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On Groups with Root System of Type ${}^2F_4$

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Abstract. Let $\tilde{\Phi}$ be a root system of type ${}^2F_4$, and let G be a group generated by non-trivial subgroups A_r, $r \in \tilde{\Phi}$, satisfying some generalized Steinberg relations, which are also satisfied by root subgroups corresponding to a Moufang octagon. These relations can be considered as a generalization of the element-wise commutator relations in Chevalley groups. The Steinberg presentation specifies the groups satisfying the Chevalley commutator relations. In the present paper some sort of generalized Steinberg presentation for groups with root system of type ${}^2F_4$ is provided. As a main result we classify the possible structures of G.

1. Introduction

Let $\mathcal{B}$ be an irreducible, spherical Moufang building of rank $l \geq 2$, $\mathcal{A}$ an apartment of $\mathcal{B}$ and Φ the set of roots of $\mathcal{A}$. Further, for $r \in \Phi$ let A_r be the root subgroup of $\mathrm{Aut}(\mathcal{B})$ in the sense of Tits (see [9, I,4 and II,5]). We call $G := \langle A_r \mid r \in \Phi\rangle$ ethe *Lie-type group of* $\mathcal{B}$, where $\mathrm{Aut}(\mathcal{B})$ denotes the group of type preserving automorphisms of $\mathcal{B}$. Using the geometric realization of the Coxeter group of $\mathcal{A}$ (see [9, I (4.6) and p. 125–126]), one can identify Φ with a root system of type A_l, B_l, C_l ($l \geq 2$), D_l ($l \geq 4$), E_l ($6 \leq l \leq 8$), F_4 in the sense of Humphreys [2] or a Coxeter system of type $I_2(m)$. Moreover, we have $m \in \{3, 4, 6, 8\}$ by [15] and [18], if $\mathcal{B}$ is of type $I_2(m)$. We stress that there is no Moufang building of type H_3 respectively H_4 by [14, (3.7)] (see also [13, p. 275]). Furthermore, Φ can be extended to a possibly non-reduced root system $\tilde{\Phi}$, and new "root subgroups" A_r

can be introduced as subgroups of the "original" ones for $r \in \tilde{\Phi} \setminus \Phi$. For $\widetilde{C}_l = BC_l$ see [1, p. 233]. Further, for the definition of $^2F_4$ see [16]. For $^2F_4$ we use the notation $\tilde{\Phi} = {}^2F_4$ and set $\Phi_1 := \{a \in \tilde{\Phi} | a/(1+\sqrt{2}) \in \tilde{\Phi}\}$ and then $\Phi := \tilde{\Phi} \setminus \Phi_1$.

We set

$$R := \begin{cases} \mathbb{N} \cup \{0\} + (\mathbb{N} \cup \{0\})\sqrt{2} & \text{for } \tilde{\Phi} = {}^2F_4 \\ \mathbb{N} & \text{otherwise} \end{cases}$$

and

$$\widehat{\Phi} := \begin{cases} \Phi & \text{for } \tilde{\Phi} = {}^2F_4 \\ \tilde{\Phi} & \text{otherwise.} \end{cases}$$

Further, we denote the reflection along r on $\tilde{\Phi}$ by w_r for $r \in \widehat{\Phi}$. Then the following hold:

(I) $[A_r, A_s] \leq \left\langle A_{\lambda r + \mu s} \mid \lambda r + \mu s \in \tilde{\Phi}, \lambda, \mu \in R, \lambda > 0, \mu > 0 \right\rangle$ for $r, s \in \tilde{\Phi}$ with $r \notin \mathbb{R}^- \cdot s$. (See [8, (3.3)] and [17, (6)].)

(II) $X_r := \langle A_r, A_{-r} \rangle$ is a rank one group with unipotent subgroups A_r and A_{-r} for $r \in \widehat{\Phi}$. (See [9, I (4.12)(3)] and [8, (3.2)].)

(III) Let $r \in \widehat{\Phi}$ and $n_r \in X_r$ with $A_r^{n_r} = A_{-r}$ respectively $A_{-r}^{n_r} = A_r$. Then $A_s^{n_r} = A_{s^{w_r}}$ for $s \in \tilde{\Phi}$. (See [9, II (5.11)], [17, (6)] respectively [16, (1.4)].) (The existence of n_r is guaranteed by 2.1(4).)

Let G be an arbitrary group generated by subgroups A_r, $r \in \tilde{\Phi}$, where $\tilde{\Phi}$ is a root system of type A_l, B_l, C_l, BC_l ($l \geq 2$), D_l ($l \geq 4$), E_l ($6 \leq l \leq 8$), F_4, G_2 or $^2F_4$. Suppose the A_r satisfy (I)–(III). Further, assume that A_{2r} is a subgroup of A_r, if $2r \in \tilde{\Phi}$ for $r \in \tilde{\Phi}$, respectively $A_{(\sqrt{2}+1)r} \leq A_r$, if $(\sqrt{2}+1)r \in \tilde{\Phi}$ for $r \in \tilde{\Phi}$. Then it has been proved in [10, Theorem 1] that there exists an irreducible, spherical Moufang building $\mathcal{B}$ with "extended" root system $\tilde{\Phi}$ and there is a surjective homomorphism $\sigma\colon G \to \overline{G}$, where $\overline{G}$ is a Lie-type group of $\mathcal{B}$, such that the A_r with $r \neq 2s$ and $r \neq (\sqrt{2}+1)s$ for all $s \in \tilde{\Phi}$ are mapped onto the root subgroups of $\overline{G}$ corresponding to some apartment of $\mathcal{B}$ and $\ker \sigma \leq Z(G)$. In this situation we call G a *group of type* $\mathcal{B}$ or $\tilde{\Phi}$. We mention that the assumptions of [10, Theorem 1] are not satisfied by $\tilde{\Phi} = {}^2F_4$, but the assertion holds in this case, too, since in the proof it has been made use of the condition, that X_α is a rank one group, only for $\alpha \in \Phi$. Now, Timmesfeld's aim was to determine the structure of groups satisfying (I) and (II). This can be considered as a generalization of the Steinberg presentation of Chevalley groups. For a survey of his research work we refer to [12, Introduction]. Before we state his main result [12, Theorem 1], we establish some notation. Let G be a group, $\{G_i \mid 1 \leq i \leq n\}$ a set of subgroups of G with $G = \langle G_i \mid 1 \leq i \leq n \rangle$ and $[G_i, G_j] = 1$ for $i \neq j$. Then we call G a *central product* of the subgroups G_i, $1 \leq i \leq n$, and use the notation $G = \ast_{i=1}^{n} G_i$. Further, we call the G_i *central divisor*s of G.

Theorem 1.1. *Let $\tilde{\Phi}$ be a root system of type A_l, B_l, C_l, BC_l ($l \geq 2$), D_l ($l \geq 4$), E_l ($6 \leq l \leq 8$) or F_4. Further, let G be a group generated by subgroups A_r, $r \in \tilde{\Phi}$, satisfying* (I) *and* (II). *Let*

$$\Psi = \{r \in \tilde{\Phi} \mid 2r \notin \tilde{\Phi}\} \cup \{s \in \tilde{\Phi} \mid 2s \in \tilde{\Phi} \text{ and } A_s \neq A_{2s}\}.$$

Then $\Psi = \dot{\cup}\, \Psi_i$, $i \in I$, *such that the following hold:*

1. Ψ_i *carries the structure of a root system of one of the types* A_n, B_n, C_n, BC_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(6 \leq n \leq 8)$ *or* F_4 *or* $\Psi_i = \{\pm\alpha\}$ *respectively* $\Psi_i = \{\pm\alpha, \pm 2\alpha\}$ *for some* $\alpha \in \Psi$. *Moreover, if* Ψ_i *is of type* E_n, *then* $\Psi = \tilde{\Phi}$ *is of type* E_l *and* $n \leq l$.
2. *Let* $G(\Psi_i) := \langle A_r \mid r \in \Psi_i \rangle$. *Then* G *is the central product of the* $G(\Psi_i)$ *and either* $G(\Psi_i) = X_\alpha$ *(if* $\Psi_i = \{\pm\alpha\}$ *respectively* $\Psi_i = \{\pm\alpha, \pm 2\alpha\}$*) or there exists a Moufang building* $\mathcal{B}_i$ *with "extended" root system* Ψ_i *such that* $G(\Psi_i)$ *is of type* $\mathcal{B}_i$.

A result for $\tilde{\Phi} = G_2$, analogous to 1.1, has been proved in [7]. In the present paper we solve the remaining problem when $\tilde{\Phi}$ is of type $^2F_4$, i.e. $\tilde{\Phi} = \{\pm r_{2i}, \pm r_{2i-1}, \pm r_{(2i-1)'} \mid i \in \{1,2,3,4\}\}$. To simplify notation, we write $\pm k$ instead of $\pm r_k$ respectively $\pm k'$ instead of $\pm r_{k'}$.

Theorem 1.2. *Suppose* G *is generated by non-trivial subgroups* A_α, $\alpha \in \tilde{\Phi}$, *satisfying* (I) *and* (II) *with* $\tilde{\Phi}$ *of type* $^2F_4$. *Let* $J := \{\pm 2i \mid i \in \{1,2,3,4\}\}$. *Moreover, assume that* $A_{\alpha'}$ *is a subgroup of* A_α *for* $\alpha \in \Phi \setminus J$. *Then one of the following holds:*

(A) G *is of type* $^2F_4$.

(B) $G = X_\alpha * C_G(X_\alpha)$ *for some* $\alpha \in \Phi$ *and* $X_\beta \leq C_G(X_\alpha)$ *for* $\beta \in \Phi \setminus \{\pm\alpha\}$ *or* $G = G(J) * G(\tilde{\Phi} \setminus J)$ *is of type* $C_2 \times C_2$.

2. Preliminaries

In this section we summarize preliminaries which are relevant to the proof of 1.2. Regarding commutators we use the notation of [3]. We will often use the Dedekind identity in the following slightly modified sense: Let G be a group, $X \leq G$, $1 \in U \subseteq X$ and $1 \in A \subseteq G$. Then $U(A \cap X) = UA \cap X$. Rank one groups have been introduced by Timmesfeld. A group X generated by two different nilpotent subgroups A and B satisfying: for each $a \in A^\sharp$ there exists a $b \in B^\sharp$ with $A^b = B^a$ and vice versa, is called a *rank one group*. We call the conjugates of A (and B) *unipotent subgroups* of the rank one group X. For the convenience of the reader, we will in the following collect some properties of rank one groups which are needed for the proof of 1.2. Proofs of these properties are given in [9, Chapter I].

Theorem 2.1. *Let* $X = \langle A, B \rangle$ *be a rank one group with unipotent subgroups* A *and* B.

(1) *Let* $\sigma\colon X \to \sigma(X)$ *be a homomorphism with* $\sigma(A) \neq \sigma(B)$. *Then*

$$\sigma(X) = \langle \sigma(A), \sigma(B) \rangle$$

is a rank one group with unipotent subgroups $\sigma(A)$ *and* $\sigma(B)$.

(2) *We have* $N_A(B) = 1 = N_B(A)$.

(3) *For* $C, D \in A^X$ *with* $C \neq D$ *and* $d \in D^\sharp$ *we have* $X = \langle C, D\rangle = \langle C, d\rangle$.

(4) X *acts doubly transitively on the set* A^X. *In particular, there exists an* $x \in X$ *with* $A^x = B$ *and* $B^x = A$. *(We use the notation* $A \overset{x}{\longleftrightarrow} B$ *for this.)*

(5) $\langle a^X \rangle$ *is not nilpotent for* $a \in A^\sharp$. *In particular,* X *is not nilpotent.*

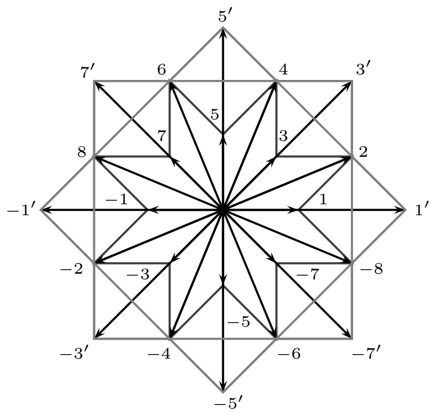
(6) *Suppose* X *acts on the group* M *such that* A *or* B *acts trivially on* M. *Then* X *acts trivially on* M.

