Binomial coefficients

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Abstract

Among some of the most interesting natural numbers are the binomial coefficients. They have uses not only in combinatorics but in other branches of mathematics such as algebra, analysis and topology. In this article we give some of the basic properties of binomial coefficients and their generalizations.

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Definitions of the binomial coefficients

We will give three different ways of defining the binomial coefficients. Each method has its own uses. One is algebraic, one is combinatorial and one is arithmetic.

Definition 1. Consider the polynomial $(1+x)^n$ in $Q[x]$ (the ring of polynomial with rational coefficients) and where $n \geq 0$ is an integer. If we expand $(1+x)^n$ then we let $\binom{n}{k}$ be the coefficient of x^k for any integer $k > 0$

We have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{n}{k}x^k$$

So easily $\binom{n}{0} = 1$, $\binom{n}{n} = 1$ and $\binom{n}{k} = 0$ if $k > n$.

We now consider the combinatorial approach. Let $n, k \geq 0$ be integers. Let $C(n, k)$ be the number of subsets A having k elements of a set X with n elements.

So, for example, let $X = \{1, 2, ..., n\}$ when $n \geq 1$. So $C(n, k)$ is the number of $A \subset X$ with $|A| = k$ ($|A|$ denotes the cardinality of A.)

Then it is clear that $C(n,0) = 1$ for any $n \geq 0$ ($A = \emptyset$ is the only possibility), that $C(n,n) = 1$ ($A = X$ is the only possibility) and that $C(n,k) = 0$ if $k > n$ (there is no such A).

Our third and arithmetic approach to defining the binomial coefficients is initially given by the formula $\dfrac{n!}{k!(n-k)!}$. But the formula only makes sense for $0 \leq k \leq n$ since we do not have a definition of $(n-k)!$ if $n-k < 0$ (but recall that $0! = 1$). But canceling we have:

$$k!\frac{n!}{(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

The formula on the right makes sense for any natural numbers $k,\ n > 0$. If we interpret an empty product as 1 we see the formula gives 1 when $k = 0$. Then the formula also gives 1 when $k = n$ and gives 0 when $k > n$ since we get a factor of 0 in the numerator in this case.

Pascal's Identity

We now argue that we have the so-called Pascal's identity for our three versions of the binomial coefficients. Then using this fact and the fact that the three definitions agree when $k = 0$, when $k = n$ and when $k > n$ we will get that they agree for all $k,\ n \geq 0$.

Given $n \geq 0$ we have

$$(1+x)^{n+1} = \sum_{k=0}^{\infty} \binom{n+1}{k} x^k$$

$$= (1+x)^n(1+x) = \left(\sum_{k=0}^{\infty} \binom{n}{k} x^k\right)(1+x).$$

But in the last product it is clear that the coefficient of x^k is $\binom{n}{k} + \binom{n}{k-1}$. Hence we have

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

for all $n \geq 0$ and $k \geq 1$.

Now we consider the numbers $C(n+1,k)$. So we want to find the number of subsets $A \subset \{1,2,...,n,n+1\}$ where $|A| = k$. These include all the $A \subset \{1,2,...,n\}$ with $|A| = k$ and there are $C(n,k)$ of these A's. If $A \not\subset \{1,2,...,n\}$ then $n+1 \in A$ and $A = B \cup \{n+1\}$ with $B \subset \{1,2,...,n\}$ and with $|B| = k-1$. There are $C(n,k-1)$ such B's. So we get

$$C(n+1,k) = C(n,k) + C(n,k-1)$$

for all $n \geq 0$ and $k \geq 1$.

Note that we are tacitly assuming $k \leq n$. We also see that the identity holds if $k = n + 1$ (we get $1 = 0 + 1$) and if $k > n + 1$ (we get $0 = 0 + 0$).

Now we consider the arithmetic version of our coefficients. The formula

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

when $1 \leq k \leq n$ is just a matter of finding a common denominator and adding fractions. But we want the identity

$$\frac{(n+1)n\cdots(n-k)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} + \frac{n(n-1)\cdots(n-k+2)}{(k-1)!}$$

to hold for all $n \geq 0$ and $k \geq 1$. If $k \leq n$ this follows from the above. If $k = n+1$ the equation becomes $1 = 0 + 1$ and if $k > n + 1$ it becomes $0 = 0 + 0$. So the equation holds for all $n \geq 0$ and $k \geq 1$.

