Some properties of solutions for the generalized thin film equation in one space dimension

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Abstract

In this paper, the author studies a generalized thin film equation in one space dimension. Some results on the finite speed of propagation of perturbations and regularity of solutions are established.

1 Introduction

In this paper, we consider the variant version of the thin film equation, namely

$$\frac{\partial u}{\partial t} + \operatorname{div}(|\nabla \Delta u|^{p-2} \nabla \Delta u) = 0, \quad x \in \Omega,\ t > 0,\ p > 2, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

The equation (1.1) is a typical higher order equations, which have a sharp physical background and a rich theoretical connotation. It was J.R.King [6] who first derived the equation. Equation (1.1) describes the surface tension driven evolution of the height $u(x,t)$ of a thin liquid film on a solid surface in lubrication approximation [6, 7, 9]. The exponent p is related to the rheological properties of the liquid: $p = 2$ corresponds to a Newtonian liquid, whereas $p \neq 2$ emerges when considering "power-law" liquids. When $p > 2$ the liquid is said to be "shear-thinning".

J.R.King [6] studied Cauchy problem of the equation for one-dimensional, exploiting local analyses about the edge of the support and special closed form solutions such as travelling waves, separable solutions, instantaneous source solutions.

On the basis of physical consideration, as usual the equation (1.1) is supplemented with the natural boundary value conditions

$$u = \Delta u = 0, \quad x \in \partial\Omega,\ t > 0. \tag{1.2}$$

The boundary value conditions (1.2) is a reasonable for the thin film equation or the Cahn-Hilliard equation, (see [1, 2, 4]) and initial value condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

This equation is something quite like the p-Laplacian equation, but many methods used in the p-Laplacian equation such as those methods based on maximun principle are no longer valid for this equation. Because of the degeneracy, the problem (1.1)-(1.3) does not admit classical solutions in general. So, we introduce weak solutions in the following sense

Definition A function u is said to be a weak solution of the problem (1.1)–(1.3), if the following conditions are satisfied:

1) $u \in L^\infty(0,T;W^{3,p}(\Omega)) \cap C(0,T;L^2(\Omega)), u, \Delta u \in W_0^{1,p}(\Omega)$,
$\frac{\partial u}{\partial t} \in L^\infty(0,T;W^{-1,p'}(\Omega))$, where p' is the conjugate exponent of p;

2) For any $\varphi \in C_0^\infty(Q_T)$, the following integral equality holds:

$$\iint_{Q_T} u\frac{\partial \varphi}{\partial t}\,dxdt + \iint_{Q_T} |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi\,dxdt = 0;$$

3) $u(x,0) = u_0(x)$, in $L^2(\Omega)$

In [8] they prove the existence and uniqueness of weak solution for dimension $N \leq 2$. This paper is a further step in the study of the properties of solutions, we proved the finite speed of propagation of perturbations and regularity of solutions of the problem (1.1)-(1.3) for one dimensional case.

In addition, through out this paper, we set $I = (0,1)$.

2 Finite speed of propagation of perturbations

In this section, we are going to prove the following theorem.

Theorem 2.1 *Assume $p > 2$, $supp\, u_0 \subset [x_1, x_2]$, $0 < x_1 < x_2 < 1$, and u is the weak solution of the problem (1.1)-(1.3), then for any fixed $t > 0$, we have*

$$supp\, u(x,\cdot) \subset [x_1(t), x_2(t)] \cap [0,1],$$

where $x_1(t) = x_1 - C_1 t^{\frac{1}{3p-2}}$, $x_2(t) = x_2 + C_2 t^{\frac{1}{3p-2}}$,

$$C_1 = C\left(\int_0^T\int_0^{x_1}|D^3u|^p\,dxd\tau\right)^{\frac{p-2}{2(3p-2)}}, \quad C_2 = C\left(\int_0^T\int_{x_2}^{1}|D^3u|^p\,dxd\tau\right)^{\frac{p-2}{2(3p-2)}},$$

$$C = 2^{3p-2}(2p+1)^p(p-1)^{p-1}(1+p^{-p}).$$

To prove the Theorem 2.1 we need the following result

Lemma 2.2 *The weak solution u of the problem (1.1)-(1.3), satisfying for any $0 \leq \rho \in C^2(\overline{I})$,*

$$\frac{1}{2}\int_0^1 \rho(x)|Du(x,t)|^2\,dx - \frac{1}{2}\int_0^1 \rho(x)|Du_0(x)|^2\,dx$$

$$= \quad -\iint_{Q_t} |D^3u|^{p-2}D^3uD^2(\rho(x)Du)\,dx,$$

where $Q_t = (0,1) \times (0,t)$, $D = \frac{\partial}{\partial x}$.