(7) *Suppose* A *and* B *are elementary Abelian* p*-groups for some prime* p, *and* A *acts on a* $\mathbb{Z}X$*-module, say* V, *with* $V = [V, A] \oplus [V, B]$. *Then* V *is an elementary Abelian* p*-group.*

In the following four sections we will prove Theorem 1.2. The proof will mainly consist of extensive commutator calculations combined with applications of the theory of rank one groups.

3. Notation and basic results

To begin with, we introduce some notation.
Let $\tilde{\Phi} = \{\pm r_{2i}, \pm r_{2i-1}, \pm r_{(2i-1)'} \mid i \in \{1, 2, 3, 4\}\}$ be a root system of type $^2F_4$.



Assume G is generated by non-trivial subgroups A_α, $\alpha \in \tilde{\Phi}$, satisfying (I) and (II) with $\tilde{\Phi} = {}^2F_4$. Further, suppose that $A_{\alpha'}$ is a subgroup of A_α for $\alpha \in \Phi \setminus J$. The set $J = \{\pm 2i \mid i \in \{1, 2, 3, 4\}\}$ is a root subsystem of $\tilde{\Phi}$ of type C_2. For $\alpha \in \Phi$ let

$$U_\alpha := \langle A_\beta \mid \beta \in \Phi, \alpha < \beta < -\alpha \rangle,$$

where $\alpha < \beta < -\alpha$ means that β is between α and $-\alpha$ clockwise. We notice that the commutator relations in (I) provide the identity $U_\alpha = \prod_{\alpha<\beta<-\alpha} A_\beta$, where the roots are ordered "from α to $-\alpha$". Notice that 2.1(4) guarantees the existence of $n_\alpha \in X_\alpha$ with $A_\alpha \overset{n_\alpha}{\longleftrightarrow} A_{-\alpha}$ for each $\alpha \in \widehat{\Phi}$.

The next lemma is a direct consequence of (I) and (II).

Lemma 3.1. *Let* $\alpha \in \Phi$. *Then:*

(1) *U_α and $U_{-\alpha}$ are X_α-invariant.*
(2) *$A_\alpha U_\alpha$ and $A_{-\alpha}U_\alpha$ are nilpotent.*
(3) *$A_\alpha \cap U_\alpha = 1 = A_{-\alpha} \cap U_\alpha$. In particular, $A_\beta \cap A_\gamma = 1$ for $\beta \in \tilde{\Phi}$ and $\tilde{\Phi} \ni \gamma \notin \mathbb{R}^+ \cdot \beta$.*

Proof. Transferring the proof of [10, (2.1)] to $^2F_4$, the result follows. □

Nilpotence argument 3.2. *Let $\alpha, \gamma \in \Phi$ and $\beta \in \tilde{\Phi}$ with $\alpha < \beta \neq \gamma < -\alpha$ and $[X_\gamma, X_\alpha] = 1$. Further, let $\widehat{A}_\beta$ be a non-empty subset of A_β with $\widehat{A}_\beta^{n_\alpha} \subseteq A_\gamma U_\gamma$. Then $\widehat{A}_\beta^{n_\alpha} \subseteq U_\gamma$.*

Proof. Transfering the proof of [7, (3.3)] to $^2F_4$, the claimed result follows. □

Remark 3.3. We can conclude the analogous assertion for $\widehat{A}_\beta^{n_\alpha} \leq A_\gamma U_{-\gamma}$.

Lemma 3.4. *Suppose there exist some roots $\alpha, \beta \in \widehat{\Phi}$ with $[X_\alpha, A_\beta] = 1$. Then there is no $a_{-\beta} \in A_{-\beta}^\sharp$ with $a_{-\beta}^{n_\alpha} \in U_\beta$ or $a_{-\beta}^{n_\alpha} \in U_{-\beta}$.*

Proof. Without loss, suppose there exists an $a_{-\beta} \in A_{-\beta}^\sharp$ with $a_{-\beta}^{n_\alpha} \in U_\beta$. Then we have $X_\beta^{n_\alpha} = \langle A_\beta, a_{-\beta}\rangle^{n_\alpha} \leq \langle A_\beta, U_\beta\rangle = A_\beta U_\beta$, a contradiction to 3.1(2) and 2.1(5). □

Let $\alpha \in J$ and $\beta', \gamma', \delta', \varepsilon' \in \tilde{\Phi} \setminus \Phi$ with $\alpha < \beta' < \gamma' < \delta' < \varepsilon' < -\alpha$. Then we set

$$W_\alpha := A_{\gamma'} A_{\delta'}. \tag{3.4.1}$$

For example, $W_{-2} = A_{7'} A_{5'}$. The group W_α lies in the center of U_α and is X_α-invariant by (I). Let $\alpha \in \Phi \setminus J$, $\beta', \delta', \eta' \in \tilde{\Phi} \setminus \Phi$ and $\gamma, \varepsilon \in J$ with $\alpha < \beta' < \gamma < \delta' < \varepsilon < \eta' < -\alpha$. Then we set

$$M_\alpha := A_{\beta'} A_\gamma A_{\delta'} A_\varepsilon A_{\eta'}. \tag{3.4.2}$$

For example, $M_{-1} = A_{7'} A_6 A_{5'} A_4 A_{3'}$. We notice that M_α is an Abelian, X_α-invariant subgroup of U_α.

Lemma 3.5. *For $\alpha \in \Phi \setminus J$ let $M_\alpha = A_{\beta'} A_\gamma A_{\delta'} A_\varepsilon A_{\eta'}$ with $\beta', \delta', \eta' \in \tilde{\Phi} \setminus \Phi$, $\gamma, \varepsilon \in J$ and $\alpha < \beta' < \gamma < \delta' < \varepsilon < \eta' < -\alpha$. Suppose $A_{\beta'} A_{\delta'}$ (respectively $A_{\delta'} A_{\eta'}$) is X_α-invariant. Then $[X_\alpha, A_{\beta'}] = 1$ (respectively $[X_\alpha, A_{\eta'}] = 1$).*

Proof. Without loss, let $\alpha = -1$. Suppose $A_{7'} A_{5'}$ is X_1-invariant. Then, using the nilpotence argument, we obtain $A_{7'}^{n_1} = A_{7'}$ and so $[X_1, A_{7'}] = 1$, since $[A_{7'}, A_{-1}] = 1$ by (I). So the claimed result follows by symmetry. □

Lemma 3.6. *Let M_α be as in* (3.4.2).

(1) *Suppose* $[X_\alpha, A_{\beta'}] = 1$. *Then* $[A_\alpha, A_\gamma] = 1$ *and* $[A_{\alpha'}, A_\varepsilon] = 1$. *If further* $[A_\gamma, A_{-\alpha'}] = 1$, *then* $[X_\alpha, M_\alpha] = 1$.

(2) *Suppose* $[X_\alpha, A_{\eta'}] = 1$. *Then* $[A_{-\alpha}, A_\varepsilon] = 1$ *and* $[A_{-\alpha'}, A_\gamma] = 1$. *If further* $[A_\varepsilon, A_{\alpha'}] = 1$, *then* $[X_\alpha, M_\alpha] = 1$.

Proof. Without loss, let $\alpha = -1$. Suppose

$$[X_1, A_{7'}] = 1. \tag{3.6.1}$$

This yields $[A_{-1}, A_6]^{n_1} \le A_{7'}^{n_1} \cap [A_1, M_{-1}] \le A_{7'} \cap U_7 = 1$ by (I) and 3.1(3). Thus,

$$[A_{-1}, A_6] = 1. \tag{3.6.2}$$

Moreover, $[A_{-1'}, A_4]^{n_1} \le A_{7'}^{n_1} \cap [A_1, M_{-1}] \le A_{7'} \cap U_7 = 1$ by (I) and 3.1(3), as $A_{7'}^{n_1} = A_{7'}$ by assumption. This means

$$[A_{-1'}, A_4] = 1. \tag{3.6.3}$$

If further,

$$[A_6, A_{1'}] = 1, \tag{3.6.4}$$

then we have $X_1 = \langle A_{-1}, A_{1'} \rangle \le C(A_6)$ by (3.6.2). This yields

$$\begin{aligned}[A_4, A_{-1}]^{n_1} &\le (A_{7'}A_6A_{5'})^{n_1} \cap [M_{-1}, A_1] = A_{7'}A_6A_{5'} \cap [A_{7'}A_6A_{5'}A_4A_{3'}, A_1] \\ &= A_{7'}A_6A_{5'} \cap [A_4, A_1] \le A_{7'}A_6A_{5'} \cap A_{3'} \le U_{-3} \cap A_{3'} = 1\end{aligned}$$

by (3.6.1), (I) and 3.1(3). This implies $X_1 = \langle A_{-1}, A_{1'} \rangle \le C(A_4)$, as $[A_4, A_{1'}] = 1$ by (I). So we have $[M_{-1}, A_{1'}] = 1$ by (I), (3.6.1) and (3.6.4). This shows $[M_{-1}, A_{1'}^{n_1}] = 1$. As $(1 \ne) A_{1'}^{n_1} \le A_{-1}$, we get $X_1 = \langle A_{1'}^{n_1}, A_1 \rangle \le C(A_{3'})$, since $[A_{3'}, A_1] = 1$ by (I). Hence, we obtain $[X_1, M_{-1}] = 1$, as required. So the claimed result follows by symmetry. □

Lemma 3.7. *Let* $U_\alpha = A_\beta A_\gamma A_\delta A_\varepsilon A_\eta A_\kappa A_\mu$ *with* $\alpha, \gamma, \varepsilon, \kappa \in J$; $\beta, \delta, \eta, \mu \in \Phi \setminus J$ *and* $\alpha < \beta < \gamma < \delta < \varepsilon < \eta < \kappa < \mu < -\alpha$.

(1) *Suppose* $[X_\alpha, A_\beta] = 1$. *Then we have* $[A_\alpha, A_\delta] = 1$. *If moreover* $G(J) = \ast_{\tau \in J} X_\tau$ *and* $[W_\alpha, X_\alpha] = 1$, *then* $A_\beta A_\gamma A_\delta A_\varepsilon A_\eta A_\kappa \subseteq C_{U_\alpha}(A_\alpha)$.

(2) *Suppose* $[X_\alpha, A_\mu] = 1$. *Then* $[A_\eta, A_{-\alpha}] = 1$. *If furthermore* $G(J) = \ast_{\tau \in J} X_\tau$ *and* $[W_\alpha, X_\alpha] = 1$, *then* $A_\gamma A_\delta A_\varepsilon A_\eta A_\kappa A_\mu \subseteq C_{U_\alpha}(A_{-\alpha})$.

