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Minkowski Minimal Surfaces in $\mathbb{R}^3_1$ with Minimal Focal Surfaces

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Abstract. In Euclidean geometry a regular point on a focal surface of a minimal surface has non negative Gauss-curvature. So a focal surface of a minimal surface can never be a minimal surface. We classify the minimal surfaces in Minkowski 3-space the focal surfaces of which are minimal surfaces again.
A (Euclidean) minimal surface $\Phi \subset \mathbb{R}^3$ only carries points with $K_e \leq 0$, where K_e denotes the Gauss-curvature. Parametrizing Φ by a C^2-immersion $f : U \subseteq \mathbb{R}^2 \rightarrow f(U) = \Phi$ and assuming $K_e < 0$, the principal curvatures (eigenvalues of the shape operator) are $k_{1,2} = \varphi(-K_e)^{1/2}, (\varphi = \pm 1)$. Then the focal surfaces Ψ_φ are parametrized by $z_\varphi = f + (k_{1,2})^{-1} n_e = f + \varphi(-K_e)^{-1/2} n_e$, where n_e is the unit normal vector. If a focal point is a regular point on Ψ_φ, then the Gauss-curvature of Ψ_φ is positive. More precisely it is $K_e(\Psi_\varphi) = -1/4\, K_e > 0$ ([10]), so a focal surface of a minimal surface cannot be a minimal surface again.

We will prove that in Minkowski (or Lorentz) 3-space $\mathbb{R}^3_1$ there are (up to scaling and Minkowski isometries) exactly two one-parameter families of minimal surfaces with this property.

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1. Preliminaries

A *Minkowski* (or *Lorentz*) 3-*space* $\mathbb{R}^3_1$ is $(\mathbb{R}^3, \langle x, y\rangle)$, where $\langle x, y\rangle$ is the scalar product

$$\langle x, y\rangle := x_1y_1 + x_2y_2 - x_3y_3,\ x = (x_1, x_2, x_3). \tag{1}$$

A vector $x \in \mathbb{R}^3_1$ is called

$$\begin{aligned} &\textit{spacelike} && \Longleftrightarrow && \langle x, x\rangle > 0, \\ &\textit{timelike} && \Longleftrightarrow && \langle x, x\rangle < 0, \\ &\textit{isotropic}(\textit{lightlike}) && \Longleftrightarrow && \langle x, x\rangle = 0, \quad x \neq 0. \end{aligned}$$

The *(Minkowski-) length* of a vector x is defined by

$$\|x\| := \sqrt[+]{|\langle x, x\rangle|} \geq 0. \tag{2}$$

The *(Minkowski-) crossproduct* is $x \times y$, where

$$\langle x \times y, z\rangle = \det(x, y, z). \tag{3}$$

A surface Φ in $\mathbb{R}^3_1$ is locally parametrized by $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. Here and in the following we assume $f \in C^\infty$. Local coordinates are denoted by (u, v) or $(u^1 := u, u^2 := v)$. Partial derivatives of a function s defined in $U \subset \mathbb{R}^2$ are denoted by

$$s_{,j} := \frac{\partial}{\partial u^j}\, s, \quad s_{,jk} := \frac{\partial^2}{\partial u^j \partial u^k}\, s.$$

Via f the scalar product (1) in $\mathbb{R}^3_1$ induces a (pseudo-)Riemannian metric on U, the *first fundamental form* (I) with components

$$g_{jk} := \left\langle \frac{\partial}{\partial u^j} f, \frac{\partial}{\partial u^k} f \right\rangle : U \to \mathbb{R}. \tag{4}$$

Denoting $\Delta := \det(g_{jk})$ a surface $\Phi = f(U)$ is called

$$\begin{aligned} &\textit{spacelike} && \Longleftrightarrow && \Delta > 0 && \text{in } U, \\ &\textit{timelike} && \Longleftrightarrow && \Delta < 0 && \text{in } U. \end{aligned}$$

Points with $\Delta = 0$ are excluded. The normal vector is

$$n := \frac{f_{,1} \times f_{,2}}{\| f_{,1} \times f_{,2} \|} = \frac{f_{,1} \times f_{,2}}{\sqrt{|\Delta|}}, \tag{5}$$

because of $\Delta = -\langle f_{,1} \times f_{,2}\, , f_{,1} \times f_{,2}\rangle$. Especially points with $\Delta \neq 0$ are regular. In the following we denote $\varepsilon := \langle n, n\rangle$. So in case of spacelike and timelike surfaces we have $\varepsilon = -1$ and $\varepsilon = 1$ and the *spherical image* $n(U)$ is part of the two-sheet hyperboloid $\langle x, x\rangle = -1$ and the one-sheet hyperboloid $\langle x, x\rangle = 1$, respectively. The *second fundamental form* (II) and the *shape operator* S are related by $\langle S(f_{,j}), f_{,k}\rangle = \varepsilon\,(\mathrm{II})(f_{,j}, f_{,k})$, where

$$(II)(f_j, f_k) \quad =: \quad h_{jk} = \varepsilon\langle n, f_{,jk}\rangle = \frac{\varepsilon}{\sqrt{|\Delta|}}\det(f_{,1}, f_{,2}, f_{jk}), \tag{6}$$

$$S(f_{,j}) = -n_{,j} \quad =: \quad h_j^s f_{,s} \text{ with } h_j^s = \varepsilon h_{jk} g^{ks}. \tag{7}$$

Mean curvature and *Gauss-curvature* are

$$H \;:=\; \frac{1}{2}\mathrm{tr}S = \frac{\varepsilon}{2\Delta}(h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}), \tag{8}$$

$$K \;:=\; \varepsilon\det S = \varepsilon\frac{\det(h_{jk})}{\Delta}. \tag{9}$$

In case of K we use the sign convention according to [6], [9] and [11]. From (6) and (9) it is easy to see, that Euclidean and Minkowski Gauss-curvature have different sign. The eigenvalues of S (principal curvatures of Φ) are

$$k_{1,2} = H \pm \sqrt{H^2 - \varepsilon K},\ \varphi = \pm 1. \tag{10}$$

The integrability conditions of Codazzi and the Theorema egregium are

$$Co \;:\; h_{js},_k - h_{jk},_s = \Gamma^p_{jk}h_{ps} - \Gamma^p_{js}h_{pk}, \tag{11}$$

$$Ga \;:\; R_{pjks} = R^t_{jks}g_{pt} = h_{jk}h_{sp} - h_{js}h_{kp}. \tag{12}$$

A regular surface $\Phi = f(U) \subset \mathbb{R}^3_1$ is called a *(Minkowski-) minimal surface* iff $H = 0$ in U. For discussions on these surfaces see for instance [1], [3], [6], [7], [8], [11], [12], [13], [15], [16].

We need some concepts from affine differential geometry. For details see for instance [14], [2]. From affine point of view a surface $f(U)$ is *nondegenerate* if

$$D := \det(D_{jk}) \neq 0,\ D_{jk} := \det(f_{,1}, f_{,2}, f_{,jk}). \tag{13}$$

By (6) and (9) we have $D \neq 0 \iff K \neq 0$. The components of the *affine metric* G of $f(U)$ are

$$G_{jk} := |D|^{-1/4}D_{jk}, \tag{14}$$

and the *affine normal vector* of $\Phi = f(U)$ is

$$n_a := (1/2)\Delta_G f, \tag{15}$$

where Δ_G is the Laplacian with respect to the affine metric. The *affine shape operator* B is defined by $B(f_{,j}) = -n_a{}_{,j}$, so the components B_j^k of B are given by

$$n_a{}_{,j} =: -B_j^k f_{,k}\,. \tag{16}$$

Affine curvature K_a and *affine mean curvature* H_a are

$$K_a := \det(B),\ H_a := (1/2)\mathrm{tr}\,B. \tag{17}$$

In case of a constant affine normal vector the surface Φ is called an *improper affine sphere* and it is $B = 0$, $K_a = H_a = 0$.

