The fundamental solutions for fractional evolution equations of parabolic type

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Abstract

In this paper, we treat the fractional integral equation of the form

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} [A(\theta)u(\theta) - f(\theta)] d\theta,$$

where $0 < \alpha \leq 1, \Gamma(\alpha)$ is the gamma function, $\{A(t) : t \geq 0\}$ is a family of linear closed operators defined on a dense set $D(A)$ in a Banach space E into E, u_0 is an element of $D(A)$ and f is a given E-valued function defined on an intervall $[0, T]$. The existence and uniqueness of the solution of the considerd integral equation is studied for a suitable class of the family of operators $\{A(t) : t\epsilon[0, T]\}$. The continuous dependence of solutions on f and u_0 is also studied. An application is given to a mixed problem of general parbolic partial differential equations with fractional order.

Keywords and phrases: Fractional integral and diferential equations, closed operators, fundamental solutions.
AMS Subject Classification: 26A 33; 45 N 05; 34 G10; 47D03; 47D06

1. Introduction

In this paper, we consider the following fractional integral evolution equation,

$$u(t) = u_0 \quad - \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} [A(\theta)u(\theta) - f(\theta)] d\theta, \tag{1.1}$$

where $0 < \alpha \leq 1, \Gamma(\alpha)$ *is the gamma function*, $\{A(t) : t \in [0, T]\}$ is a family of linear closed operators defined on dense set $D(A)$ in a Banach space E into E, u is the unknown E-valued function , $u_0 \in D(A)$ and f is a given E-valued function defined on $[0, T]$.
It is assumed that $D(A)$ is independent of t. Let $B(E)$ denote the Banach space of all linear bounded operators in E endowed with the topology defined by the operator norm.

We need the following conditions;

($\mathbf{A_1}$): The operator $[A(t)+\lambda I]^{-1}$ exists in B(E) for any λ with Re $\lambda \geq 0$ and

$$\| [A(t)+\lambda I]^{-1} \| \leq \frac{C}{\mid \lambda \mid +1}, \tag{1.2}$$

for each $t \in [0,T]$, where C is a positive constant independent both of t and λ.

($\mathbf{A_2}$): for any t_1, t_2, s $\in [0,T]$,

$$\| [A(t_2) - A(t_1)]A^{-1}(s) \| \leq C \mid t_2 - t_1 \mid^{\gamma} \tag{1.3}$$

where $0 < \gamma \leq 1$, $C > 0$ and the constants C and γ are independents of t_1, t_2 and s
($\mathbf{A_3}$): The function f satisfies a uniform Holder condition (with exponent β) in [0,T], i.e.,

$$\| f(t_2) - f(t_1) \| \leq C \mid t_2 - t_1 \mid^{\beta},$$

for all $t_1, t_2 \in [0,T]$, where C and β are positive constants and $0 < \beta \leq 1$,(The constants C and β are independent of t_1 *and* t_2)
Under condition (A_1) each operator - A(s) , s $\in [0,T]$, generates an analytic semigroup $exp(-tA(s))$, $t > 0$ and there exists a positive constant C independent both of t and s such that

$$\| A^n(s)exp(-tA(s)) \| \leq \frac{C}{t^n}, \tag{1.4}$$

where n = 0,1 , $t > 0, s \in [0,T], [1], [2]$

In section 2, we shall construct the fundamental solution of the homogeneous fractional differential equation

$$\frac{d^{\alpha}v(t)}{dt^{\alpha}} + A(t)v(t) = 0, t > 0 \tag{1.5}$$

We shall prove the existence and uniqueness of the solution of equation (1.5), with the initial condition

$$v(0) = u_0 \in D(A). \tag{1.6}$$

The continuous dependence of the solutions of equation (1.1) on the elements u_0 and the function f is proved.
In section 3, we give an application to a mixed problem of a parabolic partial differential equation of fractional order.

2. The fundamental solution

We say that u is a strong solution of the fractional integral equation (1.1), if $u(t) \in D(A)$ for each $t \in [0,T], u, u^*$are continuous in $t \in [0,T]$and u satisfies equation (1.1), where $u^*(t) = A(t)u(t)$.

Let h be an E-valued function defined on [0,T]. If $\frac{dh(t)}{dt}$ and the integral $\int_\tau^t (t-\theta)^{-\alpha} \frac{dh(\theta)}{d\theta} d\theta$ exist in the abstract sense, then we use the following definition of the fractional derivative ${}_\tau D_t^\alpha h(t)$;

$$ {}_\tau D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_\tau^t (t-\theta)^{-\alpha} \frac{dh(\theta)}{d\theta} d\theta \quad , \tag{2.1} $$

