the case for h being even and k being odd is similar. Second, suppose that both h and k are odd. Then $p \in \left[s, s + \frac{e_1-e_2}{2^n}\right]$, where $s = \left(\frac{h-1}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right) \in \left(\frac{1}{2^n}\mathbf{Z}\right)^2$. Thus, $\varphi(p) \in \left[s, s + \frac{e_1-e_2}{2^n}\right]$, by ($\mathrm{A}_{n+1}$). Also, $S(p)$ intersects some certain horizontal line segment $\left[t, t + \frac{e_1}{2^n}\right]$ at its midpoint m, and some vertical line segment $\left[t', t' + \frac{e_2}{2^n}\right]$ at its midpoint m'. By what we have just proved, m and m' are both fixed by φ. This is possible only if $\varphi(p) = p$. This proves (B_{n+1}), and so it completes the induction. So, (A_n) and (B_n) are true for all positive integers n.

Now, if $p = \left(x, \frac{k}{2^m}\right)$ is any point in $\mathbf{R} \times \left(\frac{1}{2^m}\mathbf{Z}\right)$ for some positive integer m, then by considering a shrinking segment of the form $\left[\frac{h}{2^n}, \frac{h+1}{2^n}\right] \times \left\{\frac{k}{2^m}\right\}$, $n \to \infty$, each containing p, we see that φ must also fix p. Likewise, φ fixes all points in $\left(\frac{1}{2^m}\mathbf{Z}\right) \times \mathbf{R}$, for every m. Finally, if p is any general point of H, then there exist (distinct) points $p' \in \mathbf{R} \times \left(\frac{1}{2^m}\mathbf{Z}\right)$ and $p'' \in \left(\frac{1}{2^n}\mathbf{Z}\right) \times \mathbf{R}$ such that $p \in S(p') \cap S(p'')$. But then $\varphi(p) \in S(p') \cap S(p'')$. As there are infinitely many choices of p' and p'', we conclude that $\varphi(p) = p$. □

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