2. Focal surfaces of minimal surfaces

Excluding points with $K = 0$ on the minimal surface $\Phi = f(U)$ the principal curvatures of Φ are from (10) $k_{1,2} = \varphi(-\varepsilon K)^{1/2}, (\varphi = \pm 1)$. Then the *focal surfaces* Ψ_φ of Φ are parametrized by

$$z_\varphi = f + (k_{1,2})^{-1} n = f + \varphi(-\varepsilon K)^{-1/2} n,\ \varphi = \pm 1. \tag{18}$$

In case of a spacelike minimal surface ($\varepsilon = -1$) the Gauss-curvature is positive (cf. [8, p. 298], [6, p. 518]), so we have two different real nonzero eigenvalues $\pm\sqrt{K}$. A timelike minimal surface ($\varepsilon = 1$) has real focal surfaces iff $K < 0$ that means by (6) and (9), that Φ is locally strongly convex. In the following we always assume real focal points so from (18) there are two distinct real nonzero eigenvalues of S. That means locally we can take lines of curvature (defined in the usual way) as parametric lines. These parameters can be normalized in case of minimal surfaces in the following way.

Lemma 1. (T. Weinstein [15, p. 160]) *Let Φ be a minimal surface in $\mathbb{R}^3_1$ with $K \neq 0$ and real focal surfaces. Then locally there is a parametrization $f : U \to \mathbb{R}^3_1$ of Φ, such that*

$$g := g_{11} > 0, g_{22} = -\varepsilon g, g_{12} = 0,$$
$$h_{11} = 1, h_{22} = \varepsilon, h_{12} = 0,$$

where $\varepsilon = -1$ and $\varepsilon = 1$ refers to spacelike and timelike surfaces respectively.

The coordinate functions g_{jk} and h_{jk} of (I) and (II) respectively in Lemma 1 fulfil the Codazzi condition (11) while the Theorema egregium (12) reads

$$G_a:\ g(g'' - \varepsilon\ddot{g} + 2) = g'^2 - \varepsilon\dot{g}^2, \tag{19}$$

where here as in the following the derivatives of $g : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ are denoted by

$$\dot{g} := \frac{\partial g}{du},\ \ddot{g} := \frac{\partial^2 g}{du^2},\ g' := \frac{\partial g}{dv},\ g'' := \frac{\partial^2 g}{dv^2},\ \dot{g}' := \frac{\partial^2 g}{du\,dv}.$$

The following lemma gives some affine quantities of $f(U) = \Phi$ using the parameters of Lemma 1.

Lemma 2. [10] *Let Φ be a minimal surface in $\mathbb{R}^3_1$ with $K \neq 0$. In parameters of Lemma 1 the following holds. The affine normal is*

$$n_a(\Phi) = \frac{1}{2g^{3/2}}(\varepsilon\dot{g} f_{,1} + g' f_{,2} + 2\varepsilon g\, n), \tag{20}$$

and the components B^k_j of the affine shape operator B are

$$B^1_1 = \frac{1}{g^{3/2}}\left(\frac{\varepsilon\dot{g}^2}{2g} - \frac{g'^2}{4g} - \frac{\varepsilon\ddot{g}}{2} + 1\right), \tag{21}$$

$$B^2_1 = \frac{1}{2g^{3/2}}\left(\dot{g}' - \frac{\dot{g}g'}{2g}\right), \tag{22}$$

$$B^1_2 = \frac{-\varepsilon}{2g^{3/2}}\left(\dot{g}' - \frac{\dot{g}g'}{2g}\right), \tag{23}$$

$$B^2_2 = \frac{1}{g^{3/2}}\left(\frac{-\varepsilon\dot{g}^2}{4g} + \frac{g'^2}{2g} - \frac{g''}{2} - 1\right). \tag{24}$$

The next lemma gives some quantities of the focal surfaces Ψ_1 and Ψ_{-1} of $\Phi = f(U)$ where f is a parametrization of Φ according to Lemma 1.

Lemma 3. [10] *Let $f(U) = \Phi$ be a minimal surface in $\mathbb{R}^3_1$ with $K \neq 0$. The focal surfaces Ψ_φ of $\Phi = f(U)$ are parametrized by $z_\varphi = f + \varphi(-\varepsilon K)^{-1/2}n$, ($\varphi = \pm 1$). In parameters of Lemma 1 the Gauss-curvature of Φ is $K = -\varepsilon/g^2$ such that $z_\varphi = f + \varphi g\, n$. If Ψ_φ is regular, the following holds.*

(1) *In case of spacelike surfaces ($\varepsilon = -1$) the mean curvature of Ψ_φ is*

$$H(\Psi_1) = \frac{1}{8|g|^{3/2}|g'|}(4g - g'^2 - \dot{g}^2) \tag{25}$$

$$H(\Psi_{-1}) = \frac{-1}{8|g|^{3/2}|\dot{g}|}(4g - g'^2 - \dot{g}^2). \tag{26}$$

(2) *In case of timelike surfaces ($\varepsilon = 1$) the mean curvature of Ψ_φ is*

$$H(\Psi_1) = \frac{-1}{8|g|^{3/2}|\dot{g}|}(-4g + g'^2 - \dot{g}^2) \tag{27}$$

$$H(\Psi_{-1}) = \frac{-1}{8|g|^{3/2}|g'|}(-4g + g'^2 - \dot{g}^2). \tag{28}$$

(3) *Denoting by $g^*_{jk} := \langle z_{\varphi,j}\,, z_{\varphi,k}\rangle$ the components of the metric of $\Psi_\varphi = z_\varphi(U)$ it holds*

$$\Delta^* = \det(g^*_{jk}) = 2g[\varepsilon g'^2(1 - \varepsilon\varphi) - \dot{g}^2(1 + \varepsilon\varphi)]. \tag{29}$$

From this a regular focal surface of a spacelike minimal surface ($\varepsilon = -1$) is timelike, whereas for a timelike minimal surface ($\varepsilon = 1$) one focal surface is spacelike (or singular) and one timelike (or singular).

Concerning the focal surfaces we repeat a result from [10].

Theorem 1. *Let $f(U) = \Phi \subset \mathbb{R}^3_1$ be a regular minimal surface with $K \neq 0$ in U and real focal surfaces Ψ_1, Ψ_{-1}. Then the following holds for $\varphi \in \{1, -1\}$:*

(a) *If Ψ_φ is a regular surface, then $K(\Psi_\varphi) = -1/4\, K$, where $K(\Psi_\varphi)$ denotes the Gauss-curvature of Ψ_φ.*

(b) *If Ψ_φ is a regular surface, then it is non degenerate and the affine normal of Ψ_φ intersects the affine normal of Φ orthogonally.*

(c) *If Ψ_φ is a regular surface then it is an affine minimal surface.*

(d) *If Ψ_1 and Ψ_{-1} are both regular surfaces, then $K_a(\Psi_1) : K_a(\Psi_{-1}) = (-\varepsilon)H(\Psi_1)^4 : H(\Psi_{-1})^4$, where $K_a(\Psi_\varphi)$ and $H(\Psi_\varphi)$ are the affine Gauss-curvature and the Minkowski mean curvature of Ψ_φ respectively.*