Proof. Similar to the discussion in [8], we can also easily prove that for any $0 \le \rho \in C^2(\overline{\Omega})$,

$$f_\rho(t) = \frac{1}{2}\int_\Omega \rho(x)|Du(x,t)|^2\,dx \in C([0,T]).$$

Consider the functional

$$\Phi_\rho[v] = \frac{1}{2}\int_\Omega \rho(x)|Dv(x)|^2\,dx.$$

It is easy to see that $\Phi_\rho[v]$ is a convex functional on $H_0^1(\Omega)$.

For any $\tau \in (0,T)$ and $h > 0$, we have

$$\Phi_\rho[u(\tau+h)] - \Phi_\rho[u(\tau)] \ge \langle u(\tau+h) - u(\tau), -D(\rho(x)Du(\tau))\rangle.$$

By $\frac{\delta\Phi_\rho[v]}{\delta v} = -D(\rho(x)Dv)$, for any fixed $t_1, t_2 \in [0,T], t_1 < t_2$, integrating the above inequality with respect to τ over (t_1,t_2) , we have

$$\int_{t_2}^{t_2+h}\Phi_\rho[u(\tau)]\,d\tau - \int_{t_1}^{t_1+h}\Phi_\rho[u(\tau)]\,d\tau \ge \int_{t_1}^{t_2}\langle u(\tau+h) - u(\tau), -D(\rho(x)Du(\tau))\rangle\,d\tau.$$

Multiplying both sides of the above equality by $\frac{1}{h}$, and letting $h \to 0$, we obtain

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \ge \int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t}, -D(\rho(x)Du(\tau))\rangle\,d\tau.$$

Similarly, we have

$$\Phi_\rho[u(\tau)] - \Phi_\rho[u(\tau-h)] \le \langle(u(\tau) - u(\tau-h)), -D(\rho(x)Du(\tau))\rangle.$$

Thus

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \le \int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t}, -D(\rho(x)Du(\tau))\rangle\,d\tau,$$

and hence

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] = \int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t}, -D(\rho(x)Du(\tau))\rangle\,d\tau.$$

Taking $t_1 = 0, t_2 = t$, we get from the definition of solutions that

$$\begin{aligned}\Phi_\rho[u(t)] - \Phi_\rho[u(0)] &= \int_0^t\langle -D(|D^3u|^{p-2}D^3u), -D(\rho(x)Du(\tau))\rangle\,d\tau\\ &= -\int_0^t\langle |D^3u|^{p-2}D^3u, D^2(\rho(x)Du(\tau))\rangle\,d\tau.\end{aligned}$$

This completes the proof.

Proof of Theorem 2.1 By Lemma 2.2, take $\rho(x) = (x-y)_+^s,\ y \in [x_2, 1)$, we have

$$\frac{1}{2}\int_0^1 (x-y)_+^s |Du(x,t)|^2\,dx = -\int_0^t\int_0^1 |D^3u|^{p-2}D^3uD^2[(x-y)_+^s Du)]\,dxd\tau.$$

Denote the left side of above equality by J, then we have

$$\begin{aligned}
J &= -\int_0^t\int_0^1 |D^3u|^{p-2}D^3uD^2[(x-y)_+^s Du]\,dxd\tau \\
&= -\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau \\
&\quad -2s\int_0^t\int_0^1 (x-y)_+^{s-1}D^2u|D^3u|^{p-2}D^3u\,dxd\tau \\
&\quad -s(s-1)\int_0^t\int_0^1 (x-y)_+^{s-2}|D^3u|^{p-2}D^3uDu\,dxd\tau \\
&\le -\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau + \frac{1}{4}\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau \\
&\quad +C_1\int_0^t\int_0^1 (x-y)_+^{s-p}|D^2u|^p\,dxd\tau \\
&\quad +\frac{1}{4}\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau + C_2\int_0^t\int_0^1 (x-y)_+^{s-2p}|Du|^p\,dxd\tau \\
&\le -\frac{1}{2}\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau + C_1\int_0^t\int_0^1 (x-y)_+^{s-p}|D^2u|^p\,dxd\tau \\
&\quad +C_2\int_0^t\int_0^1 (x-y)_+^{s-2p}|Du|^p\,dxd\tau,
\end{aligned}$$

using Hardy inequality [5], we have

$$\int_0^1 (x-y)_+^{s-2p}|Du|^p\,dx \le C\int_0^1 (x-y)_+^{s-p}|D^2u|^p\,dx.$$

Hence

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (x-y)_+^s|Du|^p\,dx + \frac{1}{2}\int_0^t\int_0^1 (x-y)_+^s|D^3u|^p\,dxd\tau \\
\le\ & C_1\int_0^t\int_0^1 (x-y)_+^{s-p}|D^2u|^p\,dxd\tau.
\end{aligned} \tag{2.1}$$