[3], [4], [5].
If u is a strong solution of (1.1), then the fractional derivative

$$ \frac{d^\alpha u}{dt^\alpha} = {}_0D_t^\alpha u, $$

exists and continuous in $t \in [0,T]$. In this case we notice that

$$ \frac{d}{dt} \int_0^t (t-\theta)^{-\alpha} F(\theta) d\theta = \int_0^t (t-\theta)^{-\alpha} \frac{dF(\theta)}{d\theta} d\theta, \tag{2.2} $$

where

$$ F(t) = \int_0^t (t-\theta)^{\alpha-1} [f(\theta) - u^*(\theta)] d\theta. $$

Using (1.1), (2.1) and (2.2), we get

$$ \frac{d^\alpha u(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \int_\theta^t (t-s)^{-\alpha} (s-\theta)^{\alpha-1} (f(\theta) - u^*(\theta)) ds d\theta $$

$$ = -A(t)u(t) + f(t), \tag{2.3} $$

$$ u(0) = u_0. \tag{2.4} $$

The converse is also true. In other words if $\frac{d^\alpha u(t)}{dt^\alpha}$ is continuous in $t \in [0,T]$ and u represents a solution of the Cauchy problem (2.3), (2.4), then u represents a strong solution of (1.1), (this means that the integral equation (1.1) is equivalent to the Cauchy problem (2.3), (2.4)).
We shall consider integrals of oprerator -valued functions. these integrals are defined in the sense of Riemann with respect to the strong topology. We shall denote by $\psi(t \ \ , s)$ the following integral,

$$ \psi(t,s) = \alpha \int_0^\infty \theta t^{\alpha-1} \zeta_\alpha(\theta) exp(-t^\alpha \theta A(s)) d\theta, $$

where ζ_α is a probability density function defined on $[0,\infty)$, such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x}\zeta_\alpha(\theta)d\theta = \sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+\alpha j)},$$

where $0<\alpha\le 1, x>0$, [5], [7].

Lemma 2.1. The improper integral $\int_0^\infty \theta\zeta_\alpha(\theta)A(t)exp(-\eta^\alpha\theta A(s))d\theta$ exists for $\eta>0,\ t,s\in[0,T]$ and represents a uniformly continuous function in the uniform topology (that is in the norm of B (E)) in the variables t,η,s, where $t,s\in[0,T], \epsilon\le\eta\le T$ and ϵ is any positive number.

Proof: The existence of the considered improper integral is clear for $\eta>0, t,s\in[0,T]$. If $0\le t_1<t_1+\Delta t_1=t_2\le T$, $\epsilon\le\eta_1<\eta_1+\Delta\eta_1=\eta_2\le T$, and $0\le s_1<s_1+\Delta s_1=s_2\le T$, then

$$\int_0^\infty \theta\zeta_\alpha(\theta)A(t_2)exp(-\eta_2^\alpha\theta A(s_2))d\theta - \int_0^\infty \theta\zeta_\alpha(\theta)A(t_1)exp(-\eta_1^\alpha\theta A(s_1))d\theta$$

$$= \int_0^\infty \theta\zeta_\alpha(\theta)P(t_1,t_2,s_2)A(s_2)exp(-\eta_2^\alpha\theta A(s_2))d\theta$$

$$+\int_0^\infty \theta\zeta(\theta)_\alpha A(t_1)[exp(-\nu_2\theta A(s_2)) - exp(-\nu_1\theta A(s_2))]exp(-\nu_1\theta A(s_2))d\theta$$

$$+\int_0^\infty \theta\zeta_\alpha(\theta)A(t_1)[exp(-\eta_1^\alpha\theta A(s_2)) - exp(-\eta_1^\alpha\theta A(s_1))]d\theta, \qquad (2.5)$$

where $P(t_1,t_2,t_3)=[A(t_2)-A(t_1]A^{-1}(t_3), \nu_1=\eta_1^\alpha/2, \nu_2=\eta_2^\alpha-\eta_1^\alpha/2$.
It can be proved under conditions $(A_1)$and (A_2)that

$$\parallel A(t)[exp(-\eta A(s)) - exp(-\eta A(\tau)) \parallel \le \frac{C}{\eta}\mid s-\tau\mid^\gamma, \qquad (2.6)$$

$$\parallel A(t)[exp(-\eta A(s)) - exp(-\tau A(s))]A^{-1}(s) \parallel \le \frac{C\mid\eta-\tau\mid}{Min(\eta,\tau)}, \qquad (2.7)$$

for all $\eta>0,\tau>0,t,s\in[0,t]$, where the positive constant C is independent of $t,s,\eta\ and\ \tau$.
We estimate the norm of the first term on the right of (2.5) by using condition $(A_2)$and (1.4), the norm of the second term by using (2.7) and (1.4). We estimate the norm of the last term on the right (2.5) by using (2.6). We thus find that the norm of the left side of (2.5) is bounded by

$$C[\frac{(\Delta t_1)^\gamma}{\epsilon^\alpha} + \frac{1}{\epsilon^{2\alpha}}\{(\eta_1+\Delta\eta_1)^\alpha - \eta_1^\alpha\} + \epsilon^{1-\ \alpha}(\Delta s_1)^\gamma].$$