3. Associated minimal surfaces

Using (I)-isothermal coordinates in case of a spacelike minimal surface we have

$$H = 0 \Longleftrightarrow h_{11} + h_{22} = 0 \Longleftrightarrow f,_{11} + f,_{22} = 0, \tag{30}$$

so the coordinate functions $f^\alpha : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(\alpha = 1, 2, 3)$ are harmonic. The conjugate harmonic functions

$$\bar{f}^\alpha : U \subseteq \mathbb{R}^2 \to \mathbb{R},\ (\alpha = 1, 2, 3),$$

related to f^α by

$$f,^\alpha_1 = \bar{f},^\alpha_2\,, \quad f,^\alpha_2 = -\bar{f},^\alpha_1\ (\alpha = 1, 2, 3) \tag{31}$$

determine the *adjoined minimal surface* $\bar{\Phi} := \bar{f}(U)$, and the 1-parameter family ${}^{(\lambda)}\Phi$ of *associated minimal surfaces*, parametrized by

$${}^{(\lambda)}f(u, v) := f(u, v)\cos\lambda + \bar{f}(u, v)\sin\lambda,\ \lambda \in \mathbb{R}. \tag{32}$$

A *timelike* minimal surface in $\Phi \subset \mathbb{R}^3_1$ is locally a surface of translation with isotropic generating curves $g(I)$ and $h(J)$:

$$f(u, v) = g(u) + h(v), \quad g : I \subseteq \mathbb{R} \to \mathbb{R}^3_1,\ h : J \subseteq \mathbb{R} \to \mathbb{R}^3_1, \tag{33}$$

where

$$\langle \dot{g}, \dot{g}\rangle = \langle h', h'\rangle = 0;\ \ \dot{g} := \frac{dg}{du},\ h' := \frac{dh}{dv}$$

(see [15, p. 184] or [7, p. 338]). So we have

$$g_{11} = g_{22} = 0,\ \ g_{12} = \langle \dot{g}, h'\rangle \neq 0. \tag{34}$$

The *conjugate minimal surface* $\bar{\Phi}$ is locally parametrized by

$$\bar{f}(u, v) = g(u) - h(v), \tag{35}$$

and the family of associated minimal surfaces ${}^{(\mu)}\Phi$ is parametrized by

$${}^{(\mu)}f(u, v) = f(u, v)\cosh\mu + \bar{f}(u, v)\sinh\mu,\ \mu \in \mathbb{R}. \tag{36}$$

So the conjugate surface $\bar{f}(U)$ does not belong to the family of associated surfaces ([7, p. 338]).

Associated surfaces are well known to share metric (I), normal vector and Gauss-curvature as in the Euclidean situation (see [15, p. 184]). From [10, Theorem 4] the affine normalvector is invariant too.

4. Minimal focal surfaces

In this section we will classify minimal surfaces the focal surfaces of which are minimal surfaces again. To get examples, we consider minimal surfaces of rotation with isotropic axis and their associates.

Taking the axis g of rotation $x_1 = x_3, x_2 = 0$ we can parametrize the generating curve $c(v) = (v+m(v), 0, -v+m(v))$, where $m : I \subseteq \mathbb{R} \to \mathbb{R}$. Then a parametrization of a surface of rotation with axis g is (see for instance [8, p. 305], [16, p. 348] or Section 5 in [10])

$$\begin{aligned} f(u,v) &= \begin{pmatrix} 1-\frac{u^2}{2} & u & \frac{u^2}{2} \\ -u & 1 & u \\ -\frac{u^2}{2} & u & 1+\frac{u^2}{2} \end{pmatrix} \begin{pmatrix} v+m(v) \\ 0 \\ -v+m(v) \end{pmatrix} \\ &= [v+m(v)-u^2v, -2uv, -v+m(v)-u^2v]. \end{aligned} \tag{37}$$

An easy calculation gives $H = 0 \iff m(v) = \pm a^2v^3 + b, a \neq 0$. By a translation we can take $b = 0$ and we get

$$f(u,v) = [\pm a^2v^3 + v - u^2v, -2uv, \pm a^2v^3 - v - u^2v]. \tag{38}$$

Change of coordinates by $u = u'$, $v = (a\sqrt{3})^{-1}v'$ gives the isothermic representation

$$f'(u',v') = \left[\frac{v'(3-3u'^2-\varepsilon v'^2)}{3a}, -\frac{2u'v'}{a}, \frac{v'(-3-3u'^2-\varepsilon v'^2)}{3a}\right], \tag{39}$$

where $\varepsilon = -1$ and $\varepsilon = 1$ leads to spacelike and timelike surfaces respectively. After rescaling and writing without primes the standard representation is

$$f(u,v) = \left[-\varepsilon\frac{v^3}{3} + v - u^2v, -2uv, -\varepsilon\frac{v^3}{3} - v - u^2v\right]. \tag{40}$$

These minimal surfaces are well known. In case of $\varepsilon = -1$ the surface is called *spacelike parabolic catenoid*, see [1, p. 302], [8, p. 300], [10], [11, p. 60], [16, p. 349]. In case of $\varepsilon = 1$ it is a *timelike parabolic catenoid*, see [10], [11, p. 74], [16, p. 351].

In the following we calculate the associates of the surfaces (40).

In the spacelike case ($\varepsilon = -1$) we have from (40)

$$f(u,v) = \left[\frac{v^3}{3} + v - u^2v, -2uv, \frac{v^3}{3} - v - u^2v\right], \tag{41}$$

and the adjoined surface is parametrized by

$$\bar{f}(u,v) = \left[\frac{u^3}{3} - u - uv^2, u^2 - v^2, \frac{u^3}{3} + u - uv^2\right]. \tag{42}$$

$\bar{\Phi} = \bar{f}(\mathbb{R}^2)$ is a spacelike ruled minimal surface of order three (Cayley surface) ([1, p. 305], [5, p. 15], [8, p. 301], [11, p. 62], [16, p. 351]). The surface is well known to be the orbit of a straight line applying a certain screw motion in $\mathbb{R}^3_1$ ([4, p. 311]).

In the following we denote by $\Sigma_s := \{{}^{(\lambda)}\Phi, \lambda \in \mathbb{R}\}$ the pencil of associates of surfaces (41). According to (32), ${}^{(\lambda)}\Phi$ is parametrized by

$$ {}^{(\lambda)}f(u,v) := f(u,v)\cos\lambda + \bar{f}(u,v)\sin\lambda\,,\ \lambda \in \mathbb{R}. \tag{43}$$

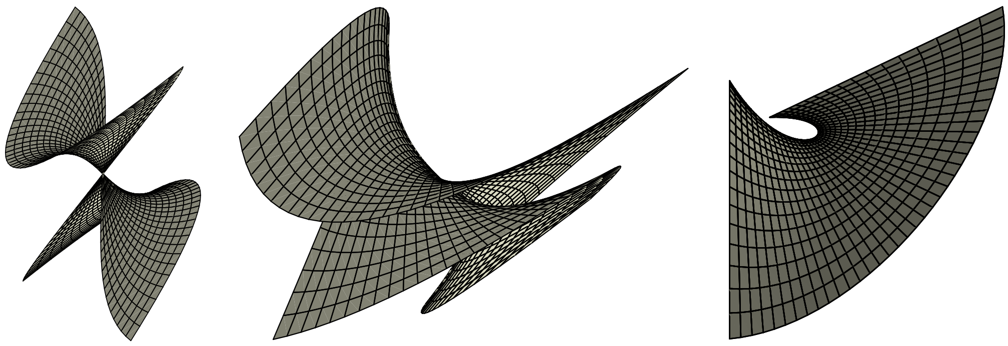

Figure 1. Associated spacelike surfaces ${}^{(\lambda)}\Phi \in \Sigma_s$: $\lambda = 0$ (surface of rotation: *spacelike parabolic catenoid*), $\lambda = 3\pi/8$, $\lambda = \pi/2$ (Cayley-ruled surface)

The surfaces are regular iff $v \neq 0$. Calculating normal n and Gauss-curvature K for surfaces (50) we get (independently of λ)

$$ n(u,v) = \left[\frac{1-u^2-v^2}{2v}, -\frac{u}{v}, -\frac{1+u^2+v^2}{2v}\right],\ \ K(u,v) = \frac{1}{4v^4}. \tag{44}$$

Inserting this according to (18) in the parametrization

$$ {}^{(\lambda)}z_\varphi(u,v) = {}^{(\lambda)}f(u,v) + \varphi(-\varepsilon K(u,v))^{-1/2} n(u,v),\ (\varphi = \pm 1) \tag{45}$$

of the focal surfaces ${}^{(\lambda)}\Psi_\varphi$, a straightforward calculation gives $H({}^{(\lambda)}\Psi_\varphi) = 0$ for regular focal surfaces ${}^{(\lambda)}\Psi_\varphi$.

