Uniqueness, nonpositivity and bounds for solutions of elliptic problems via the maximum principle

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Abstract

A class of nonlinear fourth order elliptic equations is considered. The classical maximum principle is used to deduce that certain functionals defined on solutions of the equation attain a maximum on the boundary of the domain. These maximum principles are then used to prove some uniqueness results and various a priori bounds.

1 Introduction

Several authors have used the idea to develop maximum principles for functionals defined on solutions of fourth and higher order elliptic equations(see [1]-[3],[5]-[9],[11],[12], [14],[15]-[18]).
In this paper we shall use this idea in the study of nonlinear fourth order equations of the form

$$\Delta^2 u - G(x, u, \Delta u) + F(x, u) = 0 \tag{1}$$

The maximum principle for second order elliptic equations is well known. Here (Section 2) we shall show that a similar result holds for solutions of boundary value problems involving equation (1), if F and G are selected appropriately. This is an extension of a result in [8]. Further, in Section 2, we extend some principles in [2] and [14].

In Section 3 we will be able to conclude from the elementary character of the result on maximum principles derived in Section 2, the uniqueness of solutions for some nonlinear boundary value problems. The nonpositivity of solutions of a nonlinear Dirichlet problem follows also from the maximum principle (see Section 4).
Section 5 of this paper indicates further possible applications of our maximum principles. For instance, we obtain a priori estimates for the gradient of the solution u and for Δu. Some estimates will lead bounds on quantities important in various physical problems. It is indicated in Section 6 how some results can be extended to the case when Δu is replaced by an uniformly elliptic operator.

2 Maximum Principles

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$ and let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution for the equation

$$\Delta^2 u - G(x, u, \Delta u) + F(x, u) = 0 \tag{2}$$

in Ω. We assume that $F(x, u) = \alpha(x) \cdot f(u)$, where α and f satisfy

$$\alpha \in C^2(\overline{\Omega}),\ \alpha > 0,\ \Delta\alpha \leq 0 \quad in\, \Omega \tag{3}$$

$$f \in C^1(\mathbb{R}),\ f \leq 0,\ f' \geq 0 \quad in\, \mathbb{R} \tag{4}$$

We define the function

$$\mathrm{P} = \frac{\int_0^{\Delta u} h(s)ds}{\alpha} + \int_0^u f(s)ds \tag{5}$$

where h is a smooth function to be specified later. Denoting the derivative with respect to x_i by a subscript i and using the summation convention we get

$$\mathrm{P}_{,i} = \frac{h(\Delta u) \cdot (\Delta u)_{,i}}{\alpha} - \frac{\alpha_{,i} \cdot \int_0^{\Delta u} h(s)ds}{\alpha^2} + f(u) \cdot u_{,i}$$

$$\Delta\mathrm{P} = \frac{h(\Delta u) \cdot \Delta^2 u}{\alpha} + \frac{h'(\Delta u) \cdot (\Delta u)_{,i} \cdot (\Delta u)_{,i}}{\alpha} - \frac{2\alpha_{,i} \cdot h(\Delta u)(\Delta u)_{,i}}{\alpha^2} -$$
$$- \frac{\Delta\alpha \cdot \int_0^{\Delta u} h(s)ds}{\alpha^2} + \frac{2\alpha_{,i}\alpha_{,i} \cdot \int_0^{\Delta u} h(s)ds}{\alpha^3} + f'(u)u_{,i}u_{,i} + f(u)\Delta u$$

Now using equation (2), adding and subtracting $\frac{(\Delta u)_{,i}(\Delta u)_{,i}}{\alpha}$, $\frac{\alpha_{,i}\alpha_{,i}h^2(\Delta u)}{\alpha^3}$ we obtain

$$\Delta\mathrm{P} = h(\Delta u) \cdot G(x, u, \Delta u) + \left| \frac{(\Delta u)_{,i}}{\alpha^{\frac{1}{2}}} - \frac{\alpha_{,i}h(\Delta u)}{\alpha^{\frac{3}{2}}} \right|^2 +$$
$$+ \frac{(\Delta u)_{,i}(\Delta u)_{,i}}{\alpha} \left[h'(\Delta u) - 1\right] + \frac{2\alpha_{,i}\alpha_{,i}}{\alpha^3} \left(\int_0^{\Delta u} h(s)ds - \frac{h^2(\Delta u)}{2} \right)$$
$$- \frac{\Delta\alpha}{\alpha^2} \int_0^{\Delta u} h(s)ds + f(u)\left(\Delta u - h(\Delta u)\right) + f'(u)u_{,i}u_{,i}.$$