This completes the proof.
Corollary. The operator - valued function $\psi(t-\eta,\eta)$ and $A(t)\psi(t-\eta,\eta)$are uniformly continuous in the uniform topology in the varaibles t,η, where $0 \leq \eta \leq t-\epsilon, 0 \leq t \leq T$, for any $\epsilon > 0$. Clearly

$$\parallel \psi(t-\eta,\eta) \parallel \leq C(t-\eta)^{\alpha-1}, \tag{2.8}$$

where C is a positive constant independent of t,η
Lemmma 2.2. If

$$w_1(t,\tau) = \int_\tau^t \psi(t-\eta,\eta) f(\eta) d\eta, t > \tau,$$

then

$${}_\tau D_t^\alpha w_1(t,\tau) = f(t) - \int_\tau^t A(\eta)\psi(t-\eta,\eta) f(\eta) d\eta.$$

Proof. Let $\{f_n\}$ be a sequence of functions defined by

$$f_n(t) = [I + \frac{1}{n}A(t)]^{-1} f(t), t \in [0,T].n = 1,2,...$$

Let us consider the integrals ;

$$w_{1n}(t,\tau) = \int_\tau^t \psi(t-\eta,\eta) f_n(\eta) d\eta,$$

$$w_{2n}(t,\eta) = f_n(\eta) - \frac{1}{\Gamma(\alpha)} \int_\eta^t (t-\theta)^{\alpha-1} A(\eta) w_{2n}(\theta,\eta) d\theta.$$

Since $f_n(t) \in D(A)$ for all $t \in [0,T]$, it follows from [8] that

$$w_{2n}(t,\eta) = \int_0^\infty \zeta_\alpha(\theta)[exp(-(t-\eta)^\alpha \theta A(\eta))] f_n(\eta) d\theta, \tag{2.9}$$

where $0 \leq \eta \leq t$. Thus

$${}_\eta D_t^\alpha w_{2n}(t,\eta) = \frac{1}{\Gamma(1-\alpha)} \int_\eta^t (t-s)^{-\alpha} \frac{dw_{2n}(s,\eta)}{ds} ds$$

$$= \frac{-\alpha}{\Gamma(1-\alpha} \int_\eta^t \int_0^\infty (t-s)^{-\alpha}(s-\eta)^{\alpha-1} \theta \zeta_\alpha(\theta) A(\eta)[exp(-(s-\eta)^\alpha \theta A(\eta))] f_n(\eta) d\theta ds$$

$$= -A(\eta) w_{2n}(t,\eta). \tag{2.10}$$

Using (2.9) and (2.10), we get

$$\begin{aligned} {}_\tau D_t^\alpha \omega_n(t,\tau) &= \frac{d}{dt} \int_\tau^t \int_0^\infty \zeta_\alpha(\theta)[\exp(-(t-\eta))^\alpha \theta A(\eta))] f_n(\eta) d\theta d\eta \\ &= f_n(t) - \int_\tau^t A(\eta)\psi(t-\eta,\eta) f_n(\eta) d\eta \end{aligned}$$

According to lemma (2.1), we notice that $A(\eta)\psi(t-\eta,\eta)$ is uniformly continouous function in the uniform topology in the variables $t,\eta \in [0,T]$ where $t-\eta \geq \epsilon$. Since f satisfies condition (A_3), it follows that the integral $\int_\tau^t A(\eta)\psi(t-\eta,\eta)f(\eta)d\eta$ exists (comp [8]). We notice that;

$$\| A(\eta)\psi(t-\eta,\eta) \| \leq \frac{C}{t-\eta}, \tag{2.11}$$

for all $t,\eta \in [0,T], t-\eta \geq \in$. Clearly

$$\| [I + \frac{1}{n}A(t)]^{-1} - I \| \leq C+1, \tag{2.12}$$

where as for $x \in D(A)$;

$$\| [I + \frac{1}{n}A(t)]^{-1}x - x \| \leq \frac{C}{n} \| A(t)x \| .$$

Using (2.12) and noticing that f satisfies condition (A_3), we deduce that the sequence $\{f_n\}$uniformly converges to f with respect to $t \in [0,T]$. Using (2.11), we get for any positive number ϵ, the following inequality

$$\| \int_\tau^{t-\epsilon} A(\eta)\psi(t-\eta,\eta)[f_n(\eta) - f(\eta)]d\eta \| \leq C\epsilon[ln(t-\tau) - ln\epsilon],$$

for sufficiently large n. Consequently.