In the timelike case ($\varepsilon = 1$) we have from (40)

$$ f(u,v) = \left[-\frac{v^3}{3} + v - u^2 v, -2uv, -\frac{v^3}{3} - v - u^2 v\right]. \tag{46}$$

Change of parameters $u = -(u'+v')/2, v = (-u'+v')/2$ gives a representation as a surface of translation with isotropic generating curves (u', v' nullcoordinates). If we write without primes again this reads

$$ \begin{aligned} f(u,v) \;\; &= \;\; g(u) + h(v) := \\ &\left[\frac{u^3}{6} - \frac{u}{2}, -\frac{u^2}{2}, \frac{u^3}{6} + \frac{u}{2}\right] + \left[-\frac{v^3}{6} + \frac{v}{2}, \frac{v^2}{2}, \frac{-v^3}{6} - \frac{v}{2}\right]. \end{aligned} \tag{47}$$

Then the conjugate surface is parametrized by

$$\bar{f}(u,v) = g(u) - h(v). \tag{48}$$

We denote the pencil of associates of surfaces (46) by $\Sigma_t := \{^{(\mu)}\Phi, \mu \in \mathbb{R}\}$. According to (36), $^{(\mu)}\Phi$ is parametrized by

$$^{(\mu)}f(u,v) := f(u,v)\cosh\mu + \bar{f}(u,v)\sinh\mu\,,\ \mu \in \mathbb{R}. \tag{49}$$

The surfaces are regular iff $u \neq v$. The normal and the Gauss-curvature are

$$n(u,v) = \left[\frac{uv-1}{v-u}, \frac{-v-u}{v-u}, \frac{uv+1}{v-u}\right],\ \ K(u,v) = -\frac{4}{(v-u)^4}. \tag{50}$$

As above a straight forward calculation shows that regular focal surfaces $^{(\mu)}\Psi_\varphi$ are minimal.

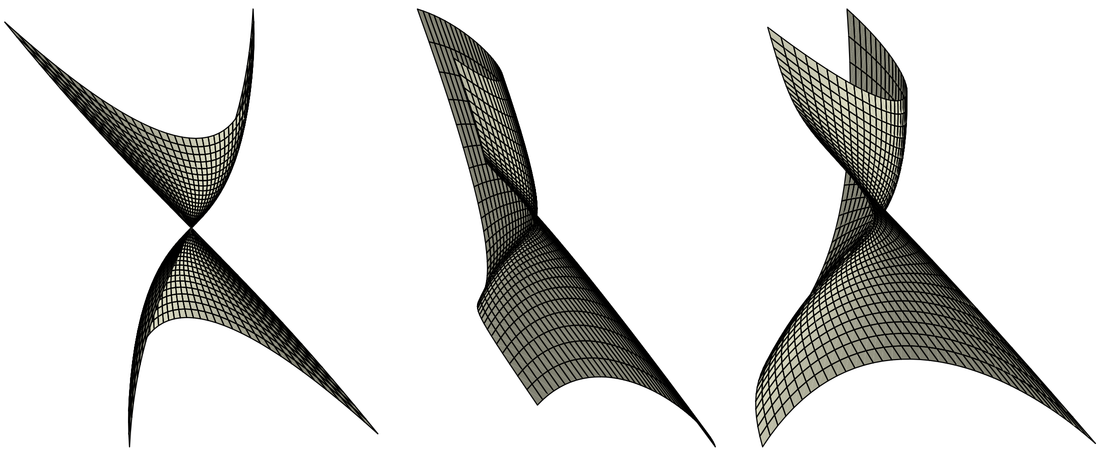

Figure 2. Associated timelike surfaces $^{(\mu)}\Phi \in \Sigma_t$: $\mu = 0$ (surface of rotation: timelike parabolic catenoid), $\mu = 3\pi/4$, $\mu = \pi/2$

Remark 1. The surface (48) is

$$\bar{f}(u,v) = \left[\frac{v^3}{6} - \frac{v}{2} + \frac{u^3}{6} - \frac{u}{2}, -\frac{1}{2}(u^2+v^2), \frac{v^3}{6} + \frac{v}{2} + \frac{u^3}{6} + \frac{u}{2}\right].$$

This fulfils the equation

$$-12x_1 + (x_1 - x_3)^3 + 6(x_1 - x_3)(x_2 + 1) = 0.$$

Replacing (x_1, x_2, x_3) by $((x_1 + x_3)/\sqrt{2}, x_2, (x_1 - x_3)/\sqrt{2})$ and rescaling gives the equation

$$x_3^3 + 3x_2x_3 - 3x_1 = 0,$$

so the surface is a Cayley ruled minimal surface again but in a timelike version. Because of $-\varepsilon K < 0$ holds for $\bar{f}(U)$, this surface and their associates have no real focal surfaces.

The following theorem contains a classification of minimal surfaces the focal surfaces of which are minimal.

Theorem 2. *Assume Φ a minimal surface in $\mathbb{R}^3_1$ and Ψ a regular focal surface of Φ. Then the following assertions are equivalent:*

(A) *Ψ is a minimal surface.*

(B) *In parameters according to Lemma 1 the first fundamental form of $\Phi = f(U)$ has the components*

$$\begin{aligned} g_{11}(u,v) &=: g(u,v) = (au+bv)^2,\ (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\},\ b^2 - \varepsilon a^2 = 1, \\ g_{12}(u,v) &= 0, \\ g_{22}(u,v) &= -\varepsilon g(u,v), \end{aligned} \tag{N}$$

where $\varepsilon = -1$ and $\varepsilon = 1$ refers to spacelike and timelike surfaces respectively.

(C) *Φ is an improper affine sphere.*

(D) *Φ is a minimal surface of rotation with isotropic axis or an associate of such a surface: $\Phi \in \{\Sigma_s \cup \Sigma_t\}$.*

Proof. (A) $\Rightarrow$ (B): Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3_1$ be a local parametrization of Φ according to Lemma 1. This means up to a Minkowski isometry in $\mathbb{R}^3_1$ the surface is determined by a C^∞-function $g := g_{11} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^+$. Using Lemma 3 a focal surface Ψ_φ ($\varphi = \pm 1$) of Φ is parametrized by $z_\varphi := f + \varphi g n$ and by (25), (26) and (27), (28) respectively we have

$$H(\Psi_\varphi) = 0 \Longleftrightarrow g'^{\,2} - \varepsilon \dot{g}^2 = 4g. \tag{51}$$

From (19) the Theorema egregium is

$$g(g'' - \varepsilon \ddot{g} + 2) = g'^2 - \varepsilon \dot{g}^2,$$

what implies together with (51)

$$g'' - \varepsilon \ddot{g} = 2. \tag{52}$$

Differentiating (51) by u and v and multiplying by g' and $\dot{g}$ respectively gives

$$\begin{aligned} g'^{\,2} \dot{g}' - \varepsilon \dot{g} g' \ddot{g} &= 2\dot{g} g' \\ \dot{g} g' g'' - \varepsilon \dot{g}^2 \dot{g}' &= 2\dot{g} g'. \end{aligned}$$