Now by a double induction on $n \geq 0$ and $k \geq 0$ we use the fact that our three versions agree in case $k = 0$, in case $k = n$ and in case $k > n$ and then use Pascal's identity to get

$$\binom{n}{k} = C(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

holds for all $n \geq 0$ and $k \geq 0$.

So we can (and will) freely use the most convenient version in any situation. Note (for example) that we immediately get that $\dfrac{n!}{k!(n-k)!}$ is an integer for any n, k with $0 \leq k \leq n$.

Computing $\binom{n}{k}$

If $0 \leq k \leq n$ we can compute $\binom{n}{k}$ using the formula $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. But we know $\binom{n}{k}$ is an integer so then we can write $\binom{n}{k}$ as a product of primes. Clearly this can be a hard and tedious procedure. But using a result of Legendre we see we can quickly write $\binom{n}{k}$ as a product of primes. Legendre's result shows us how to write $n!$ for $n \geq 0$ as a product of primes.

Let p be a prime. We want to find the largest power of p that divides $n!$. But

$$n! = (1 \cdot 2 \cdots (p-1))p((p+1\cdots(2p-1))2p((2p+1)\cdots$$

i.e. we isolate the multiples of p (the only factors among $1, 2, ..., n$ divisible by p. The last such multiple is $\left[\frac{n}{p}\right] p$ where $\left[\frac{n}{p}\right]$ is the greatest integer in the fraction

$\frac{n}{p}$. Dividing out each factor of p we get

$$n! = p^{\left[\frac{n}{p}\right]} \left[\frac{n}{p}\right]! \cdot l$$

where $p \nmid l$.

So now our problem is reduced to finding the largest power of p dividing $\left[\frac{n}{p}\right]!$. So we have the same problem with n replaced by $\left[\frac{n}{p}\right]$.

Using the same procedure again we get

$$n! = p^{\left[\frac{n}{p}\right]} \cdot p^{\left[\frac{\left[\frac{n}{p}\right]}{p}\right]} \cdot \left[\frac{\left[\frac{n}{p}\right]}{p}\right]! \cdot m$$

with $p \nmid m$.

Repeating the procedure we see that if e is the largest $e \geq 0$ such that $p^e | n!$ we have

$$e = \left[\frac{n}{p}\right] + \left[\frac{\left[\frac{n}{p}\right]}{p}\right] + \cdots$$

Example. If $n = 100,\ p = 7$ then $\left[\frac{100}{7}\right] = 14$ and $e = 14 + 2 + 0 + \cdots = 16$. We note that it is not hard to argue that $\left[\frac{\left[\frac{n}{p}\right]}{p}\right] = \left[\frac{n}{p^2}\right]$ and that in fact $e = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$. Also note that $\left[\frac{n}{p^k}\right] = 0$ if k is sufficiently large.

So now this makes it easy to write, for example, $\binom{100}{13}$ as a product of primes.

We can argue that for integers a, b, c with $c > 0$ we have

$$\left[\frac{a}{c}\right] + \left[\frac{b}{c}\right] \leq \left[\frac{a+b}{c}\right]$$

Using this and Legendre's result above we can argue that for $0 \leq k \leq n$, $\dfrac{n!}{k!(n-k)!} = \binom{n}{k}$ is an integer. We argue that for any prime p the largest power of p dividing $k!(n-k)!$ divides $n!$.

We also note that as consequence of the above we get that if $n = p$ with p a prime and if $0 < k < p$ then $p | \binom{p}{k} = \dfrac{p!}{k!(p-k)!}$ since

$$\left[\frac{p}{p}\right] = 1 \quad \text{and} \quad \left[\frac{k}{p}\right] = \left[\frac{n-k}{p}\right] = 0$$

We now use another version of our coefficients and see that this means that

$$(1+x)^p \cong 1 + x^p (\bmod p)$$

(two polynomials are congruent if the corresponding coefficients of each x^k are congruent). Then of course for any polynomial $f(x)) \in Z[x]$ with integer coefficients

$$(1+f(x))^p \cong 1 + f(x)^p (\bmod p)$$

Letting $f(x) = x^p$ we get

$$(1+x)^{p^2} = ((1+x)^p)^p \cong (1+x^p)^p \cong | + x^{p^2} (\bmod p)$$

and so that

$$(1+x)^{p^s} \cong 1 + x^{p^s} (\bmod p)$$

for any $s \geq 1$. Since $(1+x)^{p^s} = \sum_{k=0}^{\infty} \binom{p^s}{k} x^k$ we get that $p | \binom{p^s}{k}$ if $s > 0$ and if $0 < k < p^s$. This will be useful in the next section.