$$\lim_{n \to\infty} {}_\tau D_t^\alpha w_{1n}(t,\tau) = f(t) - \int_\tau^t A(\eta)\psi(t-\eta,\eta)f(\eta)d\eta,$$

uniformly with respect to $t \in [0,T], t > \tau$. This completes the proof.
Let

$$\phi_1(t,\tau) = [A(t) - A(\tau)]\psi(t-\tau,\tau),$$

$$\phi_{k+1}(t,\tau) = \int_\tau^t \phi_k(t,s)\phi_1(s,\tau)ds, k = 1,2,...$$

Using condition (A_2), we get

$$\| \phi_1(t,\tau) \| \leq \int_0^\infty \| S(t,\tau,\theta) \| \| A(\tau)exp(-(t-\tau)^\alpha\theta A(\tau)) \| d\theta \leq C(t-\tau)^{\gamma-1} \ , \tag{2.13}$$

where

$$S(t,\tau,\theta) = \alpha\theta(t-\tau)^{\alpha-1}\zeta_\alpha(\theta)P(t,\tau,\tau).$$

Using lemma (2.1), we conclude that $\phi_1$is uniformly continuous in t,τ in the uniform topology, provided that $t-\tau \geq \epsilon > 0$. Now one verifies by induction that all the functions $\phi_k, k = 1, 2, \dots$ are uniformly continuous in t, τ in the uniform topology for $t - \tau \geq\in t, \tau, \in [0, T]$, and

$$\| \phi_k(t,\tau) \| \leq \frac{C^k (t-\tau)^{\gamma k - 1}}{\Gamma(\gamma k)}. \tag{2.14}$$

Using inequalities (2.14), one can justify the relation

$$\int_\tau^t \phi(t,s)\phi_1(s,\tau)ds = \sum_{k=1}^{\infty} \int_\tau^t \phi_k(t,s)\phi_1(s,\tau)ds \quad ,$$

where

$$\phi(t,\tau) = \sum_{k=1}^{\infty} \phi_k(t,\tau),$$

It is easy to see that

$$\| \phi(t,\tau) \| \leq C(t-\tau)^{\gamma-1}. \tag{2.15}$$

The function ϕ is uniformly continuous in the uniform topology in t, τ provided that $0 \leq \tau \leq t - \epsilon, \epsilon \leq t \leq T$ for any $\epsilon > 0$. Using Fubini's theorem, we deduce that ϕ is the unique solution of the integral equation

$$\phi(t,\tau) = \phi_1(t,\tau) + \int_\tau^t \phi(t,s)\phi_1(s,\tau)ds. \tag{2.16}$$

Lemma 2.3. For any $0 < \delta < \gamma,\ 0 \leq \tau < t_1 < t_2 \leq T$;

$$\| \phi(t_2,\tau) - \phi(t_1,\tau) \| \leq C(t_2 - t_1)^{\gamma-\delta}(t_1-\tau)^{\delta-1} \tag{2.17}$$

where the positive constant C does not depend on t_1, t_2 and τ.
Proof. from (2.13), we get

$$\| \phi_1(t_2,\tau) - \phi_1(t_1,\tau) \| \leq 2C(t_1-\tau)^{\gamma-1}. \tag{2.18}$$

Writting

$$\phi_1(t_2,\tau) - \phi_1(t_1,\tau) = P(t_1,t_2,\tau)A(\tau)\psi(t_2-\tau,\tau) +$$
$$P(t_1,\tau,\tau)A(\tau)[\psi(t_2-\tau,\tau) - \psi(t_1-\tau,\tau)],$$
$$\Lambda(t_1,t_2,\tau) = P(t_1,\tau,\tau)A(\tau)[\psi(t_2-\tau,\tau) - \psi(t_1-\tau,\tau)],$$

we get

$$\| P(t_1,t_2,\tau)A(\tau)\psi(t_2-\tau,\tau) \| \leq C(t_2-t_1)^{\gamma}(t_1-\tau)^{-1},$$
$$\| \Lambda(t_1,t_2,\tau) \| \leq 2C(t_1-\tau)^{\gamma-1}.$$

We can write

$$\Lambda(t_1,t_2,\tau) = \int_0^\infty P_1(t_1,t_2,t_3,\tau,\theta)d\theta + \int_0^\infty P_2(t_1,t_2,t_3,\tau,\theta)d\theta,$$

where

$$P_1(t_1,t_2,\tau,\theta)=\alpha\theta\zeta_\alpha(\theta)[A(t_1)-A(\tau)]P_3(t_1,t_2,t_3,\tau,\theta)exp[-(t_1-\tau)^\alpha\theta A(\tau)],$$

$$P_3(t_1,t_2,t_3,\tau,\theta)=(t_2-\tau)^{\alpha-1}exp[-\{(t_2-\tau)^\alpha-(t_1-\tau)^\alpha\}\theta A(\tau)]-(t_2-\tau)^{\alpha-1}I,$$