If we assume that

$$h(s) \geq s, \quad h'(s) \geq 1 \quad in\, I = (a, b), \quad a < 0,\ b > 0 \tag{6}$$

$$\int_0^{\xi} h(s)ds \geq \frac{h^2(\xi)}{2} \quad \forall\, \xi\, \in\, I \tag{7}$$

$$h(s) \cdot G(x, t, s) \geq 0 \quad \forall\, (x, t, s)\, \in\, \Omega \times \mathbb{R} \times I \tag{8}$$

we have $\Delta\mathrm{P} \geq 0$ in Ω, and by the maximum principle for elliptic operators [4] we arrive at our first result:

THEOREM 1. *If u is a $C^4(\Omega) \cap C^2(\overline{\Omega})$ solution of (2) in Ω, where $F(x,u) = \alpha(x) \cdot f(u)$, $f \in C^1(\mathbb{R})$, $f \leq 0$, $f' \geq 0$ in $\mathbb{R}$, $\alpha \in C^2(\overline{\Omega})$, $\alpha > 0$, $\Delta\alpha \leq 0$ in Ω and if $h \in C^1(I)$ satisfies (6), (7), and (8) then*

$$\mathrm{P} = \frac{\int_0^{\Delta u} h(s)ds}{\alpha} + \int_0^u f(s)ds$$

takes its maximum on $\partial\Omega$. If $\alpha \equiv const$ in Ω then the condition (7) is not needed.

REMARK 1. If we take $h(s) = s$ (which clearly satisfies (6), (7)) the condition $f \leq 0$ in Theorem 1 can be omitted. Further, if we choose $G(x,t,s) = \beta(x) \cdot s^k$, $k = 1, 3, ...$, $\beta \geq 0$ in Ω we obtain the maximum principle derived in [5], Section 2.

REMARK 2. The special case $\mathrm{P}(x) = (\Delta u)^2 + 2\int_0^u f(s)ds$ was also used independently by the author in [2].

A consequence of Theorem 1 is the following weak maximum principle:

COROLLARY 2. *Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution for the problem*

$$\begin{cases} \Delta^2 u - G(x,u,\Delta u) + F(x,u) = 0 \quad in\ \Omega \\ \Delta u = 0 \quad on\ \partial\Omega \end{cases} \tag{9}$$

where F and G satisfy the requirements

$$F(x,u) = \alpha(x) \cdot f(u)$$

$$\alpha \in C^2(\Omega) \cap C^0(\overline{\Omega}),\ \alpha > 0\, in\, \Omega \tag{10}$$

$$f \in C^1(\mathbb{R}), \quad f > 0, \quad f' \geq 0 \quad in\, \mathbb{R} \tag{11}$$

$$s \cdot G(x,t,s) \geq 0 \quad in\, \Omega \times \mathbb{R} \times I \tag{12}$$

Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. In view of Theorem 1, the continuous function in $\overline{\Omega}$

$$\mathrm{P} = \frac{(\Delta u)^2}{2\alpha} + \int_0^u f(s)ds$$

attains its maximum on $\partial\Omega$, i.e.

$$\mathrm{P}(x) \leq \mathrm{P}(x_0)$$

for all $x \in \overline{\Omega}$ and for some $x_0 \in \partial\Omega$.
Since $\Delta u = 0$ on $\partial\Omega$ and $f > 0$ in $\mathbb{R}$ we obtain the desired result. $\square$

If the condition $f > 0$ in $\mathbb{R}$ is not satisfied, it is still possible to derive a similar maximum principle.

Corollary 3. *Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a solution for the boundary value problem*

$$\begin{cases} \Delta^2 u - G(x,u,\Delta u) + \alpha(x) \cdot u^k = 0 \quad in\, \Omega \\ \Delta u = 0 \quad on\, \partial\Omega \end{cases} \tag{13}$$

where $k = 1, 3, \ldots$, α and G satisfy the conditions (10) and (12).
Then

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Remark 3. It is of course possible to prove a strong maximum principle for solutions of the boundary value problems (9) and (13), i.e. if u is a non-constant solution of the problem (9) ((13)), then $u\,(|u|)$ cannot attain its maximum in any interior point of Ω.