Remainders

If p is a prime and $0 \leq k \leq n$ we can decide whether $p | \binom{n}{k}$ by using Legendre's procedure. In this section we will find a method for finding the remainder when we divide $\binom{n}{k}$ by p (so whether p divides $\binom{n}{k}$ or not). We will use the $C(n,k)$ version of our binomial coefficients. And we will use a special technique for computing $C(n,k)$. We think of $C(n,k)$ as the number of ways of choosing k balls but where the balls are distributed into two boxes containing n_1 and n_2 of the balls respectively. So $n = n_1 + n_2$. So we can choose k balls by choosing k_1 balls from the first box and k_2 balls from the second where $k_1 + k_2 = k$. This can be done in $C(n_1,k_1) \cdot C(n_2,k_2)$ ways. When we make a choice of some such k_1 and k_2 we will say that we have specified the form of choosing our k balls.

Clearly, this method can be generalized to the situation where we have more than two boxes. But even with two boxes we get something of interest. Namely that if $n = n_1 + n_2$ $(n_1, n_2 \geq 0$ then

$$C(n,k) = \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \geq 0}} C(n_1,k_1) \cdot C(n_2,k_2)$$

or the more familiar form

$$\binom{n_1+n_2}{k} = \sum_{k_1=0}^{k} \binom{n_1}{k_1} \binom{n_2}{k-k_1}$$

If we think of $n+1$ as the sum $n_1 + n_2 = n+1$ we recover Pascal's identity.

Now let p be a prime. Recall that any $n \geq 0$ can be written in a unique manner to the base p, i.e. we can write

$$n = a_0 + a_1 p + \cdots + a_s p^s$$

with $0 \leq a_i < p$.

Likewise for $k \geq 0$ we have

$$k = b_0 + b_1 p + + \cdots + b_s p^s$$

with $0 \leq b_i < p$. With this notation we have:

Theorem (Lucas).

$$\binom{n}{k} \cong \binom{a_0}{b_0} \cdots \binom{a_s}{b_s} \pmod{p}$$

Proof. If $k > n$ then $\binom{n}{k} = 0$. But then clearly $b_i > a_i$ for at least one $i = 0, 1, ..., s$ and so $\binom{a_i}{b_i} = 0$ for this i. Hence we assume $0 \leq k \leq n$. We now use our box technique for studying $\binom{n}{k} = C(n,k)$. Since $n = a_0 + a_1 p + \cdots + a_s p^s$ we will suppose our n balls are distributed in $a = a_0 + a_1 + \cdots + a_s$ boxes with each of the first a_0 boxes having a single ball, then each of the next a_1 boxes having p balls each and so forth (of course some a_i may be 0 and so then there are no such boxes). We now consider all the possible forms for choosing k balls from our n balls distributed inour a boxes. This corresponds to writing

$$k = k_1 + k_2 + \cdots + k_a$$

where we are required to choose k_j balls from the j-th box. If the corresponding box has p^l balls with $l \geq 1$ and $0 < k_j < p^l$, we know from the last section that $p | \binom{p^l}{k_j}$. This gives us that in this case the number of ways of choosing k balls of this particular form is divisible by p. Hence when computing the remainder when we divide $\binom{n}{k} = C(n,k)$ by p we only need concern ourselves with the special forms where from each box we choose either none of the balls or all of the balls (for the boxes with a single ball this is already necessarily so).

So choosing the balls in the special forms just means we pick the boxes from which we choose all the balls. Consider one such form. This means we pick c_0 of the first a_0 boxes, c_1 of the next a_1 boxes etc. But then $0 \leq c_i < p$ for each i and we must have

$$k = c_0 + c_1 p + \cdots + c_s p^s$$

Since $k = b_0 + b_1 p + \cdots + b_s p^s$ this means we must have $c_0 = b_0, ..., c_s = b_s$. But then $\binom{a_0}{b_0} \cdot \binom{a_1}{b_1} \cdots \binom{a_s}{b_s}$ is the number of ways of choosing our k balls from the

n balls in our special forms (all or none from each box). Then with what was noted above we get that $\binom{n}{k}$ and $\binom{a_0}{b_0}\cdots\binom{a_s}{b_s}$ have the same remainder when divided by p and so they are congruent modulo p. Note that if $b_i > a_i$ for some $i,\ 0 \le i \le s$ then $\binom{a_i}{b_i} = 0$ and so p divides $\binom{n}{k}$.