Proof. Without loss, let $\alpha = -2$. Suppose $[X_2, A_{-1}] = 1$. Then

$$[A_{-2}, A_7]^{n_2} \le A_{-1'}^{n_2} \cap [A_2, U_{-2}] = A_{-1'} \cap [A_2, U_{-2}] \le A_{-1'} \cap U_{-1} = 1$$

by (I) and 3.1(3). That is,

$$[A_{-2}, A_7] = 1. \tag{3.7.1}$$

Suppose $G(J) = \ast_{\tau\in J} X_\tau$ and $[W_{-2}, X_2] = 1$. Then repeated use of the Dedekind identity and Lemma 3.1(3) yield

$$\begin{aligned}[A_5, A_{-2}]^{n_2} &\le (A_{-1'}A_8A_{7'})^{n_2} \cap [U_{-2}, A_2] \\ &= A_{-1'}A_8A_{7'} \cap [A_{-1}A_8A_7A_6A_5A_4A_3, A_2] \\ &= A_{-1'}A_8A_{7'} \cap [A_7A_6A_5A_4A_3, A_2] \\ &\le A_{-1'}A_8A_{7'} \cap U_{-1} \cap U_8 \cap U_7 \\ &= (A_{-1'} \cap U_{-1})A_8A_{7'} \cap U_8 \cap U_7 \\ &= (A_8 \cap U_8)A_{7'} \cap U_7 = A_{7'} \cap U_7 = 1,\end{aligned}$$

since $(A_{-1'}A_8A_{7'})^{n_2} = A_{-1'}A_8A_{7'}$ by assumption. Together with (3.7.1), this implies $A_{-1}A_8A_7A_6A_5A_4 \subseteq C_{U_{-2}}(A_{-2})$, since $G(J)$ is a central product of rank one groups by assumption. Then the result follows by symmetry. □

Lemma 3.8. *Suppose $[X_\alpha, U_\alpha] = 1$ and $A_{-\beta}A_{-\gamma}A_{-\delta}A_{-\varepsilon}A_{-\eta}A_{-\kappa} \subseteq C_{U_{-\alpha}}(A_{-\alpha})$ or $A_{-\gamma}A_{-\delta}A_{-\varepsilon}A_{-\eta}A_{-\kappa}A_{-\mu} \subseteq C_{U_{-\alpha}}(A_\alpha)$ for some $\alpha \in \Phi$, where $U_\alpha = A_\beta A_\gamma A_\delta A_\varepsilon A_\eta A_\kappa A_\mu$ with $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \kappa, \mu \in \Phi$ and $\alpha < \beta < \gamma < \delta < \varepsilon < \eta < \kappa < \mu < -\alpha$. Then X_α is a central divisor of G.*

Proof. Suppose $[X_\alpha, U_\alpha] = 1$ and $A_{-\gamma}A_{-\delta}A_{-\varepsilon}A_{-\eta}A_{-\kappa}A_{-\mu} \subseteq C_{U_{-\alpha}}(A_\alpha)$. Then there exists an $a_{-\beta} \in A^\sharp_{-\beta}$ with $[a_{-\beta}, A_\alpha] = 1$. Since otherwise

$$A^{n_\alpha}_{-\beta} \le C_{U_{-\alpha}}(A_\alpha) = A_{-\gamma}A_{-\delta}A_{-\varepsilon}A_{-\eta}A_{-\kappa}A_{-\mu} \subseteq U_{-\beta},$$

contrary to 3.4.

Now let $a_{-\beta} \in A^\sharp_{-\beta}$ with $[a_{-\beta}, A_\alpha] = 1$. Then $X_\beta = \langle A_\beta, a_{-\beta}\rangle \le C(A_\alpha)$, as $[A_\beta, A_\alpha] = 1$ by (I). Thus, $[A_\alpha, U_{-\alpha}] = 1$. This implies $[X_\alpha, U_{-\alpha}] = 1$ by 2.1(6), since $U_{-\alpha}$ is X_α-invariant. Hence, X_α is a central divisor of G, as required. □

4. The structure of $G(J)$

In this section we will determine the possible structures of $G(J)$. Further, we will describe the influence of the structure of $G(J)$ on the structure of $G(\tilde{\Phi})$.

As already mentioned, J is a root subsystem of $\tilde{\Phi}$ of type C_2. Thus, $G(J)$ satisfies the assumptions of [11, Corollary 3]. This yields that $G(J) = \ast_{\alpha\in J} X_\alpha$ or $G(J)$ is of type C_2, or there exists a $\beta \in J$ such that X_β is a central divisor of $G(J)$ and $G(J \setminus \{\pm\beta\})$ is of type A_2. The following lemma will allow us to exclude the last possibility.

Lemma 4.1. *One of the following holds:*

(1) $G(J) = \ast_{\alpha\in J} X_\alpha$.

(2) $G(J)$ *is of type* C_2.

Proof. By [11, Corollary 3], it suffices to prove that (1) holds if X_α is a central divisor of $G(J)$ for some $\alpha \in J$. Without loss, let X_2 be a central divisor of $G(J)$. Then we obtain $1 = [A_8, A_6A_4A_2]$ by (I), and so $[X_8, A_6A_4A_2] = 1$ by 2.1(6), since the Abelian group $A_6A_4A_2$ is X_8-invariant. Analogously, $[A_{-6}A_{-4}A_{-2}, X_8] = 1$. Hence, X_8 is a central divisor of $G(J)$. By symmetry, X_4 is also a central divisor of $G(J)$. That is, $G(J) = \ast_{\alpha\in J} X_\alpha$, and we are done. □

Corollary 4.2. *Suppose there is no $\alpha \in J$ such that X_α is a central divisor of $G(J)$. Then all A_α, $\alpha \in J$, are elementary Abelian 2-groups.*

Proof. By the commutator relations in (I), the claimed result follows from [5, Proposition 1.4.]. □

Next, using the notation of (3.4.1), we will prove some implications starting from the assumption $G(J) = \ast_{\alpha\in J} X_\alpha$.

Lemma 4.3. *Suppose $G(J) = \ast_{\alpha\in J} X_\alpha$ and there exist "neighboring roots" β and γ in J with $[X_\beta, W_\beta] = 1 \neq [X_\gamma, W_\gamma]$. Then X_β is a central divisor of G.*

Proof. Without loss, let $\beta = -2$ and $\gamma = -4$. Then our assumptions mean $[X_2, W_{-2}] = 1 \neq [X_4, W_{-4}]$.
Step 1: We show

$$A_{-1}A_8A_7A_6A_5A_4 \subseteq C_{U_{-2}}(A_{-2}). \tag{4.3.1}$$

We have $1 \neq [A_{7'}, A_{-4}] \leq C_{A_{-1'}}(A_2)$ by assumption and by (I). This implies

$$X_1 = \langle [A_{7'}, A_{-4}], A_1\rangle \leq C(A_2). \tag{4.3.2}$$

Thus, $[X_2, A_{-1}] = 1$. An application of 3.7 to U_{-2} yields (4.3.1), as $[X_2, W_{-2}] = 1$ and $G(J) = \ast_{\alpha\in J} X_\alpha$ by assumption.
Step 2. We prove $[X_2, W_2] = 1$. Suppose $[X_2, W_2] \neq 1$.

Then $[A_2, A_{-5'}] \neq 1$. By (4.3.2), this implies $1 \neq [A_2, A_{-5'}] \leq C_{A_{-7'}}(A_{-1})$. Therefore, $X_7 = \langle [A_2, A_{-5'}], A_7\rangle \leq C_G(A_{-1'})$. By 3.6, this implies $[A_{-4}, A_{7'}] = 1$, contrary to our assumption. Thus, $[A_2, A_{-5'}] = 1$ and so $[X_2, W_2] = 1$ by 2.1(6), since W_2 is X_2-invariant and $[A_2, W_2] = 1$.
Step 3: We show

$$A_{-8}A_{-7}A_{-6}A_{-5}A_{-4}A_{-3} \subseteq C_{U_2}(A_{-2}). \tag{4.3.3}$$

By (4.3.1), we get $X_5 = \langle A_5, A_{-5'}\rangle \leq C(A_{-2})$. In particular, $[A_{-5}, A_{-2}] = 1$. Further, by Step 2 and (4.3.1), we obtain $X_7 = \langle A_7, A_{-7'}\rangle \leq C(A_{-2})$. In particular, $[A_{-7}, A_{-2}] = 1$. This implies (4.3.3), since $G(J)$ is a central product of rank one groups by assumption.
Step 4: We prove

$$[X_2, U_2] = 1. \tag{4.3.4}$$

By (4.3.3), it suffices to prove that $C_{A_1}(A_{-2}) \neq 1$. From this we namely see $X_1 = \langle C_{A_1}(A_{-2}), A_{-1}\rangle \le C(A_{-2})$, since $[A_{-1}, A_{-2}] = 1$ by (I). Thus, $[A_{-2}, U_2] = 1$, and so (4.3.4) follows by 2.1(6), since U_2 is X_2-invariant.

We assume $C_{A_1}(A_{-2}) = 1$, and lead this to a contradiction. By (4.3.3), we obtain $C_{U_2}(A_{-2}) = A_{-8}A_{-7}A_{-6}A_{-5}A_{-4}A_{-3}$ and so $A_1^{n_2} \le C_{U_2}(A_{-2}) \le U_1$. But $[X_2, A_{-1}] = 1$ by Step 1, a contradiction to 3.4. Thus, $C_{A_1}(A_{-2}) \neq 1$ and so (4.3.4) holds.

By (4.3.1) and (4.3.4), the assumptions of 3.8 are satisfied for X_2. Thus, X_2 is a central divisor of G. □

Lemma 4.4. *Suppose $G(J) = \ast_{\alpha\in J} X_\alpha$ and $[X_\alpha, W_\alpha] = 1$ for each $\alpha \in J$. Then there exists an $\alpha \in J$ such that X_α is a central divisor of G.*

Proof. Let $\alpha \in J$ and $W_\alpha = A_{\beta'}A_{\gamma'}$ with $\alpha < \beta' < \gamma' < -\alpha$ and β', γ' appropriate roots of $\tilde{\Phi} \setminus J$. To begin with, we show that either $[A_{\beta'}, A_{-\gamma'}] = 1 = [A_{\gamma'}, A_{-\beta'}]$ or X_α is a central divisor of G.

Without loss, let $\alpha = -2$. We prove that either

$$[A_{7'}, A_{-5'}] = 1 \tag{4.4.1}$$

or X_2 is a central divisor of G.

We assume that X_2 is no central divisor of G, and show that in this case (4.4.1) holds. By the commutator relations in (I), we obtain $[A_{-5'}, A_{7'}] \le A_{-3'}A_{-2}A_{-1'}$. Let $a_{-3'}a_{-2}a_{-1'} \in [A_{-5'}, A_{7'}]$ with $a_{-3'} \in A_{-3'}$, $a_{-2} \in A_{-2}$ and $a_{-1'} \in A_{-1'}$. Then $a_{-3'}a_{-2}a_{-1'} \in C(A_2)$, as $[A_{7'}, A_2] = 1 = [A_{-5'}, A_2]$ by assumption. By (I), this implies $[a_{-3'}a_{-2}a_{-1'}, A_{5'}] = [a_{-3'}, A_{5'}] \le C_{A_{-1'}A_8A_{7'}}(A_2)$, since $[X_2, W_{-2}] = 1$ by assumption.

We assume $a_{-3'} \neq 1$, and lead this to a contradiction.

Let $a^\star_{-1'}a^\star_8 a^\star_{7'} \in [a_{-3'}, A_{5'}]$ with $a^\star_{-1'} \in A_{-1'}$, $a^\star_8 \in A_8$ and $a^\star_{7'} \in A_{7'}$. Then $1 = [a^\star_{-1'}a^\star_8 a^\star_{7'}, A_2] = [a^\star_{-1'}, A_2] = 1$, as $[A_8, A_2] = 1 = [A_{7'}, A_2]$ by assumption. Suppose $a^\star_{-1'} \neq 1$. Then $X_1 = \left\langle a^\star_{-1'}, A_1\right\rangle \le C(A_2)$ and so $X_2 = \langle A_2, A_{-2}\rangle \le C(A_{-1})$. Arguing as in 4.3, we obtain that X_2 is a central divisor of G, since $[X_2, W_2] = 1$ by assumption, a contradiction. Thus, we have $[a_{-3'}, A_{5'}] \le A_8A_{7'}$. Therefore, $L := A_8A_{7'}A_6A_{5'}$ is $\langle a_{-3'}, A_3\rangle = X_3$-invariant. Conjugation with n_3 yields $[L, A_{3'}^{n_3}] = 1$, as $[L, A_{3'}] = 1$. This implies $X_3 = \langle A_{3'}^{n_3}, A_3\rangle \le C(A_{5'})$. In particular, $[A_{5'}, A_{-3}] = 1$ and so $X_5 = \langle A_{5'}, A_{-5}\rangle \le C(A_{-3'})$, since $[A_{-5}, A_{-3'}] = 1$ by (I). From this we get $[M_{-5}, A_{5'}] = 1 = [M_{-5}, A_{5'}^{n_5}]$, since $[A_{-2}, A_{5'}] = 1$ by assumption. As $(1 \neq) A_{5'}^{n_5} \le A_{-5}$, this yields $X_5 = \langle A_{5'}^{n_5}, A_5\rangle \le C(A_{7'})$. Thus, $a_{-3'}a_{-2}a_{-1'} \in [A_{-5'}, A_{7'}] = 1$ and so

$$1 \neq a_{-3'} = (a_{-2}a_{-1'})^{-1} \le A_{-3'} \cap A_{-2}A_{-1'} \le A_{-3'} \cap U_{-3} = 1$$

by 3.1(3), contradicting our assumption.