Thus

$$\sup_{0<\tau\le t}\int_0^1 (x-y)_+^s|Du|^p\,dx \le C_1\int_0^t\int_0^1 (x-y)_+^{s-p}|D^2u|^p\,dxd\tau, \tag{2.2}$$

and

$$\int_0^t \int_0^1 (x-y)_+^s |D^3u|^p \,dxd\tau \le C_1 \int_0^t \int_0^1 (x-y)_+^{s-p} |D^2u|^p \,dxd\tau. \tag{2.3}$$

For (2.2) again using Hardy inequality, we have

$$\sup_{0<\tau\le t} \int_0^1 (x-y)_+^s |Du|^p \,dx \le C \int_0^t \int_0^1 (x-y)_+^s |D^3u|^p \,dxd\tau. \tag{2.4}$$

Let

$$E_s(y) = \int_0^t \int_0^1 (x-y)_+^s |D^3u|^p \,dxd\tau, \quad E_0(y) = \int_0^t \int_y^1 |D^3u|^p \,dxd\tau.$$

In (2.3), set $s = 2p+1$ and using Nirenberg inequality [3], we have

$$\begin{aligned} & E_{2p+1}(y) \\ \le\ & C_1 \iint_{Q_t} (x-y)_+^{p+1} |D^2u|^p \,dxd\tau \\ \le\ & C \int_0^t \left(\int_0^1 (x-y)_+^{p+1} |D^3u|^p \,dx \right)^a \left(\int_\Omega (x-y)_+^{p+1} |Du|^2 \,dx \right)^{(1-a)p/2} d\tau, \end{aligned}$$

where $\frac{1}{p} = \frac{1}{p+2} + a(\frac{1}{p} - \frac{2}{p+2}) + (1-a)\frac{1}{2}$, therefore

$$0 < a = \frac{\frac{1}{p} - \frac{1}{p+2} - \frac{1}{2}}{\frac{1}{p} - \frac{2}{p+2} - \frac{1}{2}} < 1.$$

Using (2.4) we obtain

$$\begin{aligned} & E_{2p+1}(y) \\ \le\ & C \left(\iint_{Q_t} (z-z_0)_+^{p+1} |D^3u|^p \,dxd\tau \right)^{(1-a)p/2} \int_0^t \int_0^1 ((x-y)_+^{p+1} |D^3u|^p \,dx)^a \,d\tau \\ \le\ & C[E_{p+1}(y)]^{(1-a)p/2} \left(\iint_{Q_t} (x-y)_+^{p+1} |D^3u|^p \,dxd\tau \right)^a t^{1-a} \\ \le\ & C E_{p+1}(y)^{(1-a)p/2+a} t^{1-a}. \end{aligned}$$

Denote $\lambda = 1-a$, $\gamma = a + (1-a)p/2$. Applying Hölder's inequality, we have

$$\begin{aligned} & E_{2p+1}(y) \\ \le\ & C t^\lambda \left[\iint_{Q_t} (x-y)_+^{p+1} |D^3u|^p \,dxd\tau \right]^\gamma \\ \le\ & C t^\lambda \left[\iint_{Q_t} (x-y)_+^{2p+1} |D^3u|^p \,dxd\tau \right]^{\frac{(p+1)\gamma}{(2p+1)}} \left[\int_0^t \int_y^1 |D^3u|^p \,dxd\tau \right]^{\frac{p\gamma}{(2p+1)}} \\ \le\ & C t^\lambda [E_{2p+1}(y)]^{(p+1)\gamma/(2p+1)} [E_0(y)]^{p\gamma/(2p+1)}. \end{aligned}$$

Therefore

$$E_{2p+1}(y) \le Ct^{\lambda/\sigma}[E_0(y)]^{p\gamma/((2p+1)\sigma)}, \qquad \sigma = 1 - \frac{p+1}{2p+1}\gamma > 0.$$

Using Hölder's inequality again, we get

$$E_1(y) \le [E_{2p+1}(y)]^{1/(2p+1)}[E_0(y)]^{2p/(2p+1)} \le Ct^{\gamma_1}[E_0(y)]^{1+\theta},$$

where

$$\gamma_1 = \frac{\lambda}{\sigma(2p+1)}, \quad \theta = \frac{p\gamma}{\sigma(2p+1)^2} - \frac{1}{2p+1} > 0.$$