Example. If we divide $\binom{87}{31}$ by 5 we have

$$87 = 2 \cdot 2.5 + 5^2$$

$$31 = 0 + 1 \cdot 5 + 1 \cdot 5^2$$

But $\binom{2}{0}\binom{2}{1}\binom{3}{1} = 6$ and so the remainder when we divide $\binom{87}{31}$ by 5 is 1.

Exercise 1. Find a way to find the last digit of $\binom{n}{k}$ when $\binom{n}{k}$ is written as a decimal integer (use the Chinese remainder theorem).

Exercise 2. Argue that

$$\binom{pn}{pk} \cong \binom{n}{k} \pmod{p}$$

for any $k, n \ge 0$.

Exercise 3. Find all $n \ge 0$ such that all the binomial coefficients $\binom{n}{k}$ with $0 \le k \le n$ are odd.

Pascal's Formula and Discrete Derivatives

If we consider functions f defined on the natural numbers N (with $f(n)$ say any integer) then since $\delta = 1$ is the smallest of all positive integers we define the discrete derivative Δf of f to be such that

$$(\Delta f)(n) = f(n+1) - f(n)$$

for all $n \ge 0$. Then we see that $\Delta f = 0$ if and only if $f = c$ (i.e. $f(n) = c$ for all $n \ge 0$ for some constant c), that $\Delta f = \Delta g$ if and only if $f = g + c$ for some constant and then that if $k \ge 1$ and if $f(n) = \binom{n}{k}$ for all $n \ge 0$ then $(\Delta f)(n) = \binom{n}{k-1}$.

This means that

$$\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$$

which is just Pascal's identity. So by abuse of notation we write $\Delta\binom{n}{k} = \binom{n}{k-1}$. Since $\Delta\binom{n}{1} = 1$ we see that $\Delta^{k+1}\binom{n}{k} = 0$ for $k \ge 0$. Recall that from Calculus

$f^{(k+1)}(x) = 0$ for a real valued function $f(x)$ (with suitable hypotheses on $f(x)$) implies $f(x)$ is a polynomial function of degree at most k i.e. $f(x) = r_0 + r_1x + \cdots + r_kx^k$ for some $r_0, ..., r_k \in \boldsymbol{R}$. Here we get that if $\Delta^{k+1}f = 0$ then

$$f(n) = a_0 + a_1\binom{n}{1} + \cdots + a_k\binom{n}{k}$$

for all $n \geq 0$ for some $a_0, ..., a_k \in \boldsymbol{Z}$. And we see that to find the a_k we only need note that

$$f(0) = a_0 + a_1\binom{0}{1} + \cdots + a_k\binom{0}{k} = a_0,$$

$$(\Delta f)(0) = a_1 + a_2\binom{0}{1} + a_3\binom{0}{2} + \cdots + a_k\binom{0}{k-1} = a_1$$

and similarly $(\Delta^2 f)(0) = a_2, ..., (\Delta^k f)(0) = a_k$. For example if $f(0) = 0$ and $f(n) = 1+2+\cdots+n$ for $n \geq 1$ then $(\Delta f)(n) = n+1$ for all n. So $(\Delta^2 f)(n) = 1$ and $\Delta^3 f = 0$. using the above we find that

$$1 + 2 + \cdots + n = \binom{n}{1} + \binom{n}{2}$$

for all $n \geq 1$ and in fact for $n = 0$ if we interpret the empty sum as 0. In a similar manner we can find formulas for the sums $1^2 + 2^2 + \cdots + n^2$ and $1^3 + 2^3 + \cdots + n^3$. Note that $\Delta(2^n) = 2^{n+1} - 2^n = 2^n(2-1) = 2^n$. So in some sense the function 2^n is the discrete version of the function e^x of Calculus.