Hence, $[A_{7'}, A_{-5'}] \le A_{-2}A_{1'}$. Assuming that X_2 is no central divisor of G, the analogous argument with $a_{-1'}$ in place of $a_{-3'}$ yields $[A_{7'}, A_{-5'}] \le A_{-2}$. Thus, $[A_{-5'}, A_{7'}] \le C_{A_{-2}}(A_2) = 1$, since $[A_{-5'}, A_2] = 1 = [A_{7'}, A_2]$ by assumption. Analogously, either $[A_{5'}, A_{-7'}] = 1$ or X_2 is a central divisor of G.

By the above argumentation, it suffices to prove that X_β is a central divisor of G for some $\beta \in J$, if $[A_{\beta'}, A_{-\gamma'}] = 1 = [A_{\gamma'}, A_{-\beta'}]$ for each $\alpha \in J$. Lemma 3.6 yields $[X_\alpha, M_\alpha] = 1$ for each $\alpha \in J$, using that $[X_\alpha, W_\alpha] = 1$ for each $\alpha \in J$ by assumption. In particular, $[X_5, X_2] = 1 = [X_7, X_2]$ and $[X_5, X_8] = 1 = [X_3, X_8]$. By 3.8, we get that either X_2 is a central divisor of G or repeated use of the nilpotence argument yields the conjugation relations $A_{-1} \overset{n_2}{\longleftrightarrow} A_3$ and $A_{-3} \overset{n_2}{\longleftrightarrow} A_1$, as $[X_2, X_8] = [X_2, X_7] = [X_2, X_6] = [X_2, X_5] = [X_2, X_4] = 1$. In particular, $[A_1, A_8]^{n_2} = [A_{-3}, A_8] = 1$. Therefore, X_8 is a central divisor of G by 3.8. Hence, there exists an $\alpha \in J$ such that X_α is a central divisor of G, as required. □

Lemma 4.5. *Suppose $G(J) = \ast_{\alpha \in J} X_\alpha$. Then there exists an $\alpha \in J$ with $[X_\alpha, W_\alpha] = 1$.*

Proof. We assume $[X_\alpha, W_\alpha] \neq 1$ for each $\alpha \in J$ and $G(J) = \ast_{\alpha \in J} X_\alpha$, and lead this to a contradiction.

Step 1: We use the notation of 3.5. Firstly, we show $C_{M_\alpha}(A_{\alpha'}) = A_{\beta'} A_\gamma A_{\delta'}$ and $C_{M_\alpha}(A_{-\alpha'}) = A_{\delta'} A_\varepsilon A_{\eta'}$ for each $\alpha \in \Phi \setminus J$. Without loss, let $\alpha = -1$. By symmetry, it suffices to prove

$$C_{M_{-1}}(A_{-1'}) = A_{7'} A_6 A_{5'}. \tag{4.5.1}$$

By (I), we have $A_{7'} A_6 A_{5'} \leq C_{M_{-1}}(A_{-1'})$. To get the opposite inclusion, we show $C_{A_4 A_{3'}}(A_{-1'}) = 1$. Let $a_4 \in A_4$ and $a_{3'} \in A_{3'}$ with $[a_4 a_{3'}, A_{-1'}] = 1$. Then we have $1 = [a_4 a_{3'}, a_{-1'}] = [a_4, a_{-1'}]^{a_{3'}} [a_{3'}, a_{-1'}]$ for each $a_{-1'} \in A_{-1'}$. As $[A_4, A_{-1'}] \leq A_{7'}$ and $[A_{7'}, A_{3'}] = 1$, this shows $[a_{3'}, A_{-1'}] \leq A_{7'}$. Suppose $a_{3'} \neq 1$. Then $A_{-1'} A_{7'}$ is $\langle a_{3'}, A_{-3} \rangle = X_3$-invariant. Thus, $[X_3, A_{-1'}] = 1$ by 3.5. This implies $[A_{-3'}, A_6] = 1$ by 3.6, contradicting our assumption. Suppose $a_4 \neq 1$. Then $[a_4, A_{-1'}] = 1$ and so $X_4 = \langle a_4, A_{-4} \rangle \leq C(A_{-1'})$, a contradiction to $[A_4, A_{-1'}] \neq 1$ by assumption. Hence, (4.5.1) holds.

Step 2. We show

$$A_{-1}^{n_2} \subseteq A_7 A_6 A_5 A_4 A_3. \tag{4.5.2}$$

We have $A_8 A_6 A_{5'} A_4 A_3 \subseteq C_{U_{-2}}(A_2)$, since $[A_8, A_2] = 1$ by assumption. Suppose $[a_{-1} a_7 a_5, A_2] = 1$ for some $a_{-1} \in A_{-1}$, $a_7 \in A_7$ and $a_5 \in A_5$. Then we get

$$\begin{aligned} 1 &= [a_{-1} a_7 a_5, a_2] \\ &= [a_{-1}, a_2]^{a_7 a_5} [a_7 a_5, a_2] \\ &= [a_{-1}, a_2]^{a_7 a_5} [a_7, a_2]^{a_5} [a_5, a_2] \end{aligned}$$

for each $a_2 \in A_2$. As $[A_7, A_2] \leq A_{5'} A_4 A_{3'}$, $[A_{5'} A_4 A_{3'}, A_5] = 1$, $[A_5, A_2] \leq A_{3'}$ and $[A_{5'} A_4 A_{3'}, A_7] \leq A_{5'}$, this yields $[a_{-1}, A_2] \leq A_{5'} A_4 A_{3'}$. Suppose $a_{-1} \neq 1$. Then $M_{-1} A_2$ is $\langle a_{-1}, A_1 \rangle = X_1$-invariant. By Step 1, we get from this $[A_{7'}, A_2]^{n_1} \leq [C_{M_{-1}}(A_{1'}), M_{-1} A_2] = [A_{5'} A_4 A_{3'}, M_{-1} A_2] = 1$, since $A_{5'} A_4 A_{3'} \leq Z(M_{-1} A_2)$, contrary to our assumption.

Thus, $C_{U_{-2}}(A_2) \subseteq A_8A_7A_6A_5A_4A_3$, and an application of the nilpotence argument yields (4.5.2), since $[X_8, X_2] = 1$ by assumption.

Step 3: Next, we show

$$[A_{7'}, A_{-4}]^{n_2} \leq A_7A_6A_5. \tag{4.5.3}$$

We have $[[A_{7'}, A_{-4}], A_2]^{n_2} \leq [[A_{7'}A_{5'}, A_{-4}], A_{-2}] \leq A_{-1'}$ by the commutator relations in (I), since W_{-2} is X_2-invariant and $[A_{-4}, X_2] = 1$ by assumption. Further, by (4.5.2), we get $[A_{7'}, A_{-4}]^{n_2} \leq A_7A_6A_5A_4A_3$. Let $a_7a_6a_5a_4a_3 \in [A_{7'}, A_{-4}]^{n_2}$ with $a_i \in A_i$ for $i \in \{3, 4, 5, 6, 7\}$. Then

$$\begin{aligned} A_{-1'} &\geq [a_7a_6a_5a_4a_3, A_{-2}] = [a_7a_5a_4a_3, A_{-2}] \\ &= [a_7a_4a_5a_3, A_{-2}] = [a_4a_7[a_7, a_4]a_5a_3, A_{-2}] \end{aligned}$$

by (I). We have $[a_7, a_4]a_5 = a_5^\star$ for some $a_5^\star \in A_5$. Thus, we obtain $[a_7a_5^\star a_3, A_{-2}] \leq A_{-1'}$, since $[a_4, A_{-2}] = 1$ by assumption. This yields

$$\begin{aligned} A_{-1'} &\ni [a_7a_5^\star a_3, a_{-2}] \\ &= [a_7, a_{-2}]^{a_5^\star a_3}[a_5^\star a_3, a_{-2}] \\ &= [a_7, a_{-2}]^{a_3}[a_5^\star, a_{-2}]^{a_3}[a_3, a_{-2}] \end{aligned}$$

for each $a_{-2} \in A_{-2}$. Since $[A_7, A_{-2}] \leq A_{-1'}$, $[A_{-1'}, a_3] \leq A_8A_{7'}A_6A_{5'}$ and $[A_5, A_{-2}] \leq A_{-1'}A_8A_{7'}$, we get $[a_3, A_{-2}] \leq M_{-3}$. Suppose $a_3 \neq 1$. Then $A_{-2}M_{-3}$ is $\langle a_3, A_{-3}\rangle = X_3$-invariant. By Step 1, this implies $[A_{-2}, A_{5'}]^{n_3} \leq [A_{-2}M_{-3}, A_{-1'}A_8A_{7'}] = 1$, as $A_{-1'}A_8A_{7'} \leq Z(A_{-2}M_{-3})$, contradicting our assumption. This yields $[A_{7'}, A_{-4}]^{n_2} \leq A_7A_6A_5A_4$. Using the nilpotence argument, we obtain (4.5.3), since $[X_4, X_2] = 1$ by assumption.

Step 4: We show

$$[A_{7'}, A_{-4}]^{n_2} \leq A_7. \tag{4.5.4}$$

We have $[A_{7'}, A_{-4}]^{n_2} \leq [A_{7'}A_{5'}, A_{-4}] \leq U_{-5}$ by (I), since W_{-2} is X_2-invariant. We get $[A_{7'}, A_{-4}]^{n_2} \leq A_7A_6A_5 \cap U_{-5} = A_7A_6(A_5 \cap U_{-5}) = A_7A_6$ by (4.5.3) and 3.1(3), using the Dedekind identity. Using the nilpotence argument, this implies (4.5.4), as $[X_6, X_2] = 1$.

Finally, we lead our original assumption to a contradiction. We have $[[A_{-4}, A_{7'}], A_{-5'}] = 1$ by (I), and so $[[A_{-4}, A_{7'}]^{n_2}, A_{-5'}^{n_2}] = 1$. Furthermore, there exists an $a_{-5'} \in A_{-5'}^\sharp$ with $a_{-5'}^{n_2} \in A_{-7'}^\sharp$, since $[A_{-5'}, A_2] \neq 1$ by assumption. By (4.5.4), this implies $[a_7, a_{-7'}] = 1$ for some $a_7 \in A_7^\sharp$ and some $a_{-7'} \in A_{-7'}^\sharp$. Thus, $X_7 = \langle a_7, A_{-7}\rangle \leq C(a_{-7'})$ since $[a_{-7'}, A_{-7}] = 1$ by (I), a contradiction to $N_{A_{-7}}(A_7) = 1$ by 2.1(2).

Hence, there exists an $\alpha \in J$ with $[X_\alpha, W_\alpha] = 1$, if $G(J) = \ast_{\alpha\in J} X_\alpha$, as required. □

5. Proof of 1.2 – main part I

In this section we will show

Theorem 5.1. *Suppose $G(J)$ is of type C_2 and $C_{A_{\beta'}}(A_{-\alpha}) \neq 1$ or $C_{A_{\gamma'}}(A_\alpha) \neq 1$ for some $\alpha \in J$ with $W_\alpha = A_{\beta'}A_{\gamma'}$, using the notation of* (3.4.1)*. Then* 1.2 (B) *holds.*

Theorem 5.1 follows directly from the lemmas of this section.