We can find t_3 and t_4 such that $t_1<t_3<t_2, t_1<t_4<t_2$ and

$$P_3(t_1,t_2,t_3,\tau,\theta)=$$

$$(t_2-\tau)^{\alpha-1}exp[-\alpha(t_2-t_1)(t_3-\tau)^{\alpha-1}\theta A(\tau)]-(t_2-\tau)^{\alpha-1}I,$$

$$P_2(t_1,t_2,t_3,\tau,\theta)=$$

$$\alpha\theta\zeta_\alpha(\theta)[A(t_1)-A(\tau)][(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]exp[-(t_1-\tau)\theta A(\tau)]$$

$$=\alpha(\alpha-1)\theta\zeta_\alpha(\theta)(t_2-t_1)(t_4-\tau)^{\alpha-2}[A(t_1)-A(\tau)]exp[-(t_1-\tau)\theta A(\tau)].$$

We notice that

$$P_3(t_1,t_2,t_3,\tau,\theta)=$$

$$-\alpha\theta(t_2-\tau)^{\alpha-1}(t_3-\tau)^{\alpha-1}\int_0^{t_2-t_1}A(\tau)exp[-\eta\alpha\theta(t_3-\tau)^{\alpha-1}A(\tau)]d\eta.$$

Now it is easy to see that

$$\parallel\Lambda(t_1,t_2,\tau)\parallel\leq C(t_1-\tau)^{\gamma-2}(t_2-t_1).$$

Using the two bounds of $\parallel\Lambda(t_1,t_2,\tau)\parallel$ we get

$$\parallel\Lambda(t_1,t_2,\tau)\parallel=\parallel\Lambda(t_1,t_2,\tau)\parallel^\gamma\parallel\Lambda(t_1,t_2,\tau)\parallel^{1-\gamma}$$

$$\leq C(t_1-\tau)^{-1}(t_2-t_1)^\gamma$$

consequently

$$\parallel\phi_1(t_2,\tau)-\phi_1(t_1,\tau)\parallel\leq C(t_1-\tau)^{-1}(t_2-t_1)^\gamma. \tag{2.19}$$

Using (2.18) and (2.19), we get

$$\parallel\phi_1(t_2,\tau)-\phi_1(t_1,\tau)\parallel^{\delta_1+\delta_2}\leq C^{\delta_1+\delta_2}(t_2-t_1)^{\gamma\delta_1}(t_1-\tau)^{\delta_2\gamma-\delta_1-\delta_2},$$

where $\delta_1>0,\delta_2>0$. Thus

$$\parallel\phi(t_2,\tau)-\phi_1(t_1,\tau)\parallel\leq C(t_2-t_1)^{\gamma-\delta}(t_1-\tau)^{\delta-1}, \tag{2.20}$$

where $\delta=\frac{\delta_2\gamma}{\delta_1+\delta_2}<\gamma$
Using (2.16), we get

$$\phi(t_2,\tau)-\phi(t_1,\tau)=\phi_1(t_2,\tau)-\phi_1(t_1,\tau)+$$

$$+\int_{\tau}^{t_1}[\phi_1(t_2,s)-\phi_1(t_1,s)]\phi(s,\tau)ds+\int_{t_1}^{t_2}\phi_1(t_2,s)\phi(s,\tau)ds. \tag{2.21}$$

We estimate the norm of the first term on the right of (2.21) by using (2.20), the norm of the second by using (2.13) and (2.15). After simple calculations, the required result follows.
We shall make use of the inequality

$$\| A(t)A^{-1}(s) \| \leq C,$$

which follows from condition (A_2) where $t, s \in [0, T]$ and C is a positive constant independent both of t and s.
Theorem 2.1. There exists an operator - valued function Q(t) with values in B(E), defined and strongly continuous in t for $0 \leq t \leq T$ such that:
(B_1) The fractional deivative $\frac{d^\alpha Q(t)}{dt^\alpha}$ exists in the strong topology and belongs to B(E) for $0 \leq t \leq T$ and is strongly continuous in t for $0 \leq t \leq T$,
(B_2) The range of Q (t) is included in D(A) for $0 \leq t \leq T$,
(B_3) For any $u_0 \in E$, $Q(t)u_0$ satisfies the fractional differential equation

$$\frac{d^\alpha Q(t)}{dt^\alpha}u_0 + A(t)Q(t)u_0 = 0, \quad 0 < t \leq T, \tag{2.22}$$

(B_4) $\qquad Q(0) = A^{-1}(0)$
(B_5) A solution of the Cauchy problem (1.5), (1.6) is given by
$v(t) = Q(t)A(0)u_0, \;\; for \;\; any \;\; u_0 \in D(A).$
Proof. We set

$$Q(t) = A^{-1}(0) + \int_0^t \psi(t-\eta,\eta)U(\eta)d\eta. \tag{2.23}$$

We shall determine the operator valued function $U(t)$ such that $Q(t)u_0$ satisfies equation(2.22). Using formally lemma (2.2), we get

$$U(t)u_0 + \int_0^t \phi_1(t,\eta)U(\eta)u_0 d\eta = -A(t)A^{-1}(0)u_0, \tag{2.24}$$