The Binomial Polynomial $\binom{x}{k}$

Using the fact that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n+1-k)}{k!}$$

we can define polynomials $\binom{x}{k}$ where

$$\binom{x}{k} = \frac{x(x-1)\cdots(x+1-k)}{k!}$$

if $k \geq 1$ and where $\binom{x}{0} = 1$. Then the degree of $\binom{x}{k}$ is k and the coefficients of $\binom{x}{k}$ are rational numbers.

We have the identity

$$\binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1}$$

since the polynomials on each side of the equation have the same values for an infinite number of values of x, namely $x = 0, 1, 2, ...$

One advantage of now having the binomial polynomials $\binom{x}{k}$ is that we can now give a meaning to the symbol $\binom{n}{k}$ for any $n \in Z$ (so also for $n \leq 0$). So, for example, $\binom{-1}{k} = \dfrac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k$. But of course we also have a meaning for $\binom{z}{n}$ for any complex number z. So now the original binomial theorem

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

for these more general n becomes Newton's binomial theorem.

For example, if $n = -1$ we get $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$ i.e. that $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$.

Here we are operating in the ring $Z[[x]]$ of formal power series with coefficients mZ so with no concern about questions of convergence. If we consider $(1+x)^z$ for $z \in \mathbb{C}$ ($\mathbb{C}$ the field of complex numbers) then we operate in $\mathbb{C}[[x]]$.

Noting that deg $\binom{x}{k} = k$, we see that if $P(x) \in Q[x]$ is any polynomial of degree k, then for some rational number $q \in Q$, $P(x)$ and $q\binom{x}{k}$ have the same dominant coefficient, or equivalently that

$$\deg\,(P(x) - q\binom{x}{k}) \leq k - 1$$

(for $k \geq 1$).

From this it follows that we get $P(x) = a_0 + a_1\binom{x}{1} + \cdots + a_k\binom{x}{k}$ for some rational numbers $a_0, a_1, ..., a_k$.

Now noting that $P(0) = a_0$

$$P(1) = a_0 + a_1, \quad P(2) = a_0 + 2a_1 + a_2, ...$$

$$P(k) = a_0 + \binom{k}{1}a_1 + \cdots + \binom{k}{k}a_k$$

we see that if the polynomial $P(x) \in Q[x]$ is such that $P(n) \in Z$ for all $n \geq 0$ then in fact $a_0, ..., a_k \in Z$. And also if $a_0, ..., a_k \in Z$ and if $P(x) = a_0 + a_1\binom{x}{1} + \cdots + a_k\binom{x}{k}$ then $P(n) \in Z$ for all $n \geq 0$.

We let $Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$ denote the set all such polynomials

$$a_0 + a_1\binom{x}{1} + \cdots + a_k\binom{x}{k} \quad (\text{with } a_0, ..., a_k \in Z)$$

By the above $Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$ consists of all the $P(x) \in Q[x]$ such that $P(m) \in Z$ for $m = 0, 1, 2, ...$ These are called the integer valued polynomials. Using this characterization of the $P(x) \in Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$ we see that $Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$ is a ring. So, for example, if $k \geq 1$ then $\binom{x}{1} \cdot \binom{x}{k} \in Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$.

To write $\binom{x}{1}\binom{x}{k}$ as $a_0 + a_1\binom{x}{1} + \cdots +$ we revert to the viewpoint of combinatorics. If $n \geq 0$, to compute $\binom{n}{1}\binom{n}{k}$ means to compute $C(n,1) \cdot C(n,k)$. This is the number of ways to simultaneously choose two subsets of X where $|X| = n$ with the first subset T having one elements and the second subset S having n elements. The number of ways of choosing T and S with $T \subset S$ is $C(k,1)C(n,k)$ (i.e. choose S and choose one of its elementss to form T) and the number of ways with $T \not\subset S$ is $C(k+1,1) \cdot C(n,k+1)$ (so first choose $T \cup S$) then choose T). So

$$C(k,1)C(n,k) = kC(n,k) + (k+1)C(n,k+1)$$

or

$$\binom{n}{1}\binom{n}{k} = k\binom{n}{k} + (k+1)\binom{n}{k+1}.$$

This gives the polynomial identity

$$\binom{x}{1}\binom{x}{k} = k\binom{x}{k} + (k+1)\binom{x}{k+1}$$

In a similar manner $\binom{x}{k}\binom{x}{l}$ can be computed for any $k, l \geq 0$.