Noticing that $E_1'(y) = -E_0(y)$, we obtain

$$E_1'(y) \le -Ct^{-\gamma_1/(\theta+1)}[E_1(y)]^{1/(\theta+1)}.$$

If $E_1(x_2) = 0$, then supp$u \subset [0, x_2]$. If $E_1(x_2) > 0$, then there exists a maximal interval (x_2, x_2^*) in which $E_1(y) > 0$, $E_1(x_2^*) = 0$ and

$$\Big[E_1(y)^{\theta/(\theta+1)}\Big]' = \frac{\theta}{\theta+1}\frac{E_1'(y)}{[E_1(y)]^{1/(\theta+1)}} \le -Ct^{-\gamma_1/(\theta+1)}.$$

Integrating the above inequality over (x_2, x_2^*), we have

$$E_1(x_2^*)^{\theta/(\theta+1)} - E_1(x_2)^{\theta/(\theta+1)} \le -Ct^{-\gamma_1/(\theta+1)}(x_2^* - x_2),$$

which implies that

$$x_2^* \le x_2 + Ct^{\mu}(E_0(x_2))^{\theta/(\theta+1)} \equiv x_2(t), \quad \mu = \frac{\gamma_1}{\theta+1} = \frac{1}{3p-2} > 0.$$

Results in [8] imply that $E_0(y)$ can be controlled by a constant C independent of y. Therefore

$$\text{supp}u(\cdot, t) \subset [0, x_2(t)].$$

Similarly, we have

$$\text{supp}u(\cdot, t) \subset [x_1(t), 1].$$

We have thus completed the proof of Theorem 2.1.

3 Regularity of solutions

Theorem 3.1 *If u is weak solution of the problem(1.1)-(1.3), then for any (x_1, t_1), $(x_2, t_2) \in Q_T$, we have*

$$|u(x_1, t_1) - u(x_2, t_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{1/2}),$$

where C is a constant depending only on p.

Proof. Let

$$u_\varepsilon(x,t) = J_\varepsilon u(x,t) = \int_0^T \int_{|x-y|<\varepsilon} j_\varepsilon(x-y,t-s)u(y,s)\,dyds$$

where $j_\varepsilon(x,t)$ is a mollifier.

For any $x_1, x_2 \in I$, we have

$$\begin{aligned}
& u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t) \\
= & \int_0^T \int_R j_\varepsilon(x_1-y,t-s)u(y,s)dyds - \int_0^T \int_R j_\varepsilon(x_2-y,t-s)u(y,s)\,dyds \\
= & \int_0^T \int_R \frac{\partial j_\varepsilon(zx_1+(1-z)x_2-y,t-s)}{\partial z} u(y,s)\,dzdyds \\
= & \int_0^T \int_R \int_0^1 D_x j_\varepsilon(zx_1+(1-z)x_2-y,t-s)(x_1-x_2)u(y,s)\,dzdyds \\
= & -\int_0^T \int_R \int_0^1 D_y j_\varepsilon(zx_1+(1-z)x_2-y,t-s)(x_1-x_2)u(y,s)\,dzdyds \\
= & \int_0^T \int_R \int_0^1 j_\varepsilon(zx_1+(1-z)x_2-y,t-s)D_y u(y,s)\,dzdyds(x_1-x_2).
\end{aligned}$$

Therefore

$$\begin{aligned}
& |u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \\
\le & \int_0^T \int_R \int_0^1 |j_\varepsilon(zx_1+(1-z)x_2-y,t-s)||D_y u(y,s)|\,dzdyds|x_1-x_2|,
\end{aligned}$$

by $u \in L^\infty(0,T;W^{3,p}(\Omega))$, hence using Sobolev embedding theorem, we have $\frac{\partial u}{\partial x} \in L^\infty(Q_T)$ and $u \in L^\infty(Q_T)$. Thus we obtain