(Comp [9] , [10] , [11]).
The operator- valued function $U(t)$ can be obtained by successive approximations, that is we put

$$U(t) = \sum_{k=0}^{\infty} U_k(t),$$

where $U_0(t) = -A(t)A^{-1}(0)$,

$$U_{k+1}(t) = -\int_0^t \phi_1(t,s)U_k(s)ds. \tag{2.25}$$

Using the properties of ϕ_k and Fubini's theorem one easily shows by intduction that
$$U_k(t) = -\int_0^t \phi_k(t,s)A(s)A^{-1}(0)ds. \tag{2.26}$$
Using (2.15), (2.25) and (2.26) we deduce that the series $\sum_{k=0}^{\infty} U_k(t)$ uniformly coverges on $[0,T]$. It is clear that $U(t)$ is given by
$$U(t) = -A(t)A^{-1}(0) - \int_0^t \phi(t,s)A(s)A^{-1}(0)ds. \tag{2.27}$$
Using (2.15) , we get
$$\| U(t) \| \leq C + Ct^{\gamma}. \tag{2.28}$$
It is easy to see that
$$U(t_2) - U(t_1) = [A(t_1) - A(t_2)]A^{-1}(0)$$
$$-\int_0^{t_1} [\phi(t_2,s) - \phi(t_1,s)]A(s)A^{-1}(0)ds - \int_{t_1}^{t_2} \phi(t_2,s)A(s)A^{-1}(0)ds.$$
Using condition $(A_2)$and lemma (2.3), we fined
$$\| U(t_2) - U(t_1) \| \leq C(t_2-t_1)^{\gamma} + \frac{c}{\delta}(t_2-t_1)^{\gamma-\delta}t_1^{\delta} + \frac{c}{\gamma}(t_2-t_1)^{\gamma}, \tag{2.29}$$
where $t_2 > t_1, t_1, t_2 \in [0,T]$ and C is positive constant independent of t_1, t_2. Recalling that $\psi(t-\eta,\eta)$ is uniformly continuous in t,η provided $t-\eta \geq \epsilon > 0$ and using (2.8), (2.29), one can verify without difficulty that $\int_0^t \psi(t-\eta,\eta)U(\eta)d\eta$ is uniformly continuous (in the norm of B (E))in $t \in [0,T]$. Using (2.28), we get $\| Q(t) \| \leq C$, for all $t \in [0,T]$, where C is a positive constant independent of t. It is also obvious that $Q(0) = A^{-1}(0)$ and $Q(t)u_0$ is contiuous in $t \in [0,T]$ for every $u_0 \in E$. Let us prove now that the range of $Q(t)$included in $D(A)$ for $0 < t \leq T$.
Using (2.29) and lemma (2.1), we deduce that $A(t)\psi(t-\eta,\eta)U(\eta)$ is uniformly continuous in the uniform tolpology in the variables $t,\eta \in [0,T]$, provided that $t-\eta \geq \epsilon$ where ϵ is any positive number.
The operator - valued function $A(t)\psi(t-\eta,\eta)U(\eta)$ can be written in the form
$$A(t)\psi(t-\eta,\eta)U(\eta) = A(t)[\psi(t-\eta,\eta) - \psi(t-\eta,t)]U(\eta)$$
$$+A(t)\psi(t-\eta,t)[U(\eta) - U(t)] + A(t)\psi(t-\eta,t)U(t). \tag{2.30}$$
By using (2.6) and (2.28), we find that the norm of the first term on the right of (2.30) is bounded by $C(t-\eta)^{\gamma-1}$. By using (1.4) and (2.29), we find that the norm of the second term on the right of (2.30) is bounded by $C(t-\eta)^{\gamma-\delta-1}$,

(where C is a generic positive constant independent both of t and η). Using these estimations and noticing that;

$$\int_0^t A(t)\psi(t-\eta,\eta)U(\eta)u_0 d\eta = \int_0^t A(t)[\psi(t-\eta,\eta)-\psi(t-\eta,t)]U(\eta)u_0 d\eta +$$

$$\int_0^t A(t)\psi(t-\eta,t)[U(\eta)-U(t)]u_0 d\eta - \int_0^\infty \zeta_\alpha(\theta)[exp(-t^\alpha\theta A(t)]U(t)u_0 d\theta + U(t)u_0.$$

One can deduce that the integral $\int_0^t A(t)\psi(t-\eta,\eta)U(\eta)u_0 d\eta$ is continuous in $t \in [0,T]$, for every $u_0 \in E$. Consequently the range of Q (t) is included in D(A) for every $t \in [0,T]$. It can be proved that there are two positive constants C and δ such that

$$\| A(t)Q(t)u_0 \| \leq C + Ct^\delta, t \in [0,T],$$

where $0 < \delta < 1$ and C is independent of t, $u_0 \in E$. Using (2.23), (2.29) and lemma (2.2), one can easily show that $\frac{d^\alpha Q(t)}{dt^\alpha}u_0$ exists and represents a continuous function in $t \in [0,T]$ for every $u_0 \in E$.
It is clear also that $Q(t)u_0$ satisfies equation (2.22). The function $v(t) = Q(t)A(0)u_0$ represents a solution of the Cauchy problem (1.5) (1.6), if $u_0 \in D(A)$. This completes the proof of the properties $B_1, \ldots, B_5$