Remark. Given a formal sum $U(x) = a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + a_3\binom{x}{3} + \cdots$ we can make sense of the expression $U(n)$ for any $n \geq 0$ since $\binom{n}{m} = 0$ for $m > n$. So such a $U(x)$ can be used to define a function $N \to \mathbb{Z}$. In fact each such function $N \to \mathbb{Z}$ is given by a unique such $U(x)$. The functions $N \to \mathbb{Z}$ can be made into a ring, so the set of such $U(x)$ can be made into a ring. This ring is denoted

$$\mathbb{Z}\left[\left[\binom{x}{1}, \binom{x}{2}, \binom{x}{3}, ...\right]\right]$$

Then $Z\left[\binom{x}{1}, \binom{x}{2}, ...\right]$ as above is a subring of $Z\left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$.

But we also have elements such as $U(x) = 1 + \binom{x}{1} + \binom{x}{2} + \cdots$ If $n \geq 1$ then $U(n) = 1 + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$. So this $U(x)$ is denoted 2^x.

The notion of the discrete derivative Δ can easily be extended to the ring $Z\left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$. The simplest way to define it is so that

$$\Delta(a_0 + a_1\binom{x}{1}a_2\binom{x}{2} + \cdots) = a_1 + a_2\binom{x}{1} + a_3\binom{x}{2} + \cdots$$

So then $\Delta(2^x) = \Delta(1 + \binom{x}{1} + a_3\binom{x}{2} + \cdots) = 1 + \binom{x}{1} + \binom{x}{2} + \cdots = 2^x$ as expected. Note that for any $U(x) \in \mathbb{Z}\left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$ we can associate the symbol $2^{U(x)}$ with the function $N \to \mathbb{Z}$ that maps n to $2^{U(n)}$.

Such a function in turn gives us an elements $V(x) \in \left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$. So we write $2^{U(x)} = V(x)$. This raises the interesting question of the existence of a natural logarithm in this setting.

As an exercise one could try to write 0^x as a series

$$a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots \quad (\text{where } 0^0 = 1 \text{ and } 0^n = 0 \text{ if } n \geq 1)$$

Final Remarks

The first proof of the binomial theorem (in the form $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$) for $n \geq 1$ was given by Jakob Bernoulli in his posthumously published "Ars Conjectandi" (1713). In 1676 Newton had stated the more general $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$ for arbitrary n in a letter, but without proof. In 1878 Lucas gave a method for finding the remainder when $\binom{n}{k}$ is divided by a prime p. The study of integer valued polynomials with rational coefficients goes back to the seventeen century. A study of them in their own right was initiated by Pólya and Ostrowski in 1919. In 1936 Skolem began the study of the set of integer valued polynomials with rational coefficients as a ring. The association of a function defined on N with a series $a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots$ is widely used in the field of p-adic analysis and naturally leads to the extension of Skolem's approach and to the definition of the ring $Z\left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$ or in fact to $R\left[\left[\binom{x}{1}, \binom{x}{2}, ...\right]\right]$ for any ring R. A study of these rings and of the many intriguing questions about them has been initiated by Todorka Nedeva.

References

1. Paul-Jean Cahen and Jean Luc Chabert, *Integer-Valued Polynomials*, Mathematical Surveys and Monographs, volume **48**, American Mathematical Society, 1996.

This book has full treatment of ring of integer valued polynomials with rational coefficients and of many other related topics.

2. Andrew Granville, *Arithmetic Properties of Binomial Coefficients I: Binomial coefficients modulo prime powers*, Comadian American Mathematical Society Conference Proceedings, volume **20** (1977), 253-275.

This article has a nice list of references, but many more can be found on the website:

http://www.DMS.UMontreal.CA/~Andrew/Binomial/index.html

3. Kurt Mahler, *Introduction to p-adic Numbers and their Functions*, Cambridge University Press, 1973.

This beautifully written book has a treatment of the symbols

$$a_0 + a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots$$

thought of as functions of the variable $n \in N$.

4. Todorka Nedeva, *Rings of power series in the* $\binom{x}{k}$*'s.*

This is a work in progress but as far as I know is the only treatment of the ring

$$Z\left[\left[\binom{x}{1}, \binom{x}{2}, \ldots\right]\right]$$

mentioned in the last section.

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