The proof can be obtained as follows.
If $u \not\equiv const.$, then $|\nabla u|^2 = u_{,i}u_{,i} \not\equiv 0$. Since $f' \geq 0$ in $\mathbb{R}$ we see that $\Delta \mathrm{P} \not\equiv 0$ in Ω. Hence P is a non-constant function and we obtain the proof from the strong maximum principle of Hopf [13].

Remark 4. Our Corollary 3 contains the earlier result in [8].

Remark 5. We note that other maximum principles can be obtained if f is an odd, nondecreasing function (see [11]).

Remark 6. The condition $\Delta u = 0$ on $\partial\Omega$ cannot be omitted in Corollary 3. This is shown by the following one dimensional example:

$$\begin{cases} u^{(4)} + 4u = 0 \qquad in\, \Omega = (0, \frac{\pi}{3}) \\ u''(0) = -2 \\ u''(\frac{\pi}{3}) = 2e^{-\pi} \end{cases} \tag{14}$$

The function $u(x) = e^{-x} \cdot \sin(x)$ satisfies (14) and

$$\max_{\partial\Omega} |u| = u\left(\frac{\pi}{3}\right) < \max_{\overline{\Omega}} |u| = u\left(\frac{\pi}{4}\right)$$

We now consider the equation

$$\Delta^2 u - G(x,u,\Delta u) + F(x,u) = 0 \quad in\, \Omega \tag{15}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $G(x,u,\Delta u) = \varphi(u^2) \cdot \Delta u$, and show that under appropriate conditions the function

$$\mathrm{R} = 2|\nabla u|^2 - 2u\Delta u + \int_0^{u^2} \varphi(s)ds$$

takes its maximum on $\partial\Omega$.
We compute

$$\mathrm{R}_{,k} = 4u_{,i}u_{,ik} - 2u_{,k}\Delta u - 2u\Delta u_{,k} + 2uu_{,k}\varphi(u^2)$$

$$\begin{aligned}\Delta \mathrm{R} = 4u_{,i}u_{,ikk} + 4u_{,ik}u_{,ik} - 2(\Delta u)^2 - 4u_{,i}u_{,ikk} - 2u\Delta^2 u + \\ +2u\Delta u\varphi(u^2) + 2|\nabla u|^2\varphi(u^2) + 4u^2|\nabla u|^2\varphi'(u^2) = \\ = 4u_{,ik}u_{,ik} - 2(\Delta u)^2 + 2uF(x,u) + 2|\nabla u|^2\varphi(u^2) + 4u^2|\nabla u|^2\varphi'(u^2).\end{aligned}$$

Now, if φ and F satisfy

$$\varphi(0) \geq 0 \tag{16}$$

$$\varphi'(s) \geq 0 \quad for\ s \geq 0 \tag{17}$$

$$s \cdot F(x,s) \geq 0 \quad for (x,s) \in \Omega \times \mathbb{R} \tag{18}$$

we obtain that R is subharmonic in Ω, since in two dimensions, we have

$$2u_{,ij}u_{,ij} \geq (\Delta u)^2$$

Consequently, we deduce the following extension of Schaefer's result [14] which extends a classical result of Miranda (see [19], p.175).

THEOREM 4. *Let Ω be a bounded domain in $\mathbb{R}^2$. If $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of (15), where the function $\varphi \in C^1(\mathbb{R})$ satisfies (16), (17) and F satisfies (18), then*

$$\mathrm{R} = 2|\nabla u|^2 - 2u\Delta u + \int_0^{u^2} \varphi(s)ds$$

assumes its maximum on $\partial\Omega$.

REMARK 7. If $\varphi \equiv 1$ and $F(x,u) = f(u)$, where $sf(s) \geq 0$, $\forall s \in \mathbb{R}$ then Theorem 3 in [2] becomes a particular case of our theorem.

The following theorem now generalizes Theorem 1 in [2].

THEOREM 5. *Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of (2), where Ω is a bounded domain in $\mathbb{R}^N$, $N \leq 4$. If $G(x,u,\Delta u) = \gamma\Delta u$, $\gamma \geq 0$ and if $f \in C^1(\mathbb{R})$ is an increasing function, then*

$$\mathrm{S} = \gamma|\nabla u|^2 - 2\nabla u\nabla(\Delta u) + 2u_{,ij}u_{,ij}$$

attains its maximum value on the boundary of Ω.