Lemma 5.2. *Suppose $[A_\alpha, G(J)] = 1$ for each $\alpha \in \tilde{\Phi}\setminus\Phi$. Then $[G(J), G(\tilde{\Phi}\setminus J)] = 1$, and the root subgroups A_β, $\beta \in \tilde{\Phi} \setminus J$, are closed under commutators. Further, $G(\tilde{\Phi} \setminus J) = \ast_{\alpha\in\Phi\setminus J} X_\alpha$ or $G(\tilde{\Phi} \setminus J)$ is of type C_2.*

Proof.
Step 1: We have $X_3 = \langle A_{3'}, A_{-3}\rangle \leq C(A_{-2})$, since $[A_{3'}, A_{-2}] = 1$ by assumption. Analogously, we get $[A_{-2}, X_1] = 1$ and $[A_2, X_1] = 1$. By 3.7, this implies $[A_{-2}, A_7] = 1$. Thus, $X_7 = \langle A_7, A_{-7'}\rangle \leq C(A_{-2})$. Analogously, $[A_{-2}, X_5] = 1$. Thus, $[A_{-2}, G(\tilde{\Phi} \setminus J)] = 1$. By symmetry, we obtain

$$[G(J), G(\tilde{\Phi} \setminus J)] = 1. \qquad (5.2.1)$$

Step 2: Next, we show that the root subgroups A_α, $\alpha \in \tilde{\Phi} \setminus J$, are closed under commutators. For this it suffices to prove that the A_α, $\alpha \in \Phi \setminus J$, are closed under commutators. Without loss, we show the appropriate commutator relations for A_1. By (5.2.1) and 2.1(2), we have $[A_1, A_3] \leq C_{A_2}(A_{-2}) = 1$. Further, $[A_1, A_5] = 1$ by (I). We show $[A_1, A_7] \leq A_5A_3$ by a division into cases (see Lemma 4.1):
(a) Suppose $G(J)$ is of type C_2. Then we have

$$\begin{aligned}[A_1, A_7] &\leq A_6A_5A_4A_3A_2 \cap C_G(A_8) \cap C_G(A_{-8}) \cap C_G(A_{-4})\\ &\leq A_6A_5A_4A_3 \cap C_G(A_{-8}) \cap C_G(A_{-4})\\ &\leq A_5A_4A_3 \cap C_G(A_{-4}) \leq A_5A_3\end{aligned}$$

by (I) and (5.2.1)
(b) Suppose $G(J)$ is a central product of rank one groups. Then we obtain

$$\begin{aligned}[A_1, A_7] &\leq A_6A_5A_4A_3A_2 \cap C_G(A_{-2}) \cap C_G(A_{-6}) \cap C_G(A_{-4})\\ &\leq A_6A_5A_4A_3 \cap C_G(A_{-6}) \cap C_G(A_{-4})\\ &\leq A_5A_4A_3 \cap C_G(A_{-4}) \leq A_5A_3\end{aligned}$$

by (I) and (5.2.1).

Finally, we show that, under these assumptions, A_α is Abelian for each $\alpha \in \Phi \setminus J$ or $G(\tilde{\Phi} \setminus J) = \ast_{\alpha\in\Phi\setminus J} X_\alpha$. Without loss, let $\alpha = 1$. By Step 2, we get $[A_7, A_1, A_1] \leq [A_5A_3, A_1] = 1$. By (I) and the Three-Subgroup-Lemma, we obtain

$A_1' = [A_1, A_1] \leq C_{A_{1'}}(A_7)$. Suppose $A_1' \neq 1$. Arguing as in the proof of 4.1, we obtain $G(\tilde{\Phi} \setminus J) = \ast_{\alpha \in \Phi \setminus J} X_\alpha$.

Suppose $A_\alpha' = 1$ for each $\alpha \in \Phi \setminus J$. Then the root subgroups A_α, $\alpha \in \Phi \setminus J$, satisfy the assumptions of [10, Theorem 1] with respect to a root system of type C_2. As in the proof of 4.1, we see from this that $G(\tilde{\Phi} \setminus J) = G(\Phi \setminus J)$ is of type C_2 or $G(\tilde{\Phi} \setminus J) = \ast_{\alpha \in J} X_\alpha$. □

Lemma 5.3. *Suppose $G(J)$ is of type C_2 and there exist "neighboring roots" α and β in J with $C_{A_{\delta'}}(A_\alpha) \cap C_{A_{\delta'}}(A_{-\beta}) \neq 1$, where $W_\alpha = A_{\gamma'} A_{\delta'}$ and $W_\beta = A_{\delta'} A_{\eta'}$ with $\alpha < \beta < \gamma' < \delta' < \eta' < -\alpha < -\beta$ for appropriate roots $\gamma', \delta', \eta' \in \tilde{\Phi} \setminus J$. Then $[A_{\delta'}, G(J)] = 1$.*

Proof. Let, without loss, $\alpha = -4$, $\beta = -2$ and $\delta' = 7'$. Then our assumptions mean $[a_{7'}, A_{-4}] = 1 = [a_{7'}, A_2]$ for some $a_{7'} \in A_{7'}^\sharp$. From this we get the following commutator relations: $1 = [a_{7'}, A_2]^{n_4} = [a_{7'}, A_{-6}]$ and $1 = [a_{7'}, A_{-4}]^{n_2} = [a_{7'}, A_{-8}]$, since $G(J)$ is of type C_2 by assumption. This implies $X_7 = \langle a_{7'}, A_{-7} \rangle \leq C(A_{-6})$, i.e. $[X_7, A_{-6}] = 1$ and so $[A_7, X_6] = 1$ respectively $X_7 = \langle a_{7'}, A_{-7} \rangle \leq C(A_{-8})$, i.e. $[X_7, A_{-8}] = 1$ and so $[A_7, X_8] = 1$. Thus, $[A_7, A_2]^{n_8} = [A_7, A_6] = 1$ respectively $[A_7, A_{-4}]^{n_6} = [A_7, A_8] = 1$. Therefore, $[A_{7'}, G(J)] = 1$. □

Lemma 5.4. *Suppose $G(J)$ is of type C_2 and there exist three roots α, β, τ in J such that $C_{A_{\delta'}}(A_\alpha) \cap C_{A_{\delta'}}(A_{-\beta}) \neq 1$ and $[X_\tau, W_\tau] \neq 1$, where α and β respectively β and τ are "neighbors" in J with $\alpha < \beta < \tau$, and W_α respectively W_β are as in 5.3. Then X_δ is a central divisor of G.*

Proof. Let, without loss, $\alpha = -4$, $\beta = -2$ and $\tau = 8$. Then, as in the proof of 5.3, we get

$$[X_7, A_{-8}] = 1 = [X_7, A_{-6}]. \tag{5.4.1}$$

Moreover, $[A_{5'}, A_{-8}] \neq 1$ by assumption. This implies that $[a_{5'} a_4, A_{-7'}] = 1$ for some $a_{5'} \in A_{5'}^\sharp$ and some $a_4 \in A_4$, as $A_4 A_{3'} A_2 A_{1'} \subseteq C_{M_7}(A_{-8})$ by (I), $A_{-8}^{n_7} = A_{-8}$, $A_{5'}^{n_7} \leq C_{M_7}(A_{-7'})$ and $A_{3'} A_2 A_{1'} \leq C_{M_7}(A_{-7'})$. Then we obtain $1 = [a_{5'} a_4, a_{-7'}] = [a_{5'}, a_{-7'}]^{a_4} [a_4, a_{-7'}]$ for each $a_{-7'} \in A_{-7'}$. Thus, $[a_{5'}, A_{-7'}] \leq A_{1'}$, since $[A_4, A_{-7'}] \leq A_{1'}$ and $[A_4, A_{1'}] = 1$. Hence, $A_{1'} A_{-7'}$ is $\langle a_{5'}, A_{-5} \rangle = X_5$-invariant. By 3.5, this yields $[A_{-7'}, X_5] = 1$ and so $X_7 = \langle A_{-7'}, A_7 \rangle \leq C(A_{5'})$, as $[A_7, A_{5'}] = 1$ by (I). Thus,

$$[A_{-8}, A_{-5}] = 1 = [A_2, A_{-5'}] \tag{5.4.2}$$

respectively $[A_4, A_7] = 1$ by 3.6. Therefore, $X_4 = \langle A_4, A_{-4} \rangle \leq C(A_7)$, since, as in the proof of 5.3, $[A_7, A_{-4}] = 1$ holds. This implies $[A_7, A_{-2}]^{n_4} = [A_7, A_6] = 1$ and so $[X_2, A_7] = 1$, since, as in the proof of 5.3, $[A_7, A_2] = 1$ holds.

Moreover, $[A_4, A_{-7'}] = 1$, since we obtain otherwise a contradiction to 3.4, as $[X_4, A_7] = 1$. As $[X_7, A_{5'}] = 1$ and by 3.6, this implies

$$[M_7, X_7] = 1. \tag{5.4.3}$$

By (5.4.2), we get $1 = [A_{-2}, A_{-7'}]^{n_4} = [A_6, A_{-7'}]$ and by (5.4.3), we obtain $1 = [A_4, A_{-7'}]^{n_2} = [A_8, A_{-7'}]$. Thus, the next commutator relations follow: $[X_7, A_6] = 1$ and so $[X_6, A_{-7}] = 1$ respectively $[X_7, A_8] = 1$ and so $[X_8, A_{-7}] = 1$. Moreover, $[A_{-2}, A_{-7}]^{n_8} = [A_{-6}, A_{-7}] = 1$ and so $[X_7, A_{-2}] = 1$. Further,

$$[A_{-7}, A_{-5}]^{n_7} \le A_{-6}^{n_7} \cap [A_{-7}, U_{-7}]^{n_7} \le A_{-6} \cap U_{-6} = 1, \tag{5.4.4}$$

as $[X_7, A_{-6}] = 1$ by (5.4.1). Thus,

$$X_5 = \langle A_{-5}, A_{5'} \rangle \le C(A_{-7}) \tag{5.4.5}$$

by (5.4.3). Analogously as in (5.4.4), we obtain $[A_7, A_5] = 1$, since $[X_7, A_6] = 1$. By (5.4.5), this implies $[X_7, A_5] = 1$. We also have $[A_1, A_{-7}] = 1$, since $[X_7, A_{-8}] = 1$ by (5.4.1). All in all, $[U_7, A_{-7}] = 1$ and so $[U_7, X_7] = 1$ by 2.1(6), since U_7 is X_7-invariant. Therefore, $[M_{-7}, X_7] = 1$, as $A_{-5'}A_{-4}A_{-3'}A_{-2} \subseteq C_{M_{-7}}(A_{-7})$, since otherwise we would obtain a contradiction to 3.4. Thus, $X_1 = \langle A_{-1'}, A_1 \rangle \le C(A_{-7})$. Hence, we get $[U_{-7}, A_{-7}] = 1$. Then, by 2.1(6), we also have $[U_{-7}, X_7] = 1$. All in all, we obtain that X_7 is a central divisor of G. □

Lemma 5.5. *Suppose $G(J)$ is of type C_2 and there exist "neighboring roots" α and β in J such that $C_{A_{\delta'}}(A_\alpha) \cap C_{A_{\delta'}}(A_{-\beta}) \neq 1$, where δ' is as in* 5.3. *Then* 1.2 (B) *holds.*

Proof. By 5.3 and 5.4, X_ν is a central divisor of G for some $\nu \in \Phi \setminus J$, or each root subgroup A_α, $\alpha \in \tilde{\Phi} \setminus \Phi$, commutes with each root subgroup A_β, $\beta \in J$. In the latter case the assumptions of 5.2 are satisfied. So 1.2 (B) holds in both cases. □

Lemma 5.6. *Suppose $G(J)$ is of type C_2 and there exist "neighboring roots" α and β in J such that $C_{A_{\delta'}}(A_{-\beta}) \neq 1$ and $C_{A_{\delta'}}(A_\alpha) \cap C_{A_{\delta'}}(A_{-\beta}) = 1$, where W_α and W_β are as in* 5.3. *Then* 1.2 (B) *holds.*

Proof. Let, without loss, $\alpha = -4$, $\beta = -2$ and $\delta' = 7'$. Further, let $a_{7'} \in A_{7'}^\sharp$ with $[a_{7'}, A_2] = 1 \neq [a_{7'}, A_{-4}]$.

Step 1: First, we show $a_{7'}^{n_4} \in A_{-1'}$.