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le C|x_1-x_2|. \tag{3.1}$$

Set $0<\varepsilon<t_1<t_2<T$. Let $\Delta t = t_2-t_1$, $I_\rho = I_{(\Delta t)^{1/2}}(x_0) = (x_0-(\Delta t)^{1/2}, x_0+(\Delta t)^{1/2})$, $x_0 \in I$, choose ρ sufficiently small, such that $I_\rho \subset I, \varphi \in C_0^1(I_\rho)$, we

can obtain

$$\begin{aligned}
&\int_{I_\rho} \varphi(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\,dx \\
= &\int_{I_\rho} \varphi(x)\int_0^1 \frac{\partial u_\varepsilon(x,st_2+(1-s)t_1)}{\partial s}\,dsdx \\
= &\Delta t\int_{I_\rho} \varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon} u(y,\tau)\cdot \\
&\qquad \cdot j_{\varepsilon t}(x-y,st_2+(1-s)t_1-\tau)\,dyd\tau dsdx \\
= &-\Delta t\int_{I_\rho} \varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon} u(y,\tau)\cdot \\
&\qquad \cdot j_{\varepsilon \tau}(x-y,st_2+(1-s)t_1-\tau)\,dyd\tau dsdx.
\end{aligned} \tag{3.2}$$

Fixed $(x,t)\in Q_T,\ 0<\varepsilon<t<T-\varepsilon$, we have $j_\varepsilon(x-y,t-\tau)\in C_0^1(Q_T)$, from definition of weak solution

$$\begin{aligned}
&\int_0^T\int_{|x-y|<\varepsilon} j_{\varepsilon\tau}(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)\,dyd\tau \\
= &-\int_0^T\int_{|x-y|<\varepsilon} |D_y^3u|^{p-2}D_y^3uD_yj_\varepsilon(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)\,dyd\tau,
\end{aligned}$$

hence (3.2) is converted into

$$\begin{aligned}
&\int_{I_\rho} \varphi(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\,dx \\
= &\Delta t\int_{I_\rho} \varphi(x)\int_0^1\int_0^T\int_{|x-y|<\varepsilon} |D_y^3u|^{p-2}D_y^3u\cdot \\
&\qquad \cdot D_yj_\varepsilon(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)\,dyd\tau dsdx \\
= &\Delta t\int_0^1\int_{I_\rho} D_x\varphi(x)\int_0^T\int_{|x-y|<\varepsilon} |D_y^3u|^{p-2}D_y^3u\cdot \\
&\qquad \cdot j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)\,dyd\tau dxds.
\end{aligned}$$

Taking

$$\varphi(x)=\varphi_h(x)=\int_{-h}^{(\Delta t)^{1/2}-|x-x_0|-2h} \delta_h(s)\,ds,$$

where $\delta(s)\in C_0^1(R);\ \delta(s)\ge 0;\ \delta(s)=0$, as $|s|\ge 1;\int_R\delta(s)\,ds=1$. For $h>0$ define $\delta_h(s)=\frac{1}{h}\delta(\frac{s}{h})$.

Hence

$$\int_{I_\rho} \varphi_h(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))\,dx$$

$$= \quad \Delta t \int_0^1 \int_{I_\rho} \delta_h((\Delta t)^{1/2} - |x - x_0| - 2h) \frac{x_0 - x}{|x - x_0|} J_\varepsilon(|D^3 u|^{p-2} D^3 u)\, dxds,$$

Noting that for $x \in I_\rho$, $\lim\limits_{h\to 0} \varphi_h(x) = 1$, and if $|x - x_0| < (\Delta t)^{1/2} - h$, then $\delta_h((\Delta t)^{1/2} - |x - x_0| - 2h) = 0$. $\delta_h \le \frac{C}{h}$, and

$$m(I_\rho \backslash I_{(\Delta t)^{1/2} - |x - x_0| - 2h}) \le Ch.$$

By $J_\varepsilon(|D^3 u|^{p-2} D^3 u) \le C$, therefore

$$\left| \int_{I_\rho} \varphi_h(x)(u_\varepsilon(x, t_2) - u_\varepsilon(x, t_1))\, dx \right| \le C\Delta t.$$

Letting $h \to 0$, we obtain

$$\left| \int_{I_\rho} (u_\varepsilon(x, t_2) - u_\varepsilon(x, t_1))\, dx \right| \le C\Delta t.$$

Applying the mean value theorem, we see that for some $x^* \in I_\rho$ such that

$$|u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| \le C(\Delta t)^{1/2}.$$

Taking this into account and using (3.1), it follows that

$$\begin{aligned} & |u_\varepsilon(x, t_2) - u_\varepsilon(x, t_1)| \\ \le \;& |u_\varepsilon(x, t_2) - u_\varepsilon(x^*, t_2)| + |u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| + |u_\varepsilon(x^*, t_1) - u_\varepsilon(x, t_1)| \\ \le \;& C(\Delta t)^{1/2}, \end{aligned}$$

letting $\varepsilon \to 0$, we known that u is Hölder continuous. This completes the proof.

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