Adding these equations using (51) and (52) gives

$$2g\dot{g}' - \dot{g} g' = 0,$$

or

$$g(u,v) = (\alpha(u) + \beta(v))^2, \tag{53}$$

because of $g > 0$. Inserting in (51) gives

$$\beta'^{\,2} - \varepsilon \dot{\alpha}^2 = 1.$$

Using this and (53) calculating (52) gives

$$\alpha(u) = au + a_1,\ \beta(v) = bv + b_1,\ a, b, a_1, b_1 \in \mathbb{R},\ (a,b) \neq (0,0),\ b^2 - \varepsilon a^2 = 1,$$

or

$$g(u,v) = (au + bv + c)^2,\ \ c \in \mathbb{R}.$$

If for instance $a \neq 0$ rescaling $u \to u - \frac{c}{a}$ (which does not affect the other coordinate functions of the fundamental forms) gives

$$g(u,v) = (au + bv)^2,\ \ (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\},\ b^2 - \varepsilon a^2 = 1. \tag{54}$$

(B) $\Rightarrow$ (A) is straight forward from (51).

(B) $\Rightarrow$ (C): (B) is valid means that in terms of parameters according to Lemma 1 the metric of the minimal surface $\Phi = f(U)$ is given by (N). The components B_j^k of the affine shape operator B of $f(U)$ are given by (21)–(24). A short calculation shows $B = 0$, so Φ is an improper affine sphere.

(C) $\Rightarrow$ (B): (C) is valid means $B = 0$. By (21)–(24) this is equivalent to

$$\begin{aligned} 2\,\varepsilon\dot{g}^2 - g'^{\,2} - 2\,\varepsilon g\ddot{g} + 4g &= 0, && (55)\\ -\varepsilon\,\dot{g}^2 + 2g'^{\,2} - 2gg'' - 4g &= 0, && (56)\\ 2g\dot{g}' - \dot{g}g' &= 0. && (57) \end{aligned}$$

Subtracting (56) from (57) and using Theorema egregium (19) gives

$$g'^{\,2} - \varepsilon\dot{g}^2 = 4g.$$

So from (51) it is $H(\Psi_\varphi) = 0$. This means (A) is valid and from the first step it is (A)$\Rightarrow$(B).

(D) $\Rightarrow$ (B):

(a) If $\Phi \in \{\Sigma_s\}$ then by (41), (42) and (43) $\Phi = {}^{(\lambda)}\Phi = {}^{(\lambda)}f(U)$ where

$$\begin{aligned} {}^{(\lambda)}f(u,v) = &\left[\frac{v^3}{3} + v - u^2 v, -2uv, \frac{v^3}{3} - v - u^2 v\right]\cos\lambda\ + \\ &+ \left[\frac{u^3}{3} - u - uv^2, u^2 - v^2, \frac{u^3}{3} + u - uv^2\right]\sin\lambda\,, \end{aligned} \tag{58}$$

$(u,v) \in \mathbb{R}^2, v \neq 0, \lambda \in \mathbb{R}$. The change of parameters

$$\begin{aligned} u(u',v') &= \frac{\sqrt{2}}{2}\left(\cos\frac{\lambda}{2}u' - \sin\frac{\lambda}{2}v'\right)\\ v(u',v') &= \frac{\sqrt{2}}{2}\left(\sin\frac{\lambda}{2}u' + \cos\frac{\lambda}{2}v'\right) \end{aligned} \tag{59}$$

gives for every fixed $\lambda \in \mathbb{R}$ a parametrization according to Lemma 1 with

$$g(u',v') = \left(u'\sin\frac{\lambda}{2} + v'\cos\frac{\lambda}{2}\right)^2 =: (au' + bv')^2. \tag{60}$$

Because ${}^{(\lambda)}\Phi$ is spacelike we have $\varepsilon=-1$, so it holds $b^2-\varepsilon a^2=1$.
(b) If $\Phi\in\{\Sigma_t\}$ then by (47), (48) and (49) $\Phi={}^{(\mu)}\Phi={}^{(\mu)}f(U)$ where

$$\begin{aligned}
{}^{(\mu)}f(u,v) &= (g(u)+h(v))\cosh\mu+(g(u)-h(v))\sinh\mu;\\
g(u) &= \left[\frac{u^3}{6}-\frac{u}{2},-\frac{u^2}{2},\frac{u^3}{6}+\frac{u}{2}\right],\\
h(v) &= \left[-\frac{v^3}{6}+\frac{v}{2},\frac{v^2}{2},-\frac{v^3}{6}-\frac{v}{2}\right],
\end{aligned}\tag{61}$$

$(u,v)\in\mathbb{R}^2, u\neq v, \mu\in\mathbb{R}$. For every fixed μ the reparametrization

$$\begin{aligned}
u(u',v') &= \frac{\sqrt{2}}{2}\left(\cosh\frac{\mu}{2}-\sinh\frac{\mu}{2}\right)(u'-v')\\
v(u',v') &= \frac{\sqrt{2}}{2}\left(\cosh\frac{\mu}{2}+\sinh\frac{\mu}{2}\right)(u'+v')
\end{aligned}\tag{62}$$

gives a parametrization of ${}^{(\mu)}\Phi$ according to Lemma 1 with

$$g(u',v')=\left(u'\sinh\frac{\mu}{2}+v'\cosh\frac{\mu}{2}\right)^2=:(au'+bv')^2.\tag{63}$$

Because of $\varepsilon=1$ it holds $b^2-\varepsilon a^2=1$.
(B) $\Rightarrow$ (D): Assuming (B) holds, the metric of $f(U)=\Phi$ is given by (N) with suitable constants a and b; w.l.o.g. we take $b\geq 0$. From the proof (D)$\Rightarrow$(B) it is easy to see that there is a surface $\tilde{\Phi}\in\{\Sigma_s\cup\Sigma_t\}$, parametrized by $\tilde{f}:U\to\mathbb{R}^3_1$ where the first fundamental form induced by $\tilde{f}$ on U coincides with (N). Then by the theorem of H. Schwartz there is an isometry of $\mathbb{R}^3_1$ the application of which to Φ gives an associate of $\tilde{\Phi}$ (see [15, p. 185, 187]). This means (D) holds. □

The next two theorems classify the focal surfaces of the surfaces characterized in Theorem 2.

Theorem 3. *Assume $\Phi\in\Sigma_s$. Then a regular focal surfaces Ψ_φ of Φ $(\varphi=\pm1)$ is element of Σ_t.*

Proof. A representation of $\Phi={}^{(\lambda)}\Phi\in\Sigma_s$ is given by (58)

$$\begin{aligned}
{}^{(\lambda)}f(u,v)=&\left[\frac{v^3}{3}+v-u^2v,-2uv,\frac{v^3}{3}-v-u^2v\right]\cos\lambda+\\
&+\left[\frac{u^3}{3}-u-uv^2,u^2-v^2,\frac{u^3}{3}+u-uv^2\right]\sin\lambda\,.
\end{aligned}\tag{64}$$

The focal surfaces ${}^{(\lambda)}\Psi_\varphi$ of ${}^{(\lambda)}\Phi$ can be parameterized because of $\varepsilon=-1$ from (18)

$${}^{(\lambda)}z_\varphi={}^{(\lambda)}f+\varphi K^{-1/2}n,\varphi=\pm1,\tag{65}$$

where Gauss-curvature K and normal vector n are given by (44). This gives

$$
\begin{aligned}
&{}^{(\lambda)}z_{\varphi}(u,v) = \\
&\left[\left(\frac{v^3}{3}+v-u^2v\right)\cos\lambda+\left(\frac{u^3}{3}-u-uv^2\right)\sin\lambda+\varphi(v-vu^2-v^3),\right. \\
&-2\,uv\cos\lambda+(u^2-v^2)\sin\lambda+\varphi(-2uv), \qquad (66)\\
&\left.\left(\frac{v^3}{3}-v-u^2v\right)\cos\lambda+\left(\frac{u^3}{3}+u-uv^2\right)\sin\lambda+\varphi\left(-v-vu^2-v^3\right)\right].
\end{aligned}
$$

From (64) and (65) we get

$$
{}^{(\lambda+\pi)}f = -{}^{(\lambda)}f \;\Rightarrow\; {}^{(\lambda)}z_1 = -{}^{(\lambda+\pi)}z_{-1}, {}^{(\lambda+\pi)}z_1 = -{}^{(\lambda)}z_{-1}. \qquad (67)
$$

Therefore we take without restriction $\lambda \in [-\pi/2, \pi/2]$ and $\lambda \in [\pi/2, 3\pi/2]$ in case of $\varphi = 1$ and $\varphi = -1$ respectively.