Theorem 2.2. A solution of the Cauchy problem (2.3) , (2.4) is given by

$$u(t) = u_0 + \int_0^t \psi(t-\eta,\eta)U(\eta)A(0)u_0 d\eta$$

$$+ \int_0^t \psi(t-\eta,\eta)f(\eta)d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)f(s)dsd\eta, \tag{2.31}$$

or

$$u(t) = u_0 - \int_0^t \psi(t-\eta,\eta)A(\eta)u_0 d\eta - \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)u_0 dsd\eta$$

$$\int_0^t \psi(t-\eta,\eta)f(\eta)d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)f(s)ds, d\eta, \tag{2.32}$$

where $u_0 \in D(A)$ and f satisfies condition $(A_3), t \in [0,T]$.
Proof. We set $u(t) = A^{-1}(0)u_0 + \int_0^t \psi(t-\eta,\eta)V(\eta)d\eta$. Then we determine the abstract function V such that u satisfies equation (2.3). The Proof is carried out similar to Theorem 2.1.
Theorem 2.3. The strong solution of the Cauchy problem (1.5), (1.6) is unique.
Proof: We introduce the bounded operators $A_n(t) = A(t)[I + \frac{1}{n}A(t)]^{-1}$. It is known that

$$\| (A_n(t) - \lambda I)^{-1} \| \leq \frac{C}{|\lambda| + 1}, \tag{2.33}$$

$$\| (A_n(t) - A_n(\tau))A_n^{-1}(s) \| \leq C \mid t - \tau \mid^{\gamma}, \tag{2.34}$$

where $s, t, \eta \in [0, T]$ and C is a positive constant independent of t, τ, s and n. Consider the following Cauchy problem

$$\frac{d^{\alpha} v_n(t)}{dt^{\alpha}} + A_n(t)v_n(t) = 0, n = 1, 2, ... \tag{2.35}$$

$$v_n(0) = u_0. \tag{2.36}$$

The function $w_n(t) = v(t) - v_n(t)$, then satisfies

$$\frac{d^{\alpha} w_n(t)}{dt^{\alpha}} + A_n(t)w_n(t) = g_n(t), t \in [0, T] \tag{2.37}$$

$$w_n(0) = 0, \tag{2.38}$$

where $g_n(t) = [A_n(t) - A(t)]v(t)$.
The solution of the Cauchy problem (2.37), (2.38) is unique. To prove this fact, suppose $g_n(t) = 0$, then $w_n(t)$ satisfies

$$\| w_n(t) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha - 1} \| A_n(\theta) w_n(\theta) \| d\theta$$

$$\leq \frac{C_n}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha - 1} \| w_n(\theta) \| d\theta,$$

for every n, where C_n is a positive constant. It follows that $w_n(t) = 0$ for all $t \in [0, T]$.
Noticing that g_n is continuous in $t \in [0, T]$ for every $n = 1, 2, \ldots$ and $A_n(t)$ is bounded operator that varies continuously in $t \in [0, T]$ (in the uniform topology), then it is easy to see with the help of (2.3) that the unique solution of the Cauchy problem (2.37), (2.38) is given by

$$w_n(t) = \int_0^t \psi_n(t - \eta, \eta) g_n(\eta) d\eta + \int_0^t \int_0^{\eta} \psi_n(t - \eta, \eta) \phi^{(n)}(\eta, s) g_n(s) ds d\eta, \tag{2.39}$$

where

$$\psi_n(t - \eta, \eta) = \alpha \int_0^t \theta (t - \eta)^{\alpha - 1} \zeta_{\alpha}(\theta) exp[-(t - \eta)^{\alpha} \theta A_n(\eta)] d\theta,$$

$\phi^{(n)}(t, \tau)$ is the unique solution of the integral equation

$$\phi_1^{(n)}(t, \tau) = \phi_1^{(n)}(t, \tau) + \int_{\tau}^t \phi^{(n)}(t, s) \phi_1^{(n)}(s, t) ds,$$

$$\phi_1^{(n)}(t, \tau) = [A_n(t) - A_n(\tau)] \psi_n(t - \tau, \tau)$$

It can be shown that the sequence $\{g_n\}$ uniformly converges to zero in E with respect to $t \in [0, T]$.
Consequently by using (2.8), (2.15), (2.23), (2.34) and (2.39), we get $v(t) = lim_{n\to\infty} v_n(t)$ uniformly with respect to $t \in [0, T]$ since $v_n(t)$ is defined uniquely as the solution of the Cauchy problem (2.35), (2.36), also $v(t)$ is unique.
The continuous dependence of solution of the Cauchy problem (2.3), (2.4) on f and u_0 is obtained from formula (2.32), (Comp [12]).
It must be noticed that the fractional differential equations have many important applications in different branches of applied mathamatics (see [13], [14], [15]).