The proof of the preceding theorem is based on an inequality due to Payne [12] and the maximum principle. See [2] for details.
Theorem 5 may be used to derive gradient bounds (see Section 5).

3 Uniqueness results

Often we deduce uniqueness theorems for second order boundary value problems with the help of maximum principles.
A corresponding remark is true in our case.

Corollary 6. *Suppose that α, f, G satisfy the requirements of Theorem 1, except $f \le 0$ in $\mathbb{R}$. If $f(0) = 0$ and $G(x,0,0) = 0$, $\forall x \in \Omega$, then the trivial solution is the only classical solution of the problem*

$$\begin{cases} \Delta^2 u - G(x,u,\Delta u) + \alpha(x)\cdot f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

The proof is achieved exactly as that of Theorem 2, [5].

Corollary 7. *The boundary value problem*

$$\begin{cases} \Delta^2 u - A(x,\Delta u) + \alpha(x)\cdot u = \varphi(x) & \text{in } \Omega \\ u = \psi(x) & \text{on } \partial\Omega \\ \Delta u = \chi(x) & \text{on } \partial\Omega \end{cases} \tag{20}$$

where
i) $\varphi, \chi \in C^0(\overline{\Omega}), \psi \in C^2(\overline{\Omega})$,
ii) $\alpha > 0$ and $\Delta\alpha \le 0$ in Ω,
iii) the function $A = A(x,z)$ is non-increasing in z for every $x \in \Omega$,
iv) the function $A = A(x,z)$ is continuously differentiable with respect to the z variable in $\Omega \times \mathbb{R}$,
has a unique solution.

Proof. If u and v are two solutions of (20), the difference $w = u - v$ satisfies the homogeneous problem

$$\begin{cases} \Delta^2 w + \beta(x)\Delta w + \alpha(x)\cdot w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \\ \Delta w = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

where $\beta \le 0$ in Ω. Note that we have used the mean value theorem. Using Corollary 6 we obtain $w \equiv 0$ in Ω. Hence $u = v$. $\square$

With the help of the Theorem 4 we can now prove the following extensions of Theorem 4 and Theorem 5 in [14].

Corollary 8. *Let Ω be a bounded plane domain, with smooth boundary. If $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} \Delta^2 u - \varphi(u^2)\Delta u + F(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{22}$$

where φ, F satisfy the conditions of Theorem 4 and $F(x,0) = 0$ in Ω, then $u \equiv 0$.
If $F(x,0) \neq 0$ for some $x \in \Omega$, then no classical solution of (22) exists.

Proof. By Theorem 4 we have

$$2|\nabla u|^2 - 2u\Delta u + \int_0^{u^2} \varphi(s)ds \leq 0 \quad in\, \Omega.$$

Integrating over Ω, we obtain

$$4\int_\Omega |\nabla u|^2 + \int_\Omega \left(\int_0^{u^2} \varphi(s)ds \right) \leq 0$$

and hence $|\nabla u| \equiv 0$ in Ω. Consequently $u \equiv 0$ in Ω(because we seek only smooth solutions). □

If $F(x,0) = 0,\ \forall\, x \in \Omega$ and φ, F satisfy the requirements of Theorem 4 we then obtain the following result.

Corollary 9. *The only $C^4(\Omega) \cap C^2(\overline{\Omega})$ solution of the boundary value problem*

$$\begin{cases} \Delta^2 u - \varphi(u^2)\Delta u + F(x,u) = 0 & in\, \Omega \\ u = 0 & on\, \partial\Omega \\ \Delta u = 0 & on\, \partial\Omega \end{cases} \tag{23}$$

is the trivial solution. Here Ω is assumed to be a bounded smooth plane domain, with curvature K of ∂D positive.