In case of $\varphi = 1$ the reparametrization

$$
\begin{aligned}
u(u',v') &= \left(\frac{c+1}{8}\right)^{1/2}u' + \left(\frac{c-1}{8}\right)^{1/2}v' \\
v(u',v') &= \left(\frac{c-1}{8}\right)^{1/2}u' - \left(\frac{c+1}{8}\right)^{1/2}v' \qquad (68)
\end{aligned}
$$

with $c := (\cos\frac{\lambda}{2})^{-1} \in [1, \sqrt{2}\,]$ gives the coordinate functions of the fundamental forms of the focal surface ${}^{(\lambda)}z_1(U') = {}^{(\lambda)}\Psi_1$ according to Lemma 1

$$
\begin{aligned}
g^*_{11}(u',v') &= : g^*(u',v') = \left(-\left(\frac{c-1}{2}\right)^{1/2}u' + \left(\frac{c+1}{2}\right)^{1/2}v'\right)^2 =: (au'+bv')^2 \\
g^*_{22} &= -g^*_{11},\ g^*_{12} = 0,\ h^*_{11} = h^*_{22} = 1,\ h^*_{12} = 0. \qquad (69)
\end{aligned}
$$

The surfaces ${}^{(\lambda)}\Psi_1$ are timelike ($\varepsilon^* = 1$), so it is $b^2 - \varepsilon^* a^2 = 1$. Theorem 2 gives ${}^{(\lambda)}\Psi_1 \in \Sigma_t$.

In case of $\varphi = -1$ the reparametrization

$$
\begin{aligned}
u(u',v') &= \left(\frac{s+1}{8}\right)^{1/2}u' + \left(\frac{s-1}{8}\right)^{1/2}v' \\
v(u',v') &= -\left(\frac{s-1}{8}\right)^{1/2}u' + \left(\frac{s+1}{8}\right)^{1/2}v' \qquad (70)
\end{aligned}
$$

with $s := (\sin\frac{\lambda}{2})^{-1}$ leads analogously to

$$
g^*(u',v') = \left(-\left(\frac{s-1}{2}\right)^{1/2}u' + \left(\frac{s+1}{2}\right)^{1/2}v'\right)^2 \qquad (71)
$$

and as above ${}^{(\lambda)}\Psi_{-1} \in \Sigma_t$. □

Remark 2. From (69), (71) and (63), the focal surface ${}^{(\lambda)}\Psi_1$ and ${}^{(\lambda)}\Psi_{-1}$ of ${}^{(\lambda)}\Phi$ is up to a Minkowski isometry the surface ${}^{(\mu)}\Phi \in \Sigma_t$ where λ and μ are related by

$$\begin{aligned} {}^{(\lambda)}\Psi_1 \quad &: \quad \mu = \ln \frac{1-\sin\frac{\lambda}{2}}{\cos\frac{\lambda}{2}}, \quad \lambda \in [-\pi/2, \pi/2] \\ {}^{(\lambda)}\Psi_{-1} \quad &: \quad \mu = \ln \frac{1-\cos\frac{\lambda}{2}}{\sin\frac{\lambda}{2}}, \quad \lambda \in [\pi/2, 3\pi/2]. \end{aligned} \tag{72}$$

Theorem 4. *If $\Phi \in \Sigma_t$ then one of the focal surfaces Ψ_φ is an isotropic straight line or a timelike surface, the other one belongs to Σ_s.*

Proof. In case of $\Phi = {}^{(\mu)}\Phi \in \Sigma_t$ a parametrization is by (47), (48) and (49)

$$\begin{aligned} &{}^{(\mu)}f(u,v) = \\ &\left[\left(\frac{u^3}{6} - \frac{u}{2} - \frac{v^3}{6} + \frac{v}{2}\right)\cosh\mu + \left(\frac{u^3}{6} - \frac{u}{2} + \frac{v^3}{6} - \frac{v}{2}\right)\sinh\mu,\right. \\ &\left(-\frac{u^2}{2} + \frac{v^2}{2}\right)\cosh\mu + \left(-\frac{v^2}{2} - \frac{u^2}{2}\right)\sinh\mu, \\ &\left.\left(\frac{u^3}{6} + \frac{u}{2} - \frac{v^3}{6} - \frac{v}{2}\right)\cosh\mu + \left(\frac{u^3}{6} + \frac{u}{2} + \frac{v^3}{6} + \frac{v}{2}\right)\sinh\mu\right], \end{aligned} \tag{73}$$

which is an immersion iff $u \neq v$ what is assumed in the following. The focal surfaces ${}^{(\mu)}\Psi_\varphi$ are from (18) because of $\varepsilon = 1$

$${}^{(\mu)}z_\varphi = {}^{(\mu)}f + \varphi(-K)^{-1/2}n, \ \varphi = \pm 1, \tag{74}$$

where Gauss-curvature K and normal vector n are given by (50). This gives

$$\begin{aligned} {}^{(\mu)}z_\varphi(u,v) \quad = \quad &\left[\left(\frac{u^3}{6} - \frac{u}{2} - \frac{v^3}{6} + \frac{v}{2}\right)\cosh\mu + \left(\frac{u^3}{6} - \frac{u}{2} + \frac{v^3}{6} - \frac{v}{2}\right)\sinh\mu + \right. \\ &\frac{\varphi}{2}(uv-1)(-u+v), \\ &\left(-\frac{u^2}{2} + \frac{v^2}{2}\right)\cosh\mu + \left(-\frac{v^2}{2} - \frac{u^2}{2}\right)\sinh\mu + \frac{\varphi}{2}(u^2 - v^2), \\ &\left(\frac{u^3}{6} + \frac{u}{2} - \frac{v^3}{6} - \frac{v}{2}\right)\cosh\mu + \left(\frac{u^3}{6} + \frac{u}{2} + \frac{v^3}{6} + \frac{v}{2}\right)\sinh\mu) + \\ &\left.\frac{\varphi}{2}(uv+1)(-u+v)\right]. \end{aligned} \tag{75}$$

From (73) it is ${}^{(-\mu)}f(u,v) = -{}^{(\mu)}f(v,u)$ (compare [7, p. 338]). Because of $K(u,v) = K(v,u)$ and $n(u,v) = -n(v,u)$ we get

$${}^{(-\mu)}z_\varphi(u,v) = -{}^{(\mu)}z_\varphi(v,u). \tag{76}$$

So w.l.o.g. we assume $\mu \geq 0$ in the following.