We have $a_{7'}^{n_4} = a_{-1'}a_{7'}^\star$, where $a_{-1'} \in A_{-1'}$ and $a_{7'}^\star \in A_{7'}$, since W_{-4} is X_4-invariant. Thus, $\left(a_{7'}\,(a_{7'}^\star)^{-1}\right)^{n_4} = a_{-1'}$, as $a_{7'}^{n_4} \in C_{W_{-4}}(A_{-4})$ and $A_{-1'} \le C_{W_{-4}}(A_{-4})$ by (I). Further, $1 = [a_{7'}, A_2]^{n_4} = [a_{-1'}a_{7'}^\star, A_{-6}]$, since $G(J)$ is of type C_2 by assumption. This implies $1 = [a_{-1'}a_{7'}^\star, a_{-6}] = [a_{-1'}, a_{-6}]^{a_{7'}^\star}[a_{7'}^\star, a_{-6}]$ for each $a_{-6} \in A_{-6}$. Since $[A_{-1'}, A_{-6}] \le A_{-3'}$ and $[A_{-3'}, A_{7'}] = 1$, we get $[a_{7'}^\star, A_{-6}] \le A_{-3'}$. Suppose $a_{7'}^\star \neq 1$. Then $A_{-6}A_{-3'}$ is $\langle a_{7'}^\star, A_{-7} \rangle = X_7$-invariant. Further, an application of the nilpotence argument yields $A_{-6}^{n_7} = A_{-6}$, as $[X_3, X_7] = 1$ by (I). Thus, $[A_{-6}, X_7] = 1$ and so $[X_6, A_7] = 1$. This yields $[A_7, A_{-4}]^{n_6} = [A_7, A_8] = 1$, a contradiction to $[a_{7'}, A_{-4}] \neq 1$ by assumption. Thus, $a_{7'}^\star = 1$ and so $a_{7'}^{n_4} \in A_{-1'}$.

Step 2: Next, we show $a_{-1'}^{n_2} \in A_5A_4A_3$ for $a_{-1'}$ as in Step 1.

The group $[A_{7'}, A_{-4}]A_{7'}$ is X_4-invariant. By Step 1 and the Dedekind identity, we get

$$a_{-1'} = a_{7'}^{n_4} \in [A_{7'}, A_{-4}]A_{7'} \cap A_{-1'} = [A_{7'}, A_{-4}](A_{7'} \cap A_{-1'}) = [A_{7'}, A_{-4}]$$

by 3.1(3). By (I), this implies $a_{-1'}^{n_2} \in [A_{7'}, A_{-4}]^{n_2} \leq [A_{7'}A_{5'}, A_{-8}] \subseteq A_6A_5A_4A_3A_2A_1$. Therefore, $a_{-1'}^{n_2} \in C_{A_6A_5A_4A_3A_2A_1}(a_{7'})$, since $[a_{7'}, X_2] = 1$ by assumption and $[A_{7'}, A_{-1'}] = 1$ by (I). Further, $C_{A_6A_5A_4A_3A_2A_1}(a_{7'}) = A_6A_5A_4A_3A_2$, since otherwise $[X_1, A_{7'}] = 1$ and so $[A_4, A_{-1'}] = 1$ by 3.6, contrary to $[A_{-4}, A_{7'}] \neq 1$ by assumption.

On the other hand, $a_{-1'}^{n_2} \in U_{-2}$, since U_{-2} is X_2-invariant by 3.1(1). Thus,

$$a_{-1'}^{n_2} \in A_6A_5A_4A_3A_2 \cap U_{-2} = A_6A_5A_4A_3(A_2 \cap U_{-2}) = A_6A_5A_4A_3$$

by the Dedekind identity and 3.1(3). An application of the nilpotence argument yields $a_{-1'}^{n_2} \in A_5A_4A_3$, as $[X_6, X_2] = 1$.

Step 3: Next, we prove $[A_{-5'}, A_2] \neq 1$.

Suppose $[A_{-5'}, A_2] = 1$. Then, by Step 2, we get $a_{-1'}^{n_2} \in A_5A_4A_3 \cap C(A_{-5'})$, since $[A_{-1'}, A_{-5'}] = 1$. Let $a_5 \in A_5$, $a_4 \in A_4$ and $a_3 \in A_3$ with $a_{-1'}^{n_2} = a_5a_4a_3$. Then we get

$$\begin{aligned} 1 &= [a_5a_4a_3, a_{-5'}] = [a_5, a_{-5'}]^{a_4a_3}[a_4a_3, a_{-5'}] \\ &= [a_5, a_{-5'}]^{a_4a_3}[a_4, a_{-5'}]^{a_3}[a_3, a_{-5'}] \end{aligned}$$

for each $a_{-5'} \in A_{-5'}$. Therefore, $[a_5, A_{-5'}] \leq U_5$, since $[a_4, a_{-5'}]^{a_3} = [a_4, a_{-5'}][a_4, a_{-5'}, a_3] \in A_3A_2A_1A_{-8}A_{-7}A_{-6} \subseteq U_5$. Suppose $a_5 \neq 1$. Then $U_5A_{-5'}$ is $\langle a_5, A_{-5}\rangle = X_5$-invariant, as $[A_{-5}, A_{-5'}] = 1$ by (I). Thus, $X_5 = \left\langle A_{-5'}^{n_5}, A_{-5}\right\rangle \leq U_5A_{-5}$, a contradiction to 3.1(2) and 2.1(5). Hence, $a_5 = 1$ and so $a_{-1'}^{n_2} = a_4a_3$. Then $1 = [a_4a_3, a_{-6}] = [a_4, a_{-6}]^{a_3}[a_3, a_{-6}]$ for each $a_{-6} \in A_{-6}$, since $1 = [a_{7'}, A_2]^{n_4n_2} = [a_{-1'}, A_{-6}]^{n_2} = [a_4a_3, A_{-6}]$. Thus, $[a_4, A_{-6}] \leq A_{1'}A_{-8}A_{-7'}$, since $[A_3, A_{-6}] \leq A_{1'}A_{-8}A_{-7'}$ and $(A_{1'}A_{-8}A_{-7'})^{a_3} \subseteq A_{1'}A_{-8}A_{-7'}$. Suppose $a_4 \neq 1$. Then $A_{1'}A_{-8}A_{-7'}A_{-6}$ is $\langle a_4, A_{-4}\rangle = X_4$-invariant. By 3.1(3), this implies $A_2 = A_{-6}^{n_4} \leq A_2 \cap U_2 = 1$, since $G(J)$ is of type C_2 by assumption, a contradiction. Hence, $a_4 = 1$ and so $a_{-1'}^{n_2} = a_3$. Thus, $X_7 = \langle A_{-7'}, A_7\rangle \leq C(a_{-1'})$, since $1 = [A_{-7'}, a_3]^{n_2^{-1}} = [A_{-7'}, a_{-1'}]$. Therefore, $X_1 = \langle a_{-1'}, A_1\rangle \leq C(A_{-7'})$. This implies $X_7 = \langle A_{-7'}, A_7\rangle \leq C(A_{-1'})$. But, by 3.6, this yields $[A_{-4}, A_{7'}] = 1$, a contradiction to $[a_{7'}, A_{-4}] \neq 1$ by assumption. Hence, $[A_{-5'}, A_2] \neq 1$.

Step 4: Next, we show $a_{-1'}^{n_2} \in A_3$ for $a_{-1'}$ as in Step 1.

By Step 3, we have $[A_{-5'}, A_2] \neq 1$. Thus, there exists an $a_{-5'} \in A_{-5}^{\sharp}$ with $a_{-5'}^{n_2} = a_{-7'}$. By Step 2, this yields $1 = [a_{-1'}^{n_2}, a_{-7'}] = [a_5a_4a_3, a_{-7'}]$, since $[A_{-1'}, A_{-5'}] = 1$ by (I), where $a_5 \in A_5$, $a_4 \in A_4$, $a_3 \in A_3$ with $a_{-1'}^{n_2} = a_5a_4a_3$. As $[A_3, A_{-7'}] = 1$, this implies $1 = [a_5a_4a_3, a_{-7'}] = [a_5a_4, a_{-7'}] = [a_5, a_{-7'}]^{a_4}[a_4, a_{-7'}]$. Thus, $[a_5, a_{-7'}] \in A_{1'}$, since $[A_4, A_{-7'}] \leq A_{1'}$ and $[A_{1'}, A_4] = 1$. Suppose $a_5 \neq 1$. Then $\langle a_{-7'}\rangle A_{1'}$ is $\langle a_5, A_{-5}\rangle = X_5$-invariant. An application of the nilpotence argument yields $\langle a_{-7'}\rangle^{n_5} \leq \langle a_{-7'}\rangle$, since $[X_1, X_5] = 1$ by (I). Thus, $[\langle a_{-7'}\rangle, X_5] = 1$.

This implies $[X_7, A_{5'}] = 1$ and so $[X_5, A_{-7'}] = 1$. Thus, $[A_2, A_{-5'}] = 1$ by 3.6, a contradiction. Hence, $a_5 = 1$ and so $a_{-1'}^{n_2} = a_4 a_3$.

As in Step 3, we get $a_4 = 1$ and so $a_{-1'}^{n_2} = a_3$.

Step 5: Next, we show

$$C_{W_2}(A_{-2}) = A_{-5'} \stackrel{n_2}{\longleftrightarrow} C_{W_2}(A_2) = A_{-7'}. \qquad (5.6.1)$$

Let $a_{-1'}$ and a_3 be as in Step 4. By (I), we have $A_{-5'} \subseteq C_{W_2}(A_{-2})$. We show $C_{A_{-7'}}(A_{-2}) = 1$, to get the opposite inclusion. Let $a_{-7'} \in C_{A_{-7'}}(A_{-2})$. Then $1 = [a_{-7'}, a_3]^{n_2^{-1}} = [a_{-7'}, a_{-1'}]$. Suppose $a_{-7'} \neq 1$. Then $X_7 = \langle a_{-7'}, A_7\rangle \leq C(a_{-1'})$, since $[A_7, A_{-1'}] = 1$ by (I). As in Step 3, this leads to a contradiction. Hence, $C_{W_2}(A_{-2}) = A_{-5'}$. Now, we turn to the centralizer of A_2 in W_2. By (I), we have $A_{-7'} \subseteq C_{W_2}(A_2)$. We prove $C_{A_{-5'}}(A_2) = 1$, to get the opposite inclusion. Let $a_{-5'} \in C_{A_{-5'}}(A_2)$. Then $1 = [a_{-5'}, a_{-1'}]^{n_2} = [a_{-5'}, a_3]$. Suppose $a_{-5'} \neq 1$. Then we get $[X_5, A_{3'}] = 1$. By 3.6, this yields $[A_5, A_2] = 1$. Thus, $X_5 = \langle A_5, a_{-5'}\rangle \leq C(A_2)$, a contradiction to $[A_2, A_{-5'}] \neq 1$. Therefore, $C_{A_{-5'}}(A_2) = 1$. All in all, we obtain (5.6.1), since W_2 is X_2-invariant.

Step 6: Now, we prove $[A_{-5'}, A_8] = [A_{-7'}, A_4] = 1$.

By (5.6.1), we have $[A_{-5'}, A_8] \stackrel{n_2}{\longleftrightarrow} [A_{-7'}, A_4]$, since $G(J)$ is of type C_2 by assumption. We assume $[A_{-5'}, A_8] \neq 1 \neq [A_{-7'}, A_4]$, and lead this to a contradiction. Suppose $[A_{-5'}, A_8] \neq 1 \neq [A_{-7'}, A_4]$. Then $C_{W_{-2}}(A_2) = A_{5'}$. Since otherwise there exists an $\widehat{a}_{7'} \in A_{7'}^\sharp$ with $[\widehat{a}_{7'}, A_2] = 1$. This implies $1 = [[A_{-5'}, A_8], \widehat{a}_{7'}]^{n_2} = [[A_{-7'}, A_4], \widehat{a}_{7'}]$. Thus, $X_1 = \langle [A_{-7'}, A_4], A_{-1}\rangle \leq C(\widehat{a}_{7'})$ and so $X_7 = \langle \widehat{a}_{7'}, A_{-7}\rangle \leq C(A_{1'})$. By Lemma 3.6, this yields $[A_{-7'}, A_4] = 1$, contradicting our assumption. Therefore, $C_{W_{-2}}(A_2) = A_{5'}$. But this contradicts $C_{A_{7'}}(A_2) \neq 1$ by assumption.