Proof. In view of Theorem 4 the function R attains its maximum at a point P on $\partial\Omega$. We employ the Hopf maximum principle [13] to obtain $\frac{\partial \mathrm{R}}{\partial n}(P) > 0$ if R is not a constant in $\overline{\Omega}$.
Since $u = 0$ on $\partial\Omega$ we have $|\nabla u| = |\frac{\partial u}{\partial n}|$ on $\partial\Omega$, and hence

$$\frac{\partial \mathrm{R}}{\partial n} = 4 \cdot \frac{\partial u}{\partial n} \cdot \frac{\partial^2 u}{\partial n^2} \quad on\, \partial\Omega.$$

Now we follow Schaefer [14].
By the boundary conditions the relation

$$\frac{\partial^2 u}{\partial n^2} + K\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = \Delta u \quad on\, \partial\Omega \quad \text{(see [19], p.46)}$$

becomes

$$\frac{\partial^2 u}{\partial n^2} = -K\frac{\partial u}{\partial n} \quad on\, \partial\Omega.$$

Thus at P we find

$$\frac{\partial \mathrm{R}}{\partial n} = -4K\left(\frac{\partial u}{\partial n}\right)^2 \tag{24}$$

which is a contradiction. Consequently, $\mathrm{R} \equiv const$ in $\overline{\Omega}$.
If $\mathrm{R} \equiv const$ in $\overline{\Omega}$, we obtain $\frac{\partial \mathrm{R}}{\partial n} = 0$ on $\partial\Omega$.
By (24) it follows that $|\nabla u|^2 = 0$ on $\partial\Omega$ and hence

$$\mathrm{R} \equiv 0 \quad in\ \overline{\Omega}.$$

The result follows on integrating over Ω. □

4 Nonpositivity

In [5] the functional $\mathrm{P} = \frac{(\Delta u)^2}{p} + 2\int_0^u f(s)ds$ was used to deduce that $u \leq 0$ in $\overline{\Omega}$ if u is a classical solution of

$$\begin{cases} \Delta^2 u - q(x)(\Delta u)^K + p(x)f(u) = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \\ \Delta u = 0 & on\ \partial\Omega \end{cases}$$

where $f > 0$ in $\mathbb{R}$, $f' \geq 0$ in $\mathbb{R}$, $p > 0$, $\Delta p \leq 0$, $q \geq 0$, in Ω and $K = 2m-1 > 0$.
We relax here the boundary conditions and state:

Corollary 10. *If $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} \Delta^2 u - G(x, u, \Delta u) + \alpha(x)f(u) = 0 & in\ \Omega \\ u \leq 0 & on\ \partial\Omega \\ \Delta u = 0 & on\ \partial\Omega \end{cases}$$

under the conditions of Corollary 2, then $u \leq 0$ in $\overline{\Omega}$.

5 Bounds

We may use the functional S to derive bounds for the gradient of the solution of the boundary value problem

$$\begin{cases} \Delta^2 u - \gamma\Delta u + f(u) = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega \end{cases} \tag{25}$$

under the conditions of Theorem 5.
Following Payne [12], we can show that

$$\max_{\Omega} |\nabla u|^2 \leq C \cdot \max_{\partial\Omega}(\Delta u)^2 \tag{26}$$

where u is a solution of (25) and γ is a positive constant.
Note that the constant C depends only on the diameter of Ω.

Remark 8. If $\gamma = 0$ and $f(u) = -\delta$, $\delta > 0$, the problem (25) may be interpreted as the equation of a thin elastic plate under a constant load, clamped on the boundary.

From the subharmonicity of the functional P we obtain bounds for Δu for the equation:

$$\Delta^2 u - G(x,u,\Delta u) + \alpha(x) f(u) = 0 \quad in\ \Omega \tag{27}$$

under the conditions of Corollary 2 (except $f > 0$ in $\mathbb{R}$) and $f(0) = 0$.

$$\max_{\overline{\Omega}} \frac{(\Delta u)^2}{\alpha} \leq \max_{\partial\Omega} \frac{(\Delta u)^2}{\alpha} + 2 \max_{\partial\Omega} \left(\int_0^u f(s) ds \right)$$

If the nonlinearities G and f satisfy $G(x,u,\Delta u) \equiv 0$ and $f(u) = K_1 u + K_2 u^3$, the equation (27) where $\alpha \equiv 1$, K_1, K_2 are positive constants, occurs in the bending of cylindrical shells and in plate theory [10].
If u is a smooth solution of

$$\Delta^2 u - G(x,u,\Delta u) + \alpha(x) f(u) = 0 \quad in\ \Omega$$

such that $u(y) \geq 0$ for some $y \in \overline{\mathbb{R}}$, then Theorem 1 tells us that

$$\max_{\overline{\Omega}} \left(\int_0^{\Delta u} h(s) ds \right) \leq \max_{\partial\Omega} \left(\int_0^{\Delta u} h(s) ds \right) + \max_{\partial\Omega} \left(\int_0^u f(s) ds \right).$$