In case of $\varphi = 1$ the coefficients of the fundamental forms of the focal surfaces ${}^{(\mu)}\Psi_1 = {}^{(\mu)}z_1(U)$ are from (75)

$$\begin{aligned} g^*_{11} &= (u-v)^2(1-\cosh\mu-\sinh\mu) \\ g^*_{12} &= 0 \\ g^*_{22} &= (u-v)^2(1-\cosh\mu+\sinh\mu), \\ h^*_{11} &= h^*_{22} = 0,\ h^*_{12} = \sqrt{2}\sqrt{\cosh\mu-1}. \end{aligned} \tag{77}$$

From this it is

$${}^{(\mu)}\Delta^* = \det({}^{(\mu)}g^*_{jk}) = -2(u-v)^4(\cosh\mu-1) \le 0.$$

Because of $u \ne v$ it is ${}^{(\mu)}\Delta^* = 0 \Longleftrightarrow \mu = 0$. From (75) ${}^{(0)}z_1(U)$ is the isotropic axis of rotation of the timelike parabolic catenoid ${}^{(0)}f(U)$ from (73).
In case of $\mu \ne 0$ the focal surface ${}^{(\mu)}\Psi_1 = {}^{(\mu)}z_1(U)$ is timelike ($\varepsilon^* = 1$). From (77) the Gauss-curvature is $K^*(u,v) = (u-v)^{-4} > 0$. Because of $-\varepsilon^* K^* < 0$ it follows from (18) that there are no real eigenvalues of the shape operator and therefore the timelike surfaces ${}^{(\mu)}\Psi_1$ cannot belong to Σ_t.
In case of $\varphi = -1$ we calculate from (75)

$$\begin{aligned} g^*_{11} &= (u-v)^2(1+\cosh\mu+\sinh\mu) \\ g^*_{12} &= 0 \\ g^*_{22} &= (u-v)^2(1+\cosh\mu-\sinh\mu), \\ h^*_{11} &= h^*_{22} = 0,\ h^*_{12} = -\sqrt{2}\sqrt{\cosh\mu+1}. \end{aligned} \tag{78}$$

Because of $u \ne v$ it is

$${}^{(\mu)}\Delta^* = 2(u-v)^4(\cosh\mu+1) > 0,$$

so we have spacelike surfaces. Reparametrization of (75) ($\varphi = -1$)

$$\begin{aligned} u(u',v') &= a_1u' + b_1v' \\ v(u',v') &= c_1u' + d_1v', \end{aligned}$$

with

$$\begin{aligned} a_1 &= \frac{1}{2}\left(1-\tanh\frac{\mu}{2}\right)^{1/2},\ b_1 = a_1 \\ c_1 &= -\frac{1}{2}\left(\cosh\frac{\mu}{2}\left(\cosh\frac{\mu}{2}-\sinh\frac{\mu}{2}\right)\right)^{-1/2},\ d_1 = -c_1 \end{aligned}$$

gives parameters according to Lemma 1. Straight forward calculation gives

$$\begin{aligned} g^*_{11}(u',v') &= : g^*(u'v') = \left(-\left(\frac{1+d}{2}\right)^{1/2}u' + \left(\frac{1-d}{2}\right)^{1/2}v'\right)^2 =: (au'+bv')^2 \\ g^*_{22} &= g^*_{11},\ g^*_{12} = 0,\ h^*_{11} = 1,\ h^*_{12} = 0,\ h^*_{22} = -1. \end{aligned} \tag{79}$$

where $d := (\cosh\frac{\mu}{2})^{-1}$. Because of $\varepsilon^* = -1$ it yields $b^2 - \varepsilon^* a^2 = 1$, so from Theorem 2 it follows ${}^{(\mu)}\Psi_{-1} \in \Sigma_s$. □

Remark 3. From (60) and (79) the spacelike focal surface ${}^{(\mu)}\Psi_{-1}$ of ${}^{(\mu)}\Phi \in \Sigma_t$ is up to a Minkowski isometry the surface ${}^{(\lambda)}\Phi \in \Sigma_s$ where λ and μ are related by

$$\lambda = -2\arcsin\sqrt{\frac{1}{2} + \frac{1}{2\cosh\frac{\mu}{2}}},\ \lambda \in \left[-\pi, -\frac{\pi}{2}\right],$$

where surfaces ${}^{(\lambda+\pi)}\Phi$ and ${}^{(\lambda)}\Phi$ are congruent (with ${}^{(\lambda)}f(U) = {}^{(\lambda)}\Phi$ it is ${}^{(\lambda+\pi)}f = -{}^{(\lambda)}f$ from (67)). Because $\lambda = \pm\frac{\pi}{2}$ would imply $\mu = \infty$, the Cayley surface ${}^{(\pm\pi/2)}\Phi \in \Sigma_s$ does not appear as a focal surface of a surface ${}^{(\mu)}\Phi \in \Sigma_t$.

Proposition. *Every regular focal surface Ψ of a surface $\Phi \in \{\Sigma_s \cup \Sigma_t\}$ is an improper affine sphere.*

Proof. (a) In case of $\Phi \in \Sigma_s$ this is true by Theorem 3 and Theorem 2(c).
(b) In case of $\Phi \in \Sigma_t$ and Ψ spacelike the assertion is true by Theorem 4 and Theorem 2(c). If Ψ is timelike, that means $\Psi = {}^{(\mu)}\Psi_1 = {}^{(\mu)}z_1(U)$ where ${}^{(\mu)}z_1$ is given by (75) ($\varphi = 1$), a straight forward calculation gives a constant affine normal vector parallel to the isotropic direction $x_1 = x_3, x_2 = 0$. □

Remark 4. The surfaces $\Phi \in \{\Sigma_s \cup \Sigma_t\}$ are not characterized by the fact, that the focal surfaces are improper affine spheres (see for instance Chapter 4 in [10]).

Example 1. The spacelike parabolic catenoid Φ is from (41)

$$f(u,v) = \left[\frac{v^3}{3} + v - u^2 v, -2uv, \frac{v^3}{3} - v - u^2 v\right].$$

The focal surfaces are from (66) ($\lambda = 0$)

$$\begin{aligned} {}^{(0)}z_\varphi(u,v) \;\; = \;\; & \left[\left(\frac{v^3}{3} + v - u^2 v\right) + \varphi\left(v - vu^2 - v^3\right), -2\,uv + \varphi(-2uv), \right. \\ & \left. \left(\frac{v^3}{3} - v - u^2 v\right) + \varphi\left(-v - vu^2 - v^3\right)\right]. \end{aligned}$$

${}^{(0)}\Psi_{-1} = {}^{(0)}z_{-1}(U)$ is the isotropic axis of rotation, ${}^{(0)}\Psi_1 = {}^{(0)}z_1(U)$ is a timelike surface. Reparametrization of ${}^{(0)}\Psi_1$ according to (68) ($c = 1$)

$$u(u',v') = u'/2,\ v(u',v') = -v'/2$$

gives

$${}^{(0)}z_1(u',v') = \left[\frac{v'^{\,3}}{12} - v' - u'^{\,2}v', u'v', \frac{v'^{\,3}}{12} + v' + u'^{\,2}v'\right]. \tag{80}$$

and

$$g^*_{11} = u'^{\,2},\ g^*_{12} = 0,\ g^*_{22} = -u'^{\,2},\ h^*_{11} = 1,\ h^*_{12} = 0,\ h^*_{22} = 1. \tag{81}$$

On the other hand the timelike catenoid is from (47) (we write $\tilde{f}$ instead of f)

$$\tilde{f}(u,v) = \left[\frac{u^3}{6} - \frac{u}{2}, -\frac{u^2}{2}, \frac{u^3}{6} + \frac{u}{2}\right] + \left[-\frac{v^3}{6} + \frac{v}{2}, \frac{v^2}{2}, \frac{-v^3}{6} - \frac{v}{2}\right].$$