Further, $C_{A_{1'}}(A_6) = 1$ or 1.2 (B) holds by 5.5, as $[A_{1'}, A_{-4}] = 1$ by Step 6. In the first case we can repeat the above argument for $\alpha = 6$ and $\beta = 4$. Thus, $A_{-1'} \stackrel{n_4}{\longleftrightarrow} A_{7'}$ and $[A_{7'}, A_2] = 1 = [A_{-1'}, A_{-6}]$. Together with Step 6, this yields $[A_{-3'}, A_6] = 1 = [A_{-3'}, A_{-8}]$, which leads to 1.2 (B) by 5.5. □

6. Proof of 1.2 – main part II

In this section we assume that 1.2 (B) does not hold. We will show that then 1.2 (A) holds. By Section 4 and Section 5 we may, and do, suppose in view of the proof of 1.2 that $G(J)$ is of type C_2 and that not all root subgroups A_α, $\alpha \in J$, commute with all root subgroups A_β, $\beta \in \tilde{\Phi} \setminus \Phi$. Further, we assume $C_{A_{\beta'}}(A_{-\alpha}) = 1$ and $C_{A_{\gamma'}}(A_\alpha) = 1$ for $\alpha \in J$, where $W_\alpha = A_{\beta'} A_{\gamma'}$, using the notation of (3.4.1). First, we will prove that $A_\beta^{n_\alpha} = A_{\beta^{w_\alpha}}$ for each n_α, $\alpha \in J$, and for each $\beta \in \tilde{\Phi}$, where w_α denotes the reflection along α on $\tilde{\Phi}$. Further, we will show the analogous conjugation properties for all n_α, $\alpha \in \Phi \setminus J$. Hence, the assumptions of [10, Theorem 1] are satisfied for G with respect to ${}^2F_4$. Thus, G is of type ${}^2F_4$ under these circumstances.

To begin with, we show the following

Lemma 6.1. *Using the notation of* (3.4.1), *let* $W_\alpha = A_{\beta'}A_{\gamma'}$ *for* $\alpha \in J$. *Then the following hold:*

(1) $C_{W_\alpha}(A_\alpha) = A_{\beta'} \overset{n_\alpha}{\longleftrightarrow} C_{W_\alpha}(A_{-\alpha}) = A_{\gamma'}$;

(2) $[W_\alpha, A_{-\alpha}] = A_{\gamma'}$ *and* $[W_\alpha, A_\alpha] = A_{\beta'}$;

(3) $C_{A_{-\alpha}}(A_{\beta'}) = 1 = C_{A_\alpha}(A_{\gamma'})$.

Proof. We have $C_{W_\alpha}(A_\alpha) = A_{\beta'}$, since on the one hand, $[A_{\beta'}, A_\alpha] = 1$ by (I) and on the other hand, $C_{A_{\gamma'}}(A_\alpha) = 1$. Analogously, we obtain $C_{W_\alpha}(A_{-\alpha}) = A_{\gamma'}$. Thus, (1) follows, since W_α is X_α-invariant.

By the commutator relations in (I), the group $A_{\beta'}[A_{\beta'}, A_{-\alpha}]$ is X_α-invariant. Thus, $A_{\gamma'} = A_{\beta'}^{n_\alpha} \leq A_{\beta'}[A_{\beta'}, A_{-\alpha}]$ by (1). Therefore, $A_{\beta'}A_{\gamma'} = A_{\beta'}[A_{\beta'}, A_{-\alpha}]$ and so $A_{\gamma'} = [A_{\beta'}, A_{-\alpha}] = [W_\alpha, A_{-\alpha}]$. By symmetry, (2) follows.

We assume $C_{A_{-\alpha}}(A_{\beta'}) \neq 1$, and lead this to a contradiction. Suppose there exists an $a_{-\alpha} \in C_{A^\sharp_{-\alpha}}(A_{\beta'})$. Then $X_\alpha = \langle a_{-\alpha}, A_\alpha\rangle \leq C(A_{\beta'})$, since $[A_\alpha, A_{\beta'}] = 1$ by (I). But this contradicts $C_{\beta'}(A_{-\alpha}) = 1$. Thus, $C_{A_{-\alpha}}(A_{\beta'}) = 1$, and (3) holds by symmetry. □

Lemma 6.2. *For all* n_α, $\alpha \in J$, *and* $\beta \in \tilde{\Phi}$, *we have* $A_\beta^{n_\alpha} = A_{\beta^{w_\alpha}}$.

Proof. Let, without loss, $\alpha = -2$.

Step 1: By 6.1(1), we get $A_{7'} \overset{n_2}{\longleftrightarrow} A_{5'}$ respectively $A_{-5'} \overset{n_2}{\longleftrightarrow} A_{-7'}$. By 6.1(1) and (2), this implies

$$A_{-1'} = [A_{7'}, A_{-4}] \overset{n_2}{\longleftrightarrow} [A_{5'}, A_{-8}] = A_{3'} \tag{6.2.1}$$

and

$$A_{-3'} = [A_{-5'}, A_8] \overset{n_2}{\longleftrightarrow} [A_{-7'}, A_4] = A_{1'}, \tag{6.2.2}$$

since $G(J)$ is of type C_2 by assumption.

Step 2: Next, we show

$$C_{U_{-2}}(A_{-1'}) = A_{-1}A_8A_7A_6A_5. \tag{6.2.3}$$

By (I), we get $A_{-1}A_8A_7A_6A_5 \subseteq C_{U_{-2}}(A_{-1'})$. To obtain the opposite inclusion, we prove $C_{A_4A_3}(A_{-1'}) = 1$. Let $a_4 \in A_4$ and $a_3 \in A_3$ with $[a_4a_3, A_{-1'}] = 1$. Then we have $1 = [a_4a_3, a_{-1'}] = [a_4, a_{-1'}]^{a_3}[a_3, a_{-1'}]$ for each $a_{-1'} \in A_{-1'}$. Thus, $[a_3, A_{-1'}] \leq A_{7'}$, since $[A_4, A_{-1'}] \leq A_{7'}$ and $[A_{7'}, A_3] = 1$. Suppose $a_3 \neq 1$. Then $A_{-1'}A_{7'}$ is $\langle a_3, A_{-3}\rangle = X_3$-invariant. Combining 3.5 and 3.6, we obtain $[A_6, A_{-3'}] = 1$, a contradiction to $C_{A_{-3'}}(A_6) = 1$. Further, $a_4 = 1$, since otherwise we get a contradiction to 6.1(3). Hence, (6.2.3) holds.

Step 3: We prove

$$A_5 \overset{n_2}{\longleftrightarrow} A_7 \tag{6.2.4}$$

and

$$A_{-5} \overset{n_2}{\longleftrightarrow} A_{-7}. \tag{6.2.5}$$

Arguing analogously as in Step 2, we also have $C_{U_{-2}}(A_{3'}) = A_7A_6A_5A_4A_3$ and so $C_{U_{-2}}(A_{-1'}) = A_{-1}A_8A_7A_6A_5 \overset{n_2}{\longleftrightarrow} C_{U_{-2}}(A_{3'}) = A_7A_6A_5A_4A_3$ by (6.2.1). Further, $C_{C_{U_{-2}}(A_{3'})}(A_{-1'}) = A_7A_6A_5$, since otherwise the same argumentation as in (6.2.3) yields a contradiction. Analogously, $C_{C_{U_{-2}}(A_{-1'})}(A_{3'}) = A_7A_6A_5$. Therefore, $A_7A_6A_5$ is invariant under n_2 by (6.2.1). As in (6.2.3), this implies $C_{A_7A_6A_5}(A_{1'}) = A_5$ and $C_{A_7A_6A_5}(A_{-3'}) = A_7$. Thus, (6.2.4) holds by (6.2.2). By symmetry, we also have (6.2.5).

We get $C_{C_{U_{-2}}(A_{-1'})}(A_{-3'}) = A_{-1}A_8A_7 \overset{n_2}{\longleftrightarrow} C_{C_{U_{-2}}(A_{3'})}(A_{1'}) = A_5A_4A_3$, arguing as in Step 3, and so $C_{A_{-1}A_8A_7}(A_{-5'}) = A_{-1} \overset{n_2}{\longleftrightarrow} C_{A_5A_4A_3}(A_{-7'}) = A_3$. Analogously, we obtain $A_{-3} \overset{n_2}{\longleftrightarrow} A_1$.

Hence, using that $G(J)$ is of type C_2 by assumption, we obtain $A_\beta^{n_2} = A_{\beta^{w_2}}$ for each $\beta \in \tilde{\Phi}$. □

Next, we will show a result analogous to 6.2 for each root in $\Phi \setminus J$.

Lemma 6.3. *We have $A_\beta^{n_\alpha} = A_{\beta^{w_\alpha}}$ for all n_α, $\alpha \in \Phi \setminus J$, and all $\beta \in \tilde{\Phi}$.*

Proof. Without loss, we prove the appropriate conjugation relations for the root -1.

Step 1: We use the notation of (3.4.2). First, we show $C_{M_\alpha}(A_{\alpha'}) = A_{\beta'}A_\gamma A_{\delta'}$ and $C_{M_\alpha}(A_{-\alpha'}) = A_{\delta'}A_\varepsilon A_{\eta'}$ for each $\alpha \in \Phi \setminus J$. Without loss, let $\alpha = -1$. By Step 2 in the proof of 6.2, we have $C_{M_{-1}}(A_{-1'}) = A_{7'}A_6A_{5'}$. By symmetry, we also get $C_{M_{-1}}(A_{1'}) = A_{5'}A_4A_{3'}$.

Step 2: We show

$$A_{7'} \overset{n_1}{\longleftrightarrow} A_{3'} \tag{6.3.1}$$

and

$$A_{-3'} \overset{n_1}{\longleftrightarrow} A_{-7'}. \tag{6.3.2}$$

By Step 1, we have $A_{7'}^{n_1} \le C_{M_{-1}}(A_{1'}) = A_{5'}A_4A_{3'}$ respectively $A_{3'}^{n_1} \le C_{M_{-1}}(A_{-1'}) = A_{7'}A_6A_{5'}$. Thus, using the nilpotence argument, $A_{7'}^{n_1} \le A_4A_{3'}$ respectively $A_{3'}^{n_1} \le A_{7'}A_6$, as $[X_5, X_1] = 1$ by (I). Analogously, $A_{-7'}^{n_1} \le A_{-4}A_{-3'}$ respectively $A_{-3'}^{n_1} \le A_{-7'}A_{-6}$.

Let $a_{-3'}a_{-4} \in (A_{-7'}^{n_1})^\sharp$ with $a_{-3'} \in A_{-3'}$ and $a_{-4} \in A_{-4}$ and let $a_{7'}a_6 \in (A_{3'}^{n_1})^\sharp$ with $a_{7'} \in A_{7'}$ and $a_6 \in A_6$. Then

$$\begin{aligned} 1 &= [a_{-3'}a_{-4}, a_{7'}a_6] = [a_{-3'}, a_{7'}a_6]^{a_{-4}}[a_{-4}, a_{7'}a_6] \\ &= \left([a_{-3'}, a_6][a_{-3'}, a_{7'}]^{a_6}\right)^{a_{-4}} [a_{-4}, a_6][a_{-4}, a_{7'}]^{a_6} \\ &= [a_{-3'}, a_6][a_{-4}, a_6][a_{-4}, a_{7'}], \end{aligned}$$

since $[A_{-7'}, A_{3'}]^{n_1} = 1$ by (I). Therefore, $[a_{-4}, a_6] \in A_{-1'}$, as $[A_{-3'}, A_6] \leq A_{-1'}$ and $[A_{-4}, A_{7'}] \leq A_{-1'}$. On the other hand, we get $[a_{-4}, a_6] \in A_{-2}A_8$ by (I). By the Dedekind identity and Lemma 3.1(3), we obtain

$$\begin{aligned}[a_{-4}, a_6] &\in A_{-1'} \cap A_{-2}A_8 \leq (A_{-2}A_{-1'} \cap A_{-2}A_8) \cap (A_{-1'}A_8 \cap A_{-2}A_8) \\ &\leq A_{-2}(A_{-2}A_{-1'} \cap A_8) \cap A_8(A_{-1'}A_8 \cap A_{-2}) \\ &\leq A_{-2}(U_{-8} \cap A_8) \cap A_8(U_{-2} \cap A_{-2}) = A_{-2} \cap A_8 = 1.\end{aligned}$$

We show that $[a_{-4}, a_6] = 1$ implies $a_{-4} = 1 = a_6$.

