Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 1, 57-61.

# Some Small-Centralizer Properties for Rings

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**Abstract.** We characterize rings $R$ in which certain elements $x$ have the property that $C_R(x)$ (resp. the set of zero divisors in $C_R(x)$) is finite. We also explore the consequences of an assumption that certain $x$ satisfy $C_R(x) = \langle x \rangle$.

## 1. Introduction

Let $R$ be a ring with center $Z$, and let $D$ be the set of zero divisors of $R$. For $x \in R$, let $C_R(x)$ be the centralizer of $x$ in $R$. We study rings in which $C_R(x)$ is finite for all $x \in R \backslash Z$ and rings in which $C_R(x) \cap D$ is finite for all $x \in D \backslash Z$. In the first case we show that $R$ is either finite or commutative; in the second case we show that either $R$ is finite or $D \subseteq Z$.

As in [2], we call an element $x \in R$ extremely noncommutative if $C_R(x) = x\mathbb{Z}[x]$ – i.e. if $C_R(x)$ is the subring generated by $x$. Our most difficult result deals with rings such that each element of $D \backslash Z$ is extremely noncommutative.

Let us fix some additional notation and terminology. Let $N = N(R)$ denote the set of nilpotent elements of $R$, and $T = T(R)$ the set of elements of finite additive

*supported by the Natural Sciences and Engineering Research Council of Canada, Grant 3961

order. For $x \in R$, let $\langle x \rangle$ and $A(x)$ be respectively the subring generated by $x$ and the two-sided annihilator of $x$. For a subring $S$ of $R$, let $[R : S]$ denote the index of $(S, +)$ in $(R, +)$; and for a subset $X$ of $R$, let $|X|$ denote the cardinality of $X$. An element $x \in R$ is called periodic if there exist distinct positive integers $m, n$ for which $x^m = x^n$, and the ring $R$ is called periodic if each of its elements is periodic.

The following lemmas will be useful.

**Lemma 1.1.** [6] *If $R$ is a periodic ring with $N \subseteq Z$, then $R$ is commutative.*

**Lemma 1.2.** [3] *Let $R$ be a ring such that for each $x \in R$ there exist a positive integer $m$ and a polynomial $p(X) \in \mathbb{Z}[X]$ for which $x^m = x^{m+1}p(x)$. Then $R$ is periodic.*

**Lemma 1.3.** [7] *If $R$ is infinite and $x \in N$ , then $|A(x)| = |R|$. In particular, $A(x)$ is infinite.*

## 2. Finite-centralizer conditions

**Theorem 2.1.** *If $R$ is a ring such that $C_R(x)$ is finite for all $x \in R \backslash Z$, then $R$ is either finite or commutative.*

*Proof.* Suppose that $R$ is infinite. Since $A(x) \subseteq C_R(x)$, it follows by Lemma 1.3 that $N \subseteq Z$. Suppose also that $R$ is not commutative and $x \in R \backslash Z$. Since $\langle x \rangle \subseteq C_R(x)$, $\langle x \rangle$ is finite and hence $x$ is periodic; and since $Z$ is clearly finite, central elements are periodic as well. Thus, $R$ is a noncommutative periodic ring with $N \subseteq Z$, contrary to Lemma 1.1. Therefore $R$ must be commutative. □

**Theorem 2.2.** *Let $R$ be a ring such that $C_R(x) \cap D$ is finite for all $x \in D \backslash Z$. Then either $R$ is finite or $D \subseteq Z$.*

*Proof.* Note that if $S$ is any infinite subring of $R$ such that $C_S(x) = C_R(x) \cap S$ is finite for all $x \in S \backslash Z$ , $S$ is commutative by Theorem 2.1 and therefore $S \subseteq Z$. In particular, if $S$ is any infinite subring contained in $D$, $S \subseteq Z$.

Suppose that $D \backslash Z \neq \phi$, and assume without loss of generality that $xy = 0$ with $x \in D \backslash Z$ and $y \neq 0$. If $A_l(y)$ is infinite, we have $x \in A_l(y) \subseteq Z$ – a contradiction; therefore $A_l(y)$ is finite. For each $w \in A_l(y)$, consider the map $f_w : R \to A_l(y)$ given by $f_w(r) = rw$. By applying the first isomorphism theorem for additive groups, we see that $ker(f_w) = A_l(w)$ is of finite index in $R$; hence $S = A_l(A_l(y))$ is of finite index and therefore is infinite. Thus $S \subseteq Z$; and since $S \subseteq A_l(x)$, we see that $A(x)$ is an infinite subset of $C_R(x) \cap D$ – a contradiction. □

## 3. An extreme non-commutativity condition

In [2], the following theorem is proved.

**Theorem 3.1.** *If $R$ is a ring in which all noncentral elements are extremely noncommutative, then $R$ is either finite or commutative.*

In [8], we were led to consider an infinite noncentral subring $A$ with subring $B = A \cap Z$ such that $A^2 \subseteq Z$ , $A = \langle a \rangle$ for all $a \in A \backslash B$ , and $[A : B]$ is a prime. In the sections headed *Proof of Theorem* 2.1 and *Completion of proof of Theorem* 2.1, we showed that such a subring cannot exist. Thus, we proved, but did not explicitly state, the following lemma.

**Lemma 3.2.** *Let $R$ be an infinite noncommutative ring. Then $R$ contains no infinite noncentral subring $A$ such that $A^2 \subseteq Z$ , $A = \langle a \rangle$ for all $a \in A \backslash Z$, and $[A : A \cap Z]$ is a prime.*

The principal theorem of this section, which we now state, is obtained by weakening the extreme noncommutativity hypothesis in Theorem 3.1.