Reparametrization according to (62) ($\mu = 0$)

$$u(u',v') = \frac{\sqrt{2}}{2}(u'-v'),\ v(u',v') = \frac{\sqrt{2}}{2}(u'+v')$$

gives

$$\tilde{f}'(u',v') = \left[\sqrt{2}\left(-\frac{v'^{\,3}}{12} + \frac{v}{2} - \frac{1}{4}u'^{\,2}v'\right), u'v', \sqrt{2}\left(-\frac{v'^{\,3}}{12} - \frac{v}{2} - \frac{1}{4}u'^{\,2}v'\right)\right] \tag{82}$$

and as in (81)

$$g_{11} = u'^{\,2},\ g_{12} = 0,\ g_{22} = -u'^{\,2},\ h_{11} = 1,\ h_{12} = 0,\ h_{22} = 1.$$

Due to the Theorem of Schwarz there is (according to Theorem 3) a Minkowski isometry $\alpha : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ mapping ${}^{(0)}z_1(U')$ to $\tilde{f}(U')$. With respect to the standard basis the coordinate matrix of α is

$$A := \begin{pmatrix} -\frac{3\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{4} & 0 & -\frac{3\sqrt{2}}{4} \end{pmatrix} \tag{83}$$

and from (80) and (82) it yields

$$\tilde{f}'(u',v') = A\,({}^{(0)}z_1(u',v'))^T.$$

That means the (regular) focal surface of a spacelike parabolic catenoid is a timelike parabolic catenoid. An analogue consideration shows that the converse is also true. Surface ($\lambda = 0$) $\in \Sigma_s$ (first surface in Figure 1) corresponds to surface ($\mu = 0$) $\in \Sigma_t$ (first surface in Figure 2) and vice versa.

Example 2. The (spacelike) Cayley surface (conjugate to the spacelike parabolic catenoid (41)) is from (42)

$${}^{(\pi/2)}f(u,v) = \left[\frac{u^3}{3} - u - uv^2, u^2 - v^2, \frac{u^3}{3} + u - uv^2\right].$$

The focal surfaces of ${}^{(\pi/2)}\Phi = {}^{(\pi/2)}f(U)$ are the timelike surfaces ${}^{(\pi/2)}\Psi_\varphi$ parametrized by (66) ($\lambda = \pi/2$)

$$\begin{aligned} {}^{(\pi/2)}z_\varphi(u,v) \; = \; & \left[\left(\frac{u^3}{3} - u - uv^2\right) + \varphi(v - vu^2 - v^3), (u^2 - v^2) + \varphi(-2uv),\right. \\ & \left.\left(\frac{u^3}{3} + u - uv^2\right) + \varphi\left(-v - vu^2 - v^3\right)\right], \end{aligned}$$

what implies ${}^{(\pi/2)}z_1(u,v) = {}^{(\pi/2)}z_{-1}(u,-v)$.

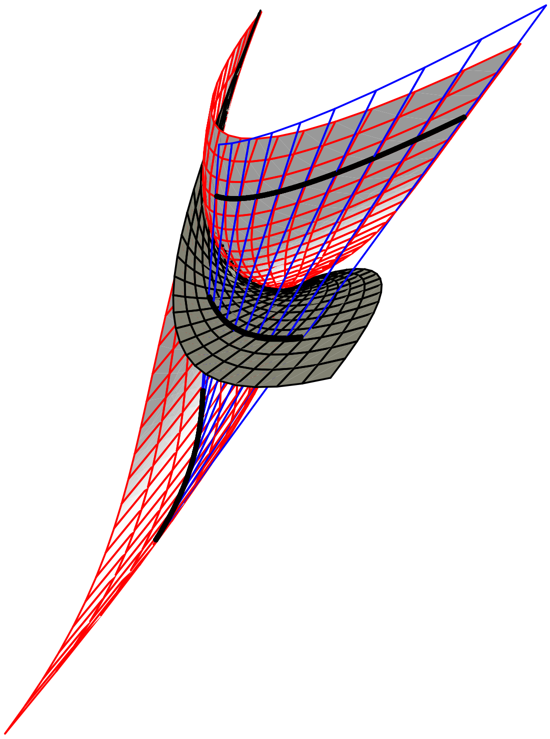

Figure 3. Cayley surface with the two (coinciding) sheets of focal surfaces.

Reparametrization of ${}^{(\pi/2)}\Psi_1$ according to (68) ($c = \sqrt{2}$) gives

$$
\begin{aligned}
{}^{(\pi/2)}z_1(u',v') = \Bigg[& -\frac{\sqrt{\sqrt{2}-1}}{12}\Big(3\,v'^2u' + 6\,u' + u'^3 - v'^3 - v'^3\sqrt{2} - 3\,v'u'^2 - \\
& 3\,v'u'^2\sqrt{2} + 6\,v'\sqrt{2} + 6\,v'\Big), u'v', -\frac{\sqrt{\sqrt{2}-1}}{12}\Big(3\,v'^2u' - 6\,u' + u'^3 - v'^3 - \\
& v'^3\sqrt{2} - 3\,v'u'^2\sqrt{2} - 3\,v'u'^2 - 6\,v'\sqrt{2} - 6\,v'\Big)\Bigg],
\end{aligned} \tag{84}
$$

where (u',v') are parameters according to Lemma 1. It is

$$
\begin{aligned}
g^*_{11} &= \left(-\frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}}u' + \frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}}v'\right)^2, \; g^*_{12} = 0, \; g^*_{22} = -g^*_{11} \\
h^*_{11} &= 1, \; h^*_{12} = 0, \; h^*_{22} = 1.
\end{aligned} \tag{85}
$$

Reparametrization of ${}^{(\pi/2)}\Psi_{-1}$ according to (70) ($c = \sqrt{2}$) gives

$$
{}^{(\pi/2)}z_{-1}(u',v') = {}^{(\pi/2)}\, z_1(u',v').
$$

On the other hand the surfaces of the timelike pencil Σ_t given by (61) can be reparametrized by (62). This gives

$$\begin{aligned} {}^{(\mu)}\tilde{f}(u',v') \;=\; & \Big[-1/12\,\sqrt{2}\,\big(\sinh(\mu/2)u^3 + 6\,\sinh(\mu/2)u + 3\,\sinh(\mu/2)uv^2 + \\ & \cosh(\mu/2)v^3 - 6\,\cosh(\mu/2)v + 3\,\cosh(\mu/2)u^2v\big)\,, uv, \qquad (86) \\ & -1/12\,\sqrt{2}\,\big(\sinh(\mu/2)u^3 - 6\,\sinh(\mu/2)u + 3\,\sinh(\mu/2)uv^2 + \\ & \cosh(\mu/2)v^3 + 6\,\cosh(\mu/2)v + 3\,\cosh(\mu/2)u^2v\big)\Big] \end{aligned}$$

where due to (63) it is

$$g(u',v') = \left(u'\sinh\frac{\mu}{2} + v'\cosh\frac{\mu}{2}\right)^2. \tag{87}$$

Comparing (87) with (85) gives

$$\mu =: \mu_0 = \ln(\sqrt{2}-1). \tag{88}$$

Inserting $\mu = \mu_0$ in (86) gives ${}^{(\mu_0)}\tilde{f}(u',v')$ and comparison with (84) shows

$${}^{(\mu_0)}\tilde{f}(u',v') = {}^{(\pi/2)}z_1(-u',-v'),$$

so the two sheets ${}^{(\pi/2)}\Psi_1$ and ${}^{(\pi/2)}\Psi_{-1}$ of focal Cayley-surfaces of a (spacelike) Cayley-surface coincide with each other and with the surface ${}^{(\mu_0)}\Phi \in \Sigma_t$ where μ_0 is given by (88).

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