Here $\alpha \equiv const. > 0$, $h(s) \leq s$, $h'(s) \geq 1$ in I, $f, f' \geq 0$ in $\mathbb{R}$ and (8) is fulfilled. Note that such a function h exists. For example: $h(s) = s - \frac{1}{s+\gamma}$, $(s > -\gamma)$, $\gamma > 0$.
As a final consequence of our maximum principles we consider the problem

$$\begin{cases} \Delta^2 u - \gamma \Delta u + f(u) = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega \end{cases} \tag{28}$$

under the conditions of Theorem 5.
Using the relation $\int_\Omega u_{,ij} u_{,ij} = \int_\Omega (\Delta u)^2$ if $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ and Theorem 5 we obtain

$$\gamma \int_\Omega |\nabla u|^2 + 2 \int_\Omega (\Delta u)^2 \leq 2\mathcal{A} \max_{\partial\Omega} (\Delta u)^2,$$

where $\mathcal{A}$ is the area(volume) of Ω.
Choosing $\gamma = 0$ and $f(u) = -c$, where c is a positive constant, we obtain a bound for the potential energy of the plate in the clamped plate problem, namely

$$\mathrm{E}_p = \int_\Omega (\Delta u)^2 \leq \mathcal{A} \max_{\partial\Omega} (\Delta u)^2 \tag{29}$$

where $\mathcal{A}$ is the area of Ω.

Remark 9. A sharper form of (29) was obtained in [12].

6 Concluding remarks

It is possible to extend Theorem 1 to the case of more general elliptic equations

$$\mathrm{L}(\mathrm{L}u) - G(x, u, \mathrm{L}u) + \alpha(x)f(u) = 0 \quad in\,\Omega \tag{30}$$

where $\mathrm{L}u = a_{ij}(x)u_{,ij}$, L is uniformly elliptic in Ω, i.e. $a_{ij}(x)\xi_i\xi_j \geq \lambda\xi_i\xi_i$ for any vector $\xi = (\xi_1, ..., \xi_N)$ and some constant $\lambda > 0$ and $a_{ij} \in C^2(\overline{\Omega})$.
The function $\mathrm{P} = \frac{\int_0^{\mathrm{L}u} h(s)ds}{\alpha} + \int_0^u f(s)ds$ can be used. P satisfies

$$\begin{array}{c} a_{ij}\mathrm{P}_{,ij} = \frac{h(\mathrm{L}u)G(x,u,\mathrm{L}u)}{\alpha} + \frac{a_{ij}(\mathrm{L}u)_{,i}(\mathrm{L}u)_{,j}}{\alpha}\left(h'(\mathrm{L}u) - 1\right) + \\ \frac{2\alpha_{,i}\alpha_{,j}a_{ij}}{\alpha^3}\left(\int_0^{\mathrm{L}u} h(s)ds - \frac{h^2(\mathrm{L}u)}{2}\right) - \frac{a_{ij}\alpha_{,ij}}{\alpha^2}\int_0^{\mathrm{L}u} h(s)ds \\ + \frac{a_{ij}}{\alpha}\left[(\mathrm{L}u)_{,i} - \frac{h(\mathrm{L}u)}{\alpha}\alpha_{,i}\right]\left[(\mathrm{L}u)_{,j} - \frac{h(\mathrm{L}u)}{\alpha}\alpha_{,j}\right] \\ + f(u)(\mathrm{L}u - h(\mathrm{L}u)) + a_{ij}u_{,i}u_{,j}f'(u) \end{array} \tag{31}$$

If $\alpha \in C^2(\Omega)$, $\alpha > 0$, $\mathrm{L}(\alpha) \leq 0$ in Ω and f, h satisfy (4),(6)-(8), then the right side of (31) can be made nonnegative.
In the paper [11], the authors obtained similar results for more equations of the form

$$\mathrm{L}(b(x)g(u)\mathrm{L}u) - G(x, u, u_{,i}, \mathrm{L}u) + \alpha(x)f(u) = 0 \quad in\,\Omega$$

but under the restriction $h(s) = s$.

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