**Theorem 3.3.** *Let $R$ be a ring in which every element of $D \backslash Z$ is extremely noncommutative. Then either $R$ is finite or $D \subseteq Z$.*

The proof will be presented as a series of lemmas, the first of which is almost obvious. In each lemma, it will be assumed without explicit mention that $R$ is a ring in which every element of $D \backslash Z$ is extremely noncommutative.

**Lemma 3.4.** *If $D \nsubseteq Z$, then $R$ is indecomposable. Hence $R$ has no nonzero central idempotent zero divisors.*

**Lemma 3.5.** *If $N \nsubseteq Z$, then $R$ is finite.*

*Proof.* Since $Z$ centralizes $N \backslash Z$, $Z \subseteq N$. We show first that all zero divisors are periodic. This is clearly true for nilpotent elements, so we consider $d \in D \backslash N$. Then $d^2 \notin N$, so $d^2 \notin Z$ and hence $d \in \langle d^2 \rangle$. Thus there exists $p(X) \in \mathbb{Z}[X]$ such that $d = d^2 p(d)$. Since each element of $D$ is in some subring of zero divisors, Lemma 1.2 shows that zero divisors are periodic.

Next we show that $D \subseteq T(R)$. Let $d \in D$ and $D' = \langle d \rangle$. By [1, Lemma 1(c)], $d = a + u$ with $u \in N$ and $a$ a power of $d$ such that $a^n = a$ for some $n > 1$. Now $e = a^{n-1}$ is an idempotent such that $a = ae$; and since $e$ is in the periodic ring $D'$ , $2e$ is periodic, hence $e \in T(R)$ and $a \in T(R)$. We now need to show that $N \subseteq T(R)$; and since $Z = Z \cap N \subseteq (N \backslash Z) - (N \backslash Z)$, it suffices to show that $N \backslash Z \subseteq T(R)$. Let $u \in N \backslash Z$ and suppose $u^k \in T(R)$ for $k \geq 2$. Since $u \notin Z$ , there exists $n \geq 2$ such that $nu \notin Z$; and it follows that $u \in \langle nu \rangle$ , so that there exist $c_1, c_2, \ldots, c_t \in \mathbb{Z}$ such that $u = c_1(nu) + c_2(nu)^2 + \cdots + c_t(nu)^t$. Multiplying by $u^{k-2}$ gives $(1 - c_1 n)u^{k-1} \in T(R)$ and hence $u^{k-1} \in T(R)$. By backward induction, $u \in T(R)$.

We now know that if $d \in D \backslash Z$ , $\langle d \rangle$ is finite and consequently $C_R(d)$ is finite. Thus, $R$ is finite by Theorem 2.2. $\square$

**Lemma 3.6.**

(i) *If $D \nsubseteq Z$ and $N \subseteq Z$ , then $d^n \in Z$ for all $d \in D$ and $n \geq 2$.*

(ii) *If $D \nsubseteq Z$, there exists a prime $p$ such that $pD \subseteq Z$.*

*Proof.* By Lemma 3.4, R has no nonzero idempotent zero divisors. Hence, we need only adapt in an obvious way the proof of Lemma 2.8 of [2]. □

**Lemma 3.7.** *If $N \subseteq Z$, then every subring of zero divisors is commutative.*

*Proof.* Let $H$ be any subring of zero divisors, and let $h \in H \backslash Z(H)$. Then $C_H(h) = \langle h \rangle$, so $H$ is either finite or commutative by Theorem 3.1. Moreover, if $H$ is finite, it is commutative by Lemma 1.1. □

**Lemma 3.8.** *Let $R$ be infinite with $D \nsubseteq Z$ and $N \subseteq Z$. Then*

(i) *$D$ is infinite;*

(ii) *$D$ is a commutative ideal and hence $D^2 \subseteq Z$;*

(iii) *$D = \langle d \rangle$ for every $d \in D \backslash Z$;*

(iv) *$[D : D \cap Z] = p$ for some prime $p$.*

*Proof.* (i) follows immediately from an old theorem of Ganesan [4,5], which asserts that any ring $R$ with $1 \leq |D \backslash \{0\}| < \infty$ must be finite.

(ii) Use the proof of Lemma 2.4 of [8], which employs Lemma 3.7.

(iii) Let $d \in D \backslash Z$. By (ii), $D \subseteq C_R(d) = \langle d \rangle$; and obviously $\langle d \rangle \subseteq D$.

(iv) Since we now know that $D$ is an additive subgroup, the result follows from (iii) and Lemma 3.6.

*Proof of Theorem* 3.3. Assume that $D \nsubseteq Z$. By Lemmas 3.8 and 3.2, we cannot have $N \subseteq Z$; hence $R$ is finite by Lemma 3.5. □

Theorem 3.3 does not provide a characterization of rings such that all $d \in D \backslash Z$ are extremely noncommutative, since we do not have complete information about the finite examples. We do, however, have partial information.

**Theorem 3.9.** *Let $R$ be a finite ring with $D \nsubseteq Z$ such that each $d \in D \backslash Z$ is extremely noncommutative. Then either $R$ is isomorphic to a matrix ring of form $GF(p)e_{11} + GF(p)e_{12}$ or $GF(p)e_{11} + GF(p)e_{21}$, or $R$ is nil.*

*Proof.* If $R = D$, the result follows by Theorem 2.11 of [2]. Otherwise, if $x \in R \backslash D$, some power of $x$ is a regular idempotent, necessarily 1. Now by Lemma 1.1, there exists $u \in N \backslash Z$; and since $1 + u \in C_R(u)$, $1 + u \in \langle u \rangle$. But this is not possible, since $\langle u \rangle$ is a nil ring and $1 + u$ is invertible. □

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Received June 26, 2008