3. Application

Let Ω be a bounded domain in the real n - dimensional Euclidean space R^n.For any
$0 < T < \infty$, denote by Q_T the cylinder $\{(x, t) : x \in \Omega, 0 < t < T\}$ and by $\partial\Omega$ the bounday of Ω.
We consider the differential operator

$$\frac{\partial^\alpha}{\partial t^\alpha} + A^*(x, t, D) = \frac{\partial^\alpha}{\partial t^\alpha} + \sum_{|q|\le 2m} a_q(x, t)D^q,$$

where $A^*(x, t, D)$ is said to be uniformly elliptic in $\overline{Q_T}$ if the coefficients $a_q(x, t)$ are bounded in $\overline{Q_T}$ and $(-1)^m Re \sum_{|q|=2m} a_q(x, t)\xi^q \ge C \mid \xi \mid^{2m}$, for all $(x, t) \in \overline{Q_T}$ and for all real ξ, where C is a positive constant independent of x, t, ξ and

$$\mid \xi \mid^2 = \xi_1^2 + ... + \xi_n^2 \ (\overline{Q_T} = \{(x, t) : x \in \Omega \cup \ \partial\ \Omega, 0 \le t \le T\}),$$

$(D^q = D_1^{q_1}...D_n^{q_n} \quad , \quad D_j = \frac{\partial}{\partial x_j} \quad , \quad |q| = q_1 + ... + q_n, q = (q_1, ..., q_n)$ is a multi-index)

We consider the Cauchy problem of the fractional evolution equation

$$\frac{d^\alpha u}{dt^\alpha} + A^*(t)u = f(t), 0 < t \le T, \tag{3.1}$$

$$u(0) = u_0 \tag{3.2}$$

in the Hilbert space $L^2(\Omega)$, where for each t , f(t) is the function f(x,t) belonging to $L_2(\Omega)$ and $A^*(t)$is the operator with domain $D(A^*) = H^{2m}(\Omega) \bigcap H_0^m(\Omega)$ given by $A^*(t) = A^*(x, t, D)$. And u_0 is a function in $H^{2m}(\Omega) \bigcap H_0^m(\Omega)$ (see [8] , [16]).
$(H^m(\Omega))$ is the completion of the space $C^m(\Omega)$ with respect to the norm

$$\parallel f \parallel_m = [\sum_{|q|\le m} \int_\Omega [D^q f(x)]^2 dx]^{\frac{1}{2}} \quad ,$$

$C^m(\Omega)$ is the set of all continuous function define on Ω , which have continuous partial derivatives of order less than or equal to m, $H_0^m(\Omega)$ is the complation of $C_0^m(\Omega)$ with respect to the norm $\| f \|_m$ and $C_0^m(\Omega)$ is the set of all function $f \in C^m(\Omega)$ with compact supports in Ω).
It is assumed that
(I) All the coefficients $a_q(x,t)$ are continuous in $\overline{Q_T}$
and $| a_q(x,t_2) - a_q(x,t_1) | \leq C | t_2 - t_1 |^\gamma$,
$0 < \gamma \leq 1, t_1, t_2 \in [0,T]$ and C is a positive constant independent of $t_1, t_2$and $x \in \Omega$.
(II) $[\int_\Omega | f(x,t_2) - f(x,t_1) |^2 \, dx]^{\frac{1}{2}} \leq C | t_2 - t_1 |^\beta, 0 < \beta \leq 1, C$ is a positive constant independent of t_1 and t_2.
Theorem 3.1. Assume that $A^*(x,t,D)$ is uniformly elliptic in $\overline{Q_T}$, that (I), (II) hold and $\partial\Omega$ is of class C^{2m}, then there exists a unique strong solution of the problem (3.1), (3.2).
Proof. Writting equation (3.1) in the form

$$\frac{d^\alpha u}{dt^\alpha} + [A^*(t) + kI]u = f(t) + ku \tag{3.3}$$

we see that for some constant k, the operator $A^*(t) + kI$ satisfies the conditions (A_1) and (A_2)
Using formula (2.32), we get

$$u(t) = u_0 - \int_0^t \psi(t-\eta,\eta)A(\eta)u_0 d\eta - \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)A(s)u_0 ds d\eta$$

$$+ \int_0^t \psi(t-\eta,\eta)[f(\eta) + ku(\eta)]d\eta + \int_0^t \int_0^\eta \psi(t-\eta,\eta)\phi(\eta,s)[f(s) + ku(s)]ds d\eta,$$

$$A(t) = A^*(t) + kI$$

It can be proved that u satisfies a uniform Holder condition, then the last integral equation has the unique required solution u(t). This completes the proof.

Acknowledgments
I would like to thank the referees and Prof. Argimiro Arratia for their careful reading of the manuscript and their valuable comments.

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