Suppose $a_{-4} \neq 1 \neq a_6$. Then $X_4 = \langle a_{-4}, A_4 \rangle \leq C(a_6)$, a contradiction, since $G(J)$ is of type C_2 by assumption. Therefore, it suffices to lead the case to a contradiction, in which one of the elements is 1 and the other one is different from 1. Without loss, let $a_{-4} = 1$ and $a_6 \neq 1$. Then $1 = [a_{-3'}, a_{7'}a_6] = [a_{-3'}, a_6]$ and so $X_6 = \langle a_6, A_{-6} \rangle \leq C(a_{-3'})$, contradicting $C_{A_{-3'}}(A_6) = 1$. (We mention that $a_{-3'} \neq 1$, because $a_{-3'}a_4 = a_{-3'} \in (A_{-7'}^{n_1})^\sharp$ by assumption.) Thus, $a_{-4} = 1 = a_6$.

Hence, $A_{3'}^{n_1} \leq A_{7'}$ and $A_{-7'}^{n_1} \leq A_{-3'}$. By symmetry, this yields (6.3.1) and (6.3.2).

Step 3: We prove

$$A_8 \overset{n_1}{\longleftrightarrow} A_2 \tag{6.3.3}$$

and

$$A_{-2} \overset{n_1}{\longleftrightarrow} A_{-8}. \tag{6.3.4}$$

By (6.3.1), we get $C_{U_{-1}}(A_{3'}) \overset{n_1}{\longleftrightarrow} C_{U_{-1}}(A_{7'})$, since U_{-1} is X_1-invariant. Further,

$$C_{U_{-1}}(A_{3'}) = A_7A_6A_5A_4A_3A_2, \tag{6.3.5}$$

since we have $A_7A_6A_5A_4A_3A_2 \subseteq C_{U_{-1}}(A_{3'})$ by (I), and the opposite inclusion follows from $C_{A_8}(A_{3'}) = 1$ by 6.1(3). Thus, (6.3.5) holds. By symmetry, we also get $C_{U_{-1}}(A_{7'}) = A_8A_7A_6A_5A_4A_3$. This implies $A_8^{n_1} \leq \big(C_{U_{-1}}(A_{7'})\big)^{n_1} = C_{U_{-1}}(A_{3'}) = A_7A_6A_5A_4A_3A_2$. Moreover, $[A_8, A_{-1}] = 1$ and so $A_8^{n_1} \leq C_{U_{-1}}(A_{1'})$. All in all, we obtain $A_8^{n_1} \leq C_{A_7A_6A_5A_4A_3A_2}(A_{1'})$. Next, we prove

$$C_{A_7A_6A_5A_4A_3A_2}(A_{1'}) = A_5A_4A_3A_2. \tag{6.3.6}$$

By (I), we have $A_5A_4A_3A_2 \subseteq C_{A_7A_6A_5A_4A_3A_2}(A_{1'})$. Therefore, it remains to show that $C_{A_7A_6}(A_{1'}) = 1$ holds. Let $a_7a_6 \in C_{A_7A_6}(A_{1'})$ with $a_7 \in A_7$ and $a_6 \in A_6$. Then we have $1 = [a_7a_6, a_{1'}] = [a_7, a_{1'}]^{a_6}[a_6, a_{1'}]$ for each $a_{1'} \in A_{1'}$. Thus, $[a_7, A_{1'}] \leq A_{3'}$, since $[A_6, A_{1'}] \leq A_{3'}$ and $[A_{3'}, A_6] = 1$. Suppose $a_7 \neq 1$. Then $A_{3'}A_{1'}$ is $\langle a_7, A_{-7} \rangle = X_7$-invariant. Combining 3.5 and 3.6, this yields $[A_4, A_{-7'}] = 1$, contradicting $C_{A_{-7'}}(A_4) = 1$. Further, $a_6 = 1$, since otherwise we get a contradiction to 6.1(3). Hence, (6.3.6) holds.

Therefore, $A_8^{n_1} \leq A_5A_4A_3A_2$, and an application of the nilpotence argument yields $A_8^{n_1} \leq A_4A_3A_2$, since $[X_5, X_1] = 1$ by (I). Thus, $A_8^{n_1} \leq C_{A_4A_3A_2}(A_{-7'})$, since $[A_8, A_{-3'}] = 1$ by (I) and $A_{-3'}^{n_1} = A_{-7'}$ by (6.3.2). This implies $A_8^{n_1} \leq A_3A_2$, as

$[A_3A_2, A_{-7'}] = 1$ by (I) and $C_{A_4}(A_{-7'}) = 1$ by 6.1(3). Finally, we will conclude $A_8^{n_1} \leq A_2$.

Suppose there exists an $a_8 \in A_8$ with $a_8^{n_1} = a_3a_2$ for some $a_2 \in A_2$ and for some $a_3 \in A_3^\sharp$. Then $A_{-7'} = A_{-3'}^{n_1} \geq [a_8, A_{-5'}]^{n_1} = [a_3a_2, A_{-5'}]$. This implies $[a_3, A_{-5'}] \leq A_{-7'}$, since $[A_2, A_{-5'}] \leq A_{-7'}$ and $[A_{-7'}, A_2] = 1$. Therefore, $A_{-7'}A_{-5'}$ is $\langle a_3, A_{-3}\rangle = X_3$-invariant and so $[X_3, A_{-5'}] = 1$ by 3.5. Thus, $[A_{-8}, A_{-3'}] = 1$ by 3.6, contradicting $C_{A_{-3'}}(A_{-8}) = 1$. Therefore, $A_8^{n_1} \leq A_2$. Analogously, $A_2^{n_1} \leq A_8$. Hence, (6.3.3) holds, and, by symmetry, we also get (6.3.4).

Step 4: We prove

$$A_6 \overset{n_1}{\longleftrightarrow} A_4 \tag{6.3.7}$$

and

$$A_{-4} \overset{n_1}{\longleftrightarrow} A_{-6}. \tag{6.3.8}$$

We obtain $C_{M_{-1}}(A_8) = A_{7'}A_6A_{5'}A_4 \overset{n_1}{\longleftrightarrow} C_{M_{-1}}(A_2) = A_6A_{5'}A_4A_{3'}$, since $C_{A_{3'}}(A_8) = 1 = C_{A_{7'}}(A_2)$, $A_{7'}A_6A_{5'}A_4 \leq C_{M_{-1}}(A_8)$ by (I), $A_6A_{5'}A_4A_{3'} \leq C_{M_{-1}}(A_2)$ by (I) and $A_8 \overset{n_1}{\longleftrightarrow} A_2$ by Step 3. From this we analogously obtain $C_{C_{M_{-1}}(A_8)}(A_2) = A_6A_{5'}A_4 \overset{n_1}{\longleftrightarrow} C_{C_{M_{-1}}(A_2)}(A_8) = A_6A_{5'}A_4$, i.e. $A_6A_{5'}A_4$ is invariant under n_1. Finally, we show that $C_{A_6A_{5'}A_4}(A_{-2}) = A_6$ and $C_{A_6A_{5'}A_4}(A_{-8}) = A_4$. By (I), we get $A_6 \subseteq C_{A_6A_{5'}A_4}(A_{-2})$. To obtain the opposite conclusion, we show $C_{A_{5'}A_4}(A_{-2}) = 1$. Let $a_{5'}a_4 \in C_{A_{5'}A_4}(A_{-2})$ with $a_{5'} \in A_{5'}$ and $a_4 \in A_4$. Then we have $1 = [a_{5'}a_4, a_{-2}] = [a_{5'}, a_{-2}]^{a_4}[a_4, a_{-2}]$ for each $a_{-2} \in A_{-2}$. This implies $[a_4, A_{-2}] \leq A_{7'}$, since $[A_{5'}, A_{-2}] \leq A_{7'}$ and $[A_{7'}, A_4] = 1$. Suppose $a_4 \neq 1$. Then $A_{-2}A_{-1'}A_{7'}$ is $\langle a_4, A_{-4}\rangle = X_4$-invariant. Thus, $A_{-2}^{n_4} \leq A_6 \cap U_{-6} = 1$ by 3.1(3), a contradiction. Moreover, $a_{5'} = 1$, since $C_{A_{5'}}(A_{-2}) = 1$. Analogously, $C_{A_6A_{5'}A_4}(A_{-8}) = A_4$. Hence, we get (6.3.7), as $A_6A_{5'}A_4$ is invariant under n_1 and by (6.3.4). By symmetry, we also have (6.3.8).

Step 5: By Step 2, Step 4 and 6.1(2), we have

$$A_{-1'} = [A_6, A_{-3'}] \overset{n_1}{\longleftrightarrow} [A_4, A_{-7'}] = A_{1'}.$$

Finally, we show

$$A_7 \overset{n_1}{\longleftrightarrow} A_3 \tag{6.3.9}$$

and

$$A_{-7} \overset{n_1}{\longleftrightarrow} A_{-3}. \tag{6.3.10}$$

By Step 3, we have $C_{U_{-1}}(A_{7'}) = A_8A_7A_6A_5A_4A_3$ and $C_{U_{-1}}(A_{3'}) = A_7A_6A_5A_4A_3A_2$.

Further, we obtain from this $C_{C_{U_{-1}}(A_{3'})}(A_{7'}) = C_{C_{U_{-1}}(A_{7'})}(A_{3'}) = A_7A_6A_5A_4A_3$, i.e. $M := A_7A_6A_5A_4A_3$ is invariant under n_1. Thus, arguing as in Step 3, we get $C_M(A_{-1'}) = A_7A_6A_5$ and $C_M(A_{1'}) = A_5A_4A_3$. Using the nilpotence argument, this implies $A_7A_6 \overset{n_1}{\longleftrightarrow} A_4A_3$. By (6.3.2), we obtain $A_7 = C_{A_7A_6}(A_{-3'}) \overset{n_1}{\longleftrightarrow}$

$C_{A_4A_3}(A_{-7'}) = A_3$, because $[A_7, A_{-3'}] = 1 = [A_3, A_{-7'}]$ by (I) and $C_{A_6}(A_{-3'}) = 1 = C_{A_4}(A_{-7'})$ by 6.1(3). Thus, (6.3.9) holds. By symmetry, we also obtain (6.3.10).

Hence, we have $A_\beta^{n_1} = A_{\beta^{w_1}}$ for each $\beta \in \tilde{\Phi}$. □

By 6.2 and 6.3, condition (III) is satisfied, if 1.2 (B) does not hold. Hence, G is in this case of type ${}^2F_4$ by [10, Theorem 1].

Finally, we analyse 1.2 (A) in detail. We have already seen in Section 4 that the A_α, $\alpha \in J$, are all elementary Abelian 2-groups in this case. By 6.1, an application of 2.1(7) to the operation of X_α on W_α, $\alpha \in J$, yields that the A_β, $\beta \in \tilde{\Phi} \setminus \Phi$, are also elementary Abelian 2-groups. The restrictions of the surjective homomorphism $\sigma\colon G \to \overline{G}$ described on page 26 to root subgroups A_r are injective (since $\ker \sigma \leq H$, where $H := \left\langle H_r | r \in \tilde{\Phi} \right\rangle$ with $H_r := N_{X_r}(A_r) \cap N_{X_r}(A_{-r})$). Since a root subgroup of a Moufang octagon is of exponent at most 4, we have $a_\gamma^4 = 1$ for each $a_\gamma \in A_\gamma$ with $\gamma \in \Phi \setminus J$. All in all, we get a far reaching conformity with the properties of root subgroups corresponding to Moufang octagons (